A MATHEMATICAL COMPENDIUM WITH APPLICATIONS

NICOLAS LAOS

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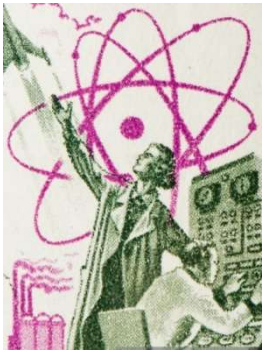

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Foreword

The word “mathematics” comes from the Greek word “manthānein,” which means “to learn.” Mathematics is mainly about forming ways to see problems in order to solve them by combining logical rigor, imagination, and intuition. Furthermore, mathematics is a peculiar sense that enables us to perceive realities that would otherwise be inaccessible to us. In fact, mathematics is our sense for patterns, relations, and logical connections.

The development of mathematical intuition depends on learning the basic concepts (thus, creating a powerful intellectual toolbox), using your intellectual toolbox in order to solve problems, and creative thinking (rather than simply memorizing mathematical tools).

Regarding my competence in mathematics, mathematical modelling, and epistemology, I would like to acknowledge the contribution of the following persons to my scientific education: Dr. Themistocles M. Rassias (Fellow of the Royal Astronomical Society of London and Accademico Ordinario of the Accademia Tiberina in Rome), who taught me Calculus I, II & III, Advanced Calculus, Linear Algebra, Differential Equations, and Number Theory, and supervised my research work in the foundations of mathematical analysis and differential dynamics at the University of La Verne, where I completed my studies in mathematics (a part of the research work and the dissertation that I completed at the University of La Verne under the supervision of Dr. Th. M. Rassias was published in 1998 as the volume no. 24 of the scientifically advanced Series in Pure Mathematics of the World Scientific Publishing Company); Dr. Christos Koutsogeorgis, who taught me Discrete Mathematics, Abstract Algebra, and Probability Theory (University of La Verne, 1994–96); Dr. Chamberlain Foes, who taught me PASCAL (programming language) and introduced me to mathematical informatics and management information systems (University of La Verne, 1995); and Dr. Giuliano Di Bernardo, who held the Chair of Philosophy of Science and Logic at the Faculty of Sociology of the University of Trento from 1979 until 2010, and with whom I cooperate in the context of several philosophical projects.

Furthermore, regarding my philosophical education and my interest in studying the interplay between philosophy and mathematics as well as the history of civilization, I would like to acknowledge the following persons: my colleagues at the Faculty of Philosophy at the Theological Academy of Saint Andrew (Academia Teológica de San Andrés), Veracruz, Mexico, where I completed a series of Ph.D. courses (specifically, Methodology of Philosophical Investigation I & II, Theology and Philosophy I–IV, Selected Topics in Christian Philosophy I–IV, Seminar on Investigation in

Christian Philosophy I–IV, and Interpretation of Philosophical Texts, I & II), and the Dean of that Theological Academy, Metropolitan Dr. Daniel de Jesús Ruiz Flores of Mexico and All Latin America of the Ukrainian Orthodox Church (Iglesia Ortodoxa Ucraniana en México), who helped me to explore and appreciate the interdisciplinary nature of Patristics and Philosophy and signed my Doctoral Degree in Christian Philosophy; the historian Dr. Vassilios Christides (affiliated with the Institute for Advanced Study, Princeton, U.S.A.), who taught me a comprehensive set of courses on the history of world civilization during my studies at the University of La Verne; the historian Dr. Paul Angelides, who taught me the courses "U.S. Intellectual History" and "Development of American Democracy" during my studies at the University of La Verne; the political scientist Dr. Blanca Ananiadis, who taught me European politics and political institutions during my studies at the University of La Verne; the political scientist Dr. Hazel Smith (Professor Emerita in International Security at Cranfield University, UK, and Fellow of the Royal Society of Arts, London) and the economist and epistemologist Dr. Michael Nicholson (University of Sussex), who supervised my research work in the epistemology and the mathematical modelling of International Relations and Political Economy during 1997-99 at the University of Kent's London Centre of International Relations; as well as Roman V. Romachev, the Founder and C.E.O. of the Moscow-based, international private intelligence company R-Techno Ltd, who has encouraged my scholarly endeavors and has honorably appointed me as a Partner in this prestigious corporation. My work as a mathematician and philosopher of science has focused on the mathematical study of a variety of practical real-world problems and on the investigation of specific problems and questions ranging from mathematical physics and mathematical biology to mathematical economics, mathematical psychology, and strategy, but also includes a broader effort to use mathematical thinking to identify and analyze structures and patterns, in general.

This book is the fruit of my work as a mathematical modelling consultant and as an analyst in various companies and academic/research institutions. It expresses my efforts to educate various groups of people in mathematical thinking and epistemology, starting with the basics. When I am given a practical problem, I transform it into a conceptual problem (that is, I create a corresponding conceptual model), and I often proceed to transform this conceptual problem (conceptual model) into a mathematical problem (mathematical model). Thus, I approach and tackle practical problems in a logical-mathematical way.

The study of mathematics is hard and takes a lot of effort, but this book may even enable one who has little or no mathematics background to learn mathematics on his/her own. Do not feel defeated or stupid if you do not understand everything immediately—it is normal. This book is self-contained; so think deeply and persist in studying it. At bottom, self-study is fun and an enriching experience, because you study for the sake of learning and at your own pace.

In addition, this book enables one to understand the significance of mathematical modelling both in the context of the natural sciences and in the context of the social sciences. Science builds knowledge through logic and testable explanations and predictions. Thus, it contrasts prejudice, superstition, personal opinion, subjective political beliefs, and, generally, irrational passions.

As Steve Halperin (*Introduction to Proof in Analysis*, p. 9) has pointedly argued, by the term "mathematical proof," one should understand "a sequence of statements which establish that certain assumptions (the hypotheses) imply that a certain statement (the conclusion) is true," and the statements that constitute a proof must satisfy the following requirements: (i) "each is clear and unambiguous"; (ii) "each is true, and its truth follows immediately from the truth of the preceding statements and the hypotheses"; and (iii) "the final statement is the conclusion." Thus, in the context of a mathematical proof, we may use several techniques, such as direct proof (involving arguing step by step, starting from what we know until we have demonstrated the truth of some conclusion), mathematical induction, counterexamples (since a single counterexample suffices to prove that a statement claiming necessity and universality is wrong), *reductio ad absurdum* (i.e., the form of argument that attempts to prove a statement by proving that the negation of the given statement leads to absurdity or contradiction), proof by contraposition (i.e., inferring a conditional statement from its contrapositive; the contrapositive of the statement "if A, then B" being the statement "if not B, then not A"), etc. In addition, the concept of a mathematical proof is inextricably linked to the concept of a definition, that is, to a deep and rigorous understanding of the essence of the object under consideration.

The primary purpose of this book is to equip the reader with a mathematical compendium that contains and explains all the basic concepts and methods of mathematics. Moreover, this book aims to equip every aspiring person with a mathematical textbook for self-study and a self-contained reference work.

Introduction:
Mathematical Philosophy

Every scientific activity is based on consciousness, thinking, perception, memory, judgment, imagination, volition, emotion, attention, as well as intuition.

Consciousness can be construed as an existential state that allows one to develop the functions that are necessary in order to know both one's existential environment as well as the events that take place around oneself and within oneself. Thinking is based on symbols, which represent various objects and events, and it is a complex mental faculty characterized by the creation and the manipulation of symbols, their meanings, and their mutual relations. Perception is a process whereby a living organism organizes and interprets sensory-sensuous data by relating them to the results of previous experiences. In other words, perception is not static, but a developing attribute of living organisms; it is active in the sense that it affects the raw material of scattered and crude sensory-sensuous data in order to organize and interpret them; and it is completed with the reconstruction of the present (present sensory-sensuous data) by means of the past (data originating from previous experiences). Therefore, perception is intimately related to memory and judgment. Judgment is one's ability to compare and contrast ideas or events, to perceive their relations with other ideas or events, and to extract correct conclusions through comparison and contrast. Memory is one's ability to preserve the past within oneself—or, equivalently, the function whereby one retains and accordingly mobilizes preexisting impressions. Imagination is a mental faculty that enables one to form mental images, representations, that do not (directly) derive from the senses. Imagination is not subject to the principle of reality, as the latter is formed by the established institutions, and it develops because consciousness cannot conceive the absolute being in an objective way. Volition, or will, is one's ability to make decisions and implement them kinetically. Emotion or affect is the mental faculty that determines one's mood. Attention is a mental faculty that focuses conscious functions on particular stimuli in a selective way, and it operates as a link between perception and consciousness. Intuition means that consciousness conceives a truth (that is, it formulates a judgment about the reality of an object) according to a process of conscious processing that starts from a minimum empirical or logical datum and rises to a whole abstract system with which consciousness realizes that it is connected or to which consciousness realizes that it belongs (logical intuition is intimately related to a type of subconscious thinking).

By the term "knowledge," we mean: (i) the mental action through which an object is recognized as an object of consciousness; (ii) the mental action through which consciousness conceives the substance of its object; (iii) the object whose image or idea is contained in consciousness; and (iv) that conscious content which is identified with the substance of the object of knowledge. Therefore, the term "knowledge" can be construed as a firm consideration of an object as something that corresponds to reality.

Logical knowledge, in particular, is a form of knowledge that derives from the rational faculty of consciousness, and it is characterized by indisputable and logically grounded truths (judgments about the reality of things). Rationality means the use of logical knowledge to attain goals. Logic is a theory of correct reasoning. Any relation between concepts is formulated by means of propositions. According to Aristotle's *Organon*, the backbone of any science is a set of propositions, so that, starting from the very primitive principles and causes, one can proceed to learn the rest. Aristotle's logic is focused on the notion of deduction (syllogism), which was defined by Aristotle, in the first book of his work entitled *Prior Analytics*, as follows: "A deduction is speech (*logos*) in which, certain things having been supposed, something different from those supposed results of necessity because of their being so"; each of the things "supposed" is a premise of the argument, and what "results of necessity" is the conclusion.

By the term "concept," we mean the set of all predicates of a thing (or of a set of conspecific things) that express the substance of the given thing (or of the given set of conspecific things). In the scholarly discipline of logic, the "intension" of a concept is the set of all predicates of the given concept, or the set of all those elements due to which and by means of which the given concept can be known and distinguished from every other concept. In other words, the intension of a concept is its formal definition. For instance, the properties of the three angles and the three sides of a geometric figure constitute the intension of the concept of a triangle. Moreover, in the scholarly discipline of logic, "extension" indicates a concept's range of applicability by naming the particular objects that it denotes. In other words, the extension of a concept encompasses all those things to which the given concept refers. For instance, the extension of the concept of a tree consists of all particular trees; the extension of the concept of a human being consists of all particular humans, etc.

By the term "genus" (plural: "genera"), we mean a concept whose extension includes other concepts, known as "species" or "kinds," which fall within it. In other words, "genera" are concepts with an extension bigger than that of other concepts, whereas "species" or "kinds" are

concepts with an extension smaller than that of other concepts. For instance, the concept of a geometric figure is a genus with regard to the concept of a triangle, whereas the concept of a triangle, which appertains to the concept of a geometric figure, is a kind with regard to the concept of a geometric figure.

Through the process of "abstraction," we decrease the intension of concepts and increase their extension. Thus, due to abstraction, the concept of a human being can be gradually generalized into the following concepts: "vertebrate," "mammal," "animal," "living being," and "being"; "being" is the most general concept, in the sense that its intension is minimum and its extension is maximum. "Being," to which every other concept is reducible, cannot be further analyzed into other concepts. Concepts of such general type, which are not susceptible to further analysis into simpler concepts, and to which other concepts are reducible, are called "categories." Aristotle, in his book *Categories*, attempted to enumerate the most general species, or kinds, into which beings in the world are divided. In particular, in *Categories* (1b25), Aristotle lists the following as the ten highest categories of things "said without any combination": "substance" (for instance, man, horse), "quantity" (for instance, four-foot, five-foot), "quality" (for instance, white, grammatical), "relation" (for instance, double, half), "place" (for instance, in the Lyceum, in the market-place), "date" (for instance, yesterday, last year), "posture" (for instance, is lying, is sitting), "state" (for instance, has shoes on, has armor on), "action" (for instance, cutting, burning), and "passion" (for instance, being cut, being burned).

No material object or system of objects—nor any connection or interaction that exists between them in material reality—is the direct object of mathematical study. In order for mathematical tools to be used to study the processes, the phenomena, and the individual objects that exist in reality, it is necessary to construct the corresponding mathematical models. By the term "mathematical model," we mean a system of mathematical relations that symbolically describes the processes or the phenomena under study. For the construction of mathematical models, a variety of mathematical tools are used—such as: equations (algebraic, differential, and integral ones), graphs, matrices and determinants, relations of mathematical logic, geometric constructions, etc. In fact, the basic type of mathematical activity, the fundamental problem of mathematics, is the construction, the study, and the application of mathematical models.

No model can represent all the properties and all the relations of the original object. In other words, a model is a simplification, an approximate representation of the original object, but, simultaneously, a model

highlights and describes an important pattern of the properties and the relations of the original object. The dialectical process of knowledge of reality consists of two processes: firstly, the replacement of existing models by others in which a more complete representation of the properties of the original object is achieved; and, secondly, the combined application of various models.

As I have already mentioned, mathematics is concerned with the construction of such models of objects (namely, of things, processes, and phenomena) that reflect the corresponding objects' quantitative and/or qualitative attributes as well as their spatial and structural peculiarities. For instance, geometry is the scientific study of the quantitative and the qualitative properties of spatial forms and relations (the criteria for equality of triangles provide instances of qualitative geometric knowledge, and the computation of lengths, areas, and volumes exemplifies quantitative geometric knowledge).

The constituent elements of a model are symbols and signs. Symbols are forms that express commonly accepted intentions and actions, and they can be organized into particular systems that are called codes, and the elements of such a code are called signs. In the context of mathematical modelling, the character of these signs can vary, since these signs can be schematic images (namely, shapes, drawings, and graphs), collections of numerical symbols, and elements of artificial or natural languages. Furthermore, symbols are subject to transformations according to specific symbol transformation rules. The symbols and their transformations are definitely interpreted in terms of the original objects. The combinations of symbols used and their transformations are dictated and determined by the properties of the original objects and by the associations selected and included in the corresponding model.

Mathematical models—which, with the help of the human senses, are directly extracted from material objects—usually express the primary simplest abstractions of a quantitative and spatial character, such as, for instance, enumeration, dimensions, form, position in space, etc. If a human being relies only on the sense organs, then he/she cannot achieve deep knowledge of the world around him/her nor of his/her inner world. Nature, acting on the sense organs, can only produce in humans a limited set of sensations, impressions—namely, that type of knowledge which we call "empirical."

The accumulation of empirical data constitutes the basis of generalizations and abstractions. The formulation of generalizations and abstractions provides the intellectual setting in which the application of mathematical tools becomes possible and meaningful. In the course of the historical

development of mathematics, the construction of models of increasingly complex systems has been achieved, including systems that consist of multiple abstractions. With regard to its theoretical essence, mathematics can be construed as a science of modelling; and, therefore, both the reality of the world and the reality of consciousness are fundamental to mathematics.

The demand for computational precision goes hand in hand with the demand for conceptual precision and logical rigor. As the renowned French mathematician and philosopher René Thom has pointed out—in mathematics and, generally, in science—in addition to descriptive accuracy, explanatory accuracy is also required. Furthermore, Cybernetics and Systems Science have given rise to a transdisciplinary approach to scientific modelling, since they are characterized by an attempt to build general, domain-independent theories. The scientifically rigorous conception of mathematical modelling is based on the concepts of homomorphism and isomorphism. In mathematics, the term "homomorphism" describes the transformation of one data set (or "system") into another while preserving relationships between elements in both sets. In other words, a homomorphism is a structure-preserving mapping. The mathematical term "isomorphism" is more specialized, since it refers to a structure-preserving mapping between two systems of the same type that can be reversed by an inverse mapping. In other words, an isomorphism is a special type of homomorphism, a bijective homomorphism. In fact, homomorphisms can lose some information about the object, but isomorphisms always preserve all the information. In view of the foregoing, a general definition of a mathematical model can be formulated as follows: Given two data sets, or systems A and B, each is a model of the other if there exist a homomorphism from data set A to a data subset A′ (read "A prime") of B and a homomorphism from data set B to a data subset B′ (read "B prime") of A, where systems A′ and B′ are isomorphic to each other.

According to such renowned mathematicians and logicians as Jacques Hadamard, Nicolas Bourbaki, René Thom, Hermann Weyl, Ljubomir Iliev, Andrey Kolmogorov, and Leonid Kantorovich the order of operations involved in the construction of mathematical models can be summarized as follows:

1. Determining and formulating the problem as clearly as possible.
2. Identification of the variable quantities that determine the process under study or are chosen for the study of the given problem.
3. Definition of the relations between these variables and the parameters on which the state of the process under study depends.

4. Formulation of a hypothesis (or hypotheses) about the nature of the conditions under study.
5. Construction of the model so that its properties coincide with the initially defined ones.
6. Conducting experimental tests.
7. Checking the hypothesis accepted for the construction of the model, and evaluating it according to the outcome of experimental tests.
8. Acceptance, rejection, or modification of the hypothesis on the basis of repeated experimental tests and conclusions.

The symbolic language of mathematics is equipped with rules for handling concepts. In addition, the logical construction of mathematical models is rigorously determined in the context of, and my means of, a hypothetico-deductive system. In a "hypothetico-deductive" (or "axiomatic") system, there are two requirements that must be met in order that we agree that a proof is correct: (i) acceptance of certain statements, called "axioms," without proof, on the basis of their intrinsic merit, or because they are regarded as self-evident; and (ii) agreement on how and when one statement "follows logically" from another, that is, agreement on certain rules of reasoning. Inextricably linked to the aforementioned two requirements is the requirement that every person who applies hypothetico-deductive reasoning to a particular discourse understands the meaning of the words and the symbols that are used in that discourse. The more consistent and the more complete a hypothetico-deductive system is, the more its imposition is safeguarded. By the term "consistency," we mean that the axioms of a hypothetico-deductive system neither contain nor produce contradictions. By the term "completeness," we mean that the truth value of any proposition that belongs to a hypothetico-deductive system can be determined within the given hypothetico-deductive system (that is, according to the terms and the rules of the given hypothetico-deductive system). All these are philosophical questions.

In general, there is a close affinity between mathematics and philosophy. Mathematics, like philosophy, is done by consciousness. Mathematics provides a model of knowledge of a particular kind, and, in fact, philosophers have highlighted the particular nature of mathematical knowledge and have argued that all knowledge could possibly aspire to the particular nature of mathematical knowledge. According to the German mathematician and philosopher Friedrich Ludwig Gottlob Frege, unlike other kinds of knowledge, mathematical knowledge is characterized by rigor and objectivity, because mathematics is constituted as a logical system.

The model of knowledge that is provided by mathematics has the following characteristics: (i) certainty (in the sense that, if something is true and known in mathematics, then it is undoubted), (ii) incorrigibility (in the sense that the development of mathematical knowledge is internally consistent), (iii) eternity (in the sense that mathematical knowledge is not subject to time), and (iv) necessity (in the sense that mathematical truths are not contingently true but necessarily true). Being aware of these attributes of mathematical knowledge, Plato had the phrase "Let no one ignorant of geometry enter" engraved at the door of his Academy. In the context of Plato's philosophy, geometry is concerned with the understanding of the reason ("logos") of the world. Thus, *Plato*, in his *Republic*, 527c, argues that "geometry is the knowledge of the eternally existent," and that, therefore, geometry "would tend to draw the soul to truth, and create the spirit of philosophy, and would be productive of a philosophical attitude of mind."

One of the reasons why ancient Greeks regarded geometry, rather than arithmetic, as the more foundational and superior branch of mathematics is the crisis that erupted in the foundations of Pythagoras's mathematical theory. Pythagoras is famous for finding out that, for any right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides. However, this geometric theorem, which is known as the Pythagorean theorem, presented a problem to Pythagoras and his disciples. Pythagoras was a philosopher, a mathematician, and a sort of cult leader in ancient Greece. Pythagoras and his school (the so-called "Pythagoreans") were dedicated to the mysticism of numbers, and they maintained that everything in the world could be expressed as a number. But they had a rather inadequate understanding of the concept of a number.

From the Pythagorean perspective of mathematics, the relations between the objects of the world (e.g., magnitudes) correspond to the relations between natural (and, generally, integral) numbers. However, it was soon realized that things are not so simple, since it was realized that there exist magnitudes that do not have a common measure.

According to the Pythagoreans, two objects (magnitudes) are "commensurable," that is, they have a common measure, if and only if there is a magnitude of the same kind contained an integral number of times in both of them. In other words, two magnitudes are "commensurable" if and only if their ratio is a rational number. However, the Pythagoreans encountered "incommensurable" magnitudes, namely, magnitudes whose ratio is an irrational number. For instance, the length of a diagonal of a unit square, namely, of a square whose sides have length 1,

is, according to the Pythagorean Theorem, equal to $\sqrt{2}$, which is an irrational number; similarly, a circle's circumference and its diameter are incommensurable. Pythagoras swore his disciples to secrecy about the existence of irrational numbers. Nevertheless, the awareness that there exist incommensurable magnitudes compelled ancient Greek mathematicians to inquire into the relations between incommensurable magnitudes. This event marked a major crisis in ancient mathematics.

According to ancient Greek mathematicians, quantities (magnitudes) are continuous and uniform objects, which are best represented by straight line segments, whereas their division into parts, namely, their measurement in terms of a "unit of measurement" (that is, a definite magnitude of a quantity), represents the notion of discreteness. Ancient Greek mathematicians used the term "ratio of magnitudes" in order to refer to the relation between two magnitudes that can be measured in terms of a common unit of measurement, and, thus, the ancient Greek concept of a ratio is most similar to the more abstract modern concept of a number. In the context of ancient Greek mathematics, the objects of mathematics were quantities (represented by straight line segments), and the ratio between two quantities was a meta-object, namely, something that was used in order to study mathematical objects without being treated as a mathematical object itself. In other words, in the context of ancient Greek mathematics, a ratio (namely, a number) was construed as a measuring relationship between two quantities, and such a measuring relationship could be built up (and, hence, proved) in finitely many steps, using a common unit of measurement. Nevertheless, the discovery of incommensurable ratios demonstrated that a ratio could not be interpreted as a measuring relationship in the aforementioned way. In fact, as a result of the discovery of incommensurable ratios, the concept of a ratio (or a number) acquired its conceptual autonomy, and, instead of being treated as a meta-object, it started being treated as an object of mathematics. Therefore, ancient Greek mathematicians had to transcend the system of mathematics that was based on commensurable ratios (notice that a commensurable ratio could easily become an object of mathematical theory, since it is a rational number, and, therefore, it can be constructed in finitely many steps, whereas the decimal representation of an irrational number neither terminates nor infinitely repeats but extends forever without regular repetition).

The discovery of irrational numbers undermined the faith in numbers as the foundational conceptual system of mathematics, and led to the belief in the superiority of geometry. The belief in the superiority of geometry was reinforced by the realization that, in geometry, an irrational number can be

constructed—as is the case, for example, when we draw a diagonal of a unit square, namely, $\sqrt{2}$—whereas, in arithmetic, an irrational number, such as $\sqrt{2}$, cannot be expressed as a ratio of two whole numbers. The aforementioned crisis in the foundations of mathematics was ultimately overcome by Eudoxus's theory of proportions and by the method of exhaustion, which derives from Eudoxus's theory of proportions, and it was used by Archimedes.

The overcoming of crises in the foundations of mathematics is intimately related to set-theoretical concepts and axioms and to the concept of infinity in particular. I would like you to address these issues in your answer regarding the relationship between philosophy and mathematics.

The concept of modern mathematics that is semantically most similar to Aristotle's concept of a "potential infinity" is the convergence of a sequence of natural numbers. Thus, from the perspective of ancient Greek mathematics, infinity is not a being (specifically, it is not an actual state)—namely, it cannot be simultaneously considered in its whole extension—but it can only be considered as a becoming (specifically, as a process). In this way, the concept of an infinite approach helps us to overcome the contradiction between incommensurable ratios and commensurable ratios, since we can think of an incommensurable ratio infinitely approaching a commensurable ratio and vice versa. Similarly, the concept of an infinite approach helps us to overcome the contradiction between broken lines and curves as well as the contradiction between continuity and discreteness. This reasoning is endorsed by Euclid, and, therefore, in his *Elements*, he does not consider infinitely extended straight lines, but he always works with straight line segments, which, as he says, can be extended as much as one needs. Moreover, in view of the foregoing, Archimedes was very careful in the use of infinite processes, and, therefore, he approximated the irrational number π (i.e., the ratio of a circle's circumference to its diameter) by using the fact that the circumference of a circle is bounded by the perimeter of an *inscribed* polygon and the perimeter of a *circumscribed* polygon. According to Eudoxus's theory of proportions and Archimedes's method of exhaustion—which incorporate the Aristotelian concept of a "potential infinity" and the modern mathematical concept of a convergent sequence—there is always a ratio between any two magnitudes, and we can always make any magnitude smaller or greater than a given magnitude.

Whereas, from the perspective of ancient mathematicians, numbers are things by means of which we count, Cartesianism (as the intellectual "school" of Descartes is known) is based on the algebraization of geometry, thus giving rise to the idea that numbers can be thought of as

positions on the number line. Fusing geometry and arithmetic is an arduous task. In order to understand the difficulties that originate from fusing geometry and arithmetic, let us consider, for instance, the famous irrational number $\sqrt{2}$, which was discovered by Pythagoreans when they attempted to compute the length of a diagonal of a unit square. The Pythagoreans realized that the diagonal of a unit square is not commensurable with the side of the given square, but, by keeping geometry and arithmetic separate from each other (that is, by refusing to identify numbers with lengths of straight line segments), ancient Greek mathematicians could argue as follows: given a straight line segment whose length is one, we can construct a straight line segment whose length is $\sqrt{2}$, and, in general, irrational numbers are geometrically constructible (and, hence, geometrically explicable and manageable), even though, from the perspective of arithmetic, irrational numbers are ideal quantities, in the sense that the calculation of irrational numbers (such as $\sqrt{2}$) is an infinite process (since irrational numbers have infinitely many decimal digits). On the other hand, in the nineteenth century, having endorsed the Cartesian approach to mathematics, mathematicians realized that they had to clarify some still ambiguous fundamental concepts (such as that of a real number), to formulate new methods of doing mathematics in a logically rigorous way, and to create a rigorous theory of the arithmetic continuum, specifically, a rigorous theory of real numbers and their arithmetic.

Another important crisis in the foundations of mathematics broke out in the seventeenth century. Whereas ancient Greek mathematics (as it is expounded and systematized by Euclid in his *Elements*) is based on a geometric way of thinking, modern European mathematics is more inclined to an algebraic way of thinking (and, hence, it tends to give primacy to arithmetic over geometry). This shift was typified by the reduction of geometry to arithmetic in the context of analytic geometry, which is characterized by the use of coordinates and by the correspondence between curves and equations. The use of coordinate systems implies that space itself is encoded by n-tuples (namely, by sequences, or ordered lists, of n numbers), and, specifically, that the 2-dimensional space, the "plane," is encoded by pairs of numbers, so that the conception of space becomes subordinate to the conception of arithmetic.

In the seventeenth century, mathematicians (primarily the initiators and developers of calculus) were preoccupied with such geometric problems as the computation of areas and volumes of arbitrary geometric figures and the construction of tangents to curves as well as with such physical problems as the formulation of the law that determines the rate of change of velocity and of acceleration (with respect to time) when one knows the

law that determines the rate of change of displacement of an object (namely, its velocity) and vice versa. The tendency towards the study of the aforementioned types of geometric and physical problems was reinforced by Galileo's physical theory, which constrained Aristotle's theory of motion (according to which the term "motion" referred to any kind of change, development, and growth) to the study of change in the relative position of physical objects.

The most prominent seventeenth-century mathematicians realized that, when we treat geometric figures and the motions of physical bodies as "wholes," we cannot demonstrate significant apparent similarities between them, but, when we analyze them into (sufficiently) "small" pieces, they display great similarities to each other. Hence, the major problem of seventeenth-century mathematics consisted of determining the proper processes for dividing the "whole" into "small" parts, which would be more easily and more rigorously studied than the "whole," as well as of determining the proper processes for resynthesizing the behavior of the "whole" from the behavior of its "small" parts. However, the "small" parts into which an object of scientific research is divided are similar to the "small" parts into which another object of scientific research is divided, and, thus, they give rise to generalizations (such as natural laws), only when the dimensions of such "small" parts tend to zero, and, thus, only when the number (namely, the population size) of such "small" parts tends to infinity. Therefore, the need for the use of infinite processes, specifically, infinitesimals, became prominent again.

Even though "infinitesimal methods" could lead to correct results and useful applications, they lacked the logical rigor that characterized ancient Greek mathematics, particularly, Euclid's *Elements*, and they were susceptible to contradictions. Some mathematicians argued that lengths consisted of (infinitely many) infinitesimal lengths, areas consisted of (infinitely many) infinitesimal areas, and volumes consisted of (infinitely many) infinitesimal volumes, while other mathematicians argued that lines consisted of an infinite number of points, surfaces consisted of an infinite number of lines, and solid bodies consisted of an infinite number of surfaces. In that era, namely, in the seventeenth century, the mathematical concept of a limit was not yet clarified. It is worth pointing out that the famous French Enlightenment scholar Voltaire described infinitesimal calculus as "the art of measuring exactly a thing whose existence cannot be conceived," thus expressing his bewilderment at the fact that the seventeenth-century infinitesimal calculus was a useful and powerful scientific instrument, but the actual things that it was talking about were

almost beyond conception (quoted in: Andrew Simoson, *Voltaire's Riddle*, U.S.A.: The Mathematical Association of America, 2010, p. 51).

In infinitesimal calculus, the usual derivative, denoted by $\frac{dy}{dx}$, is an operator (actually, a function) that describes how a function $y = f(x)$ changes relative to its argument x. Newton defined the derivative of a function as the "ultimate ratio" of "vanishing quantities," and Leibniz argued that the quantities dy and dx, which appear in the definition of the derivative of a function, are infinitely small yet non-zero quantities. These ambiguities ignited heated debates regarding the foundations of infinitesimal calculus. In fact, the major problem pertaining to the development of infinitesimal calculus in the seventeenth and the eighteenth centuries was the reduction of a continuous entity, namely, a "whole," to discrete entities, namely, infinitesimals (meaning infinitely small parts of the corresponding "whole"), by means of a non-well-defined concept, namely, the concept of infinity. However, the effectiveness of the application of infinitesimal methods to physics and astronomy played a significant role in the acceptance and the further development of infinitesimal calculus. In general, many eighteenth-century mathematicians drew their subject matter from many branches of physics, astronomy, navigation, cartography, commerce, and finance. Infinitesimal calculus was put in a rigorous conceptual setting by the French mathematician Augustin-Louis Cauchy (1789–1857), who explained the concept of a limit of a function in a clear, formal, and arithmetic, rather than geometric, way by arguing as follows: "when the successive values attributed to a variable approach indefinitely a fixed value so as to end by differing from it by as little as one wishes, this last is called the limit of all the others" (quoted in: Carl B. Boyer, *The History of Calculus and Its Conceptual Development*, New York: Dover, 1959, p. 272).

In view of the peculiar model of knowledge that is provided by mathematics, as I have delineated it, we have to answer the following fundamental questions: Which are the underpinnings of mathematical knowledge, and what exactly is it that endows mathematical knowledge with the characteristics that I have already mentioned—namely, certainty, incorrigibility, eternity, and necessity? Moreover, why do other types of knowledge differ from mathematical knowledge and cannot be adapted to the requirements of the mathematical model of knowledge?

Firstly, we have to consider mathematical Platonism. According to mathematical Platonism, numbers are forms, specifically, abstract, objectively existing objects. This thesis seems to be corroborated by the fact that numbers are not intrinsic characteristics of objects, but they are applicable to objects, and they seem to be the contents of objective truths,

irrespective of any contingency and any particular object of the sensible world. From this perspective, numbers are objects themselves. In particular, according to mathematical Platonism, numbers are a peculiar kind of objects, since they exist objectively, but they cannot be grasped by the senses, they are not part of the material space-time, and they are not subject to the laws of material space-time. Far from negating the thesis that numbers are objects, the fact that numbers are not subject to the spatio-temporal structure of our sensible world corroborates the Platonic thesis that the world of forms is the reality *par excellence*, which underpins the logical constitution of our sensible world, which, in Platonic parlance, can be regarded as a "shadow" of the world of forms. This reasoning underpins the Platonic argument that, whereas the knowledge that is provided by the senses is subject to revision, the knowledge that is provided by forms, such as numbers, is incorrigible; and, therefore, reason ("logos"), which consists of thought and language, is superior to the senses. This is how mathematical Platonism explains the peculiar characteristics of the mathematical truth—namely, the certainty, the incorrigibility, the eternity, and the necessity of the mathematical truth.

Mathematical Platonism is a variety of dualistic realism. In philosophy, the term "realism" refers to a philosophical model that is based on objectively existing objects, thus giving primacy to a consciousness-independent world, as opposed to "idealism," which gives primacy to the reality of consciousness. According to philosophical realism, the fact that experience furnishes consciousness with images—even unrelated to each other—of a reality that seems to lie outside the dominion of consciousness implies that the reality of the world is the cause of the particular images of the world that are present within consciousness. From the realist perspective, the principle of causality points us in the direction of the claim that the autonomous existence of reality is naturally and logically necessary. Even though the aforementioned reasoning is sound, dualistic realism, with its doubling of the world, leads to contradictions and logical gaps, especially regarding the existence of, and the relationship between, the world of forms and the world of "shadows," namely, their sensible images.

Aristotle attempted to overcome the contradictions and the logical gaps of Plato's dualistic realism by reformulating dualistic realism in a way that does not depend on a Platonic doubling of the world and bridges the gap between the world of forms and the human mind. In particular, Aristotelianism highlights the structural mode of being.

The cohesive bond between substance and form is the structure of a being. The deepest reality of a being is its substance, the external aspect and the existential otherness of that reality are the form of the given being—

namely, an element that animates the given being—and these two elements (modes of being) concur with each other in the context of the structural mode of being. From the perspective of structuralism, Platonic realism corresponds to the *ante rem* structuralism ("before the thing"), in the sense that, according to Platonism, the ideational structure of mental life is a real but transcendent principle vis-à-vis the mind itself and the sensible world, and philosophical consciousness tries to partake of and progress in the world of forms, while Aristotelian realism corresponds to the *in re* structuralism ("in the thing"), in the sense that, according to Aristotelianism, structures are held to exist inasmuch as they are exemplified by some concrete system, and the mind itself, not the world of forms, is a real and transcendent principle vis-à-vis the sensible world, and it conceives forms as abstractions. According to Plato's dualistic realism, forms are objectively existing objects, of which the objects of the sensible world are images, or "shadows." According to Aristotle's dualistic realism, forms are mental abstractions, the objects of the sensible world are material exemplifications, materializations of forms, forms are conceived by the mind, and the mind, rather than the world of forms itself, is transcendent to the sensible world. For this reason, Aristotle argued that the mind is the "entelechy"—that is, the program of actualization—of the body, generally, of the human organism.

According to mathematical Aristotelianism, mathematics refers to truths of the sensible world, in the sense that, even though numbers are not sensible things, they are properties of sensible things—specifically, abstract entities which can be predicated of sensible things. In other words, numbers are not objects themselves, they do not exist independently of objects, but they are features of objects, and they exist within objects. For instance, when we see ten people, the number ten is a property of the given collection of people that we see.

In the context of mathematical Aristotelianism, numbers are not self-subsistent forms, objects, but still numbers are properties of other things in an objective way. In general, according to Aristotle and according to Thomas Aquinas's variety of Aristotelianism (in the context of medieval scholasticism), consciousness is a passive mirror of reality, and truth refers to an objective correspondence between thinking consciousness and its object. But Descartes reversed the aforementioned relation between the intellect and its object, arguing that understanding (or intellection) is the basic reality, and that understanding is activated by conceiving itself; hence, Descartes's famous *dictum*: "cogito ergo sum," meaning "I think therefore I am." By assigning this active role to consciousness, Descartes emerged as the rigorous initiator and founder of modern philosophy.

Gradually, modern philosophy gave rise to a new general model, which is known as idealism. According to modern philosophical terminology, there are two general models whereby philosophers interpret the world: one gives primacy to the reality of the world, and it is known as philosophical realism, whereas the other gives primacy to the reality of consciousness, and it is known as philosophical idealism. According to idealism, the nature of consciousness is not totally different from or opposite to the nature of extra-conscious reality. The idealists' way of thinking can be summarized as follows: if the nature of reality were totally different from the nature of consciousness, then the human being would be unable to know reality. Thus, ultimately, idealism construes and studies the world not as something reflected in consciousness, but as an extension and a projection of consciousness outside itself and as part of consciousness.

In the nineteenth century, the German mathematician and philosopher Friedrich Ludwig Gottlob Frege departed from the traditional realist philosophy of mathematics, and, in contrast to mathematical Aristotelianism, he argued that, even though mathematical knowledge is objective, numbers are not objective, consciousness-independent properties of other things. According to Frege, any number n can be used in order to count any n-membered set, but the formulation of a claim concerning which number belongs to a set is determined by the way in which mathematical consciousness conceptualizes that set. For instance, consider the Tarot. The Tarot consists of 78 cards. Moreover, it has two distinct parts: the Major Arcana, consisting of 22 cards without suits, and the Minor Arcana, consisting of 56 cards divided into 4 suits of 14 cards each. Depending of whether we are thinking in terms of Tarot cards in general, or in terms of the Major Arcana Tarot cards, or in terms of the Minor Arcana Tarot cards, or in terms of the suits of the Minor Arcana Tarot cards, different numbers will belong to that particular set of cards. Hence, we have to decide if that particular set has the property 78, or the property 22, or the property 56, or the property 4. Similarly, a pair of shoes is one pair of shoes, but it consists of two shoes, and, therefore, we have to decide which number belongs to this physical object: the number one or the number two. Thus, according to Frege, numbers are not objective properties of objects, but objects acquire numbers as properties according to the ways in which consciousness thinks of the corresponding objects. Frege's argument about the active role of consciousness in mathematical creation—especially in the light of Kant's philosophy—may lead one to the conclusion that we have to do away with mathematical objectivity completely. Before explaining the way in which Frege prevented

mathematical philosophy from sinking into arbitrary idealism, it is important to summarize Kant's theses.

Immanuel Kant—who wrote the seminal book *Critique of Pure Reason* (1781/1787) and is one of the paradigmatic representatives of the European Enlightenment—formulated a theory of mathematical philosophy that is focused on the following question: given that mathematical knowledge is necessarily, intrinsically true, and, simultaneously, it is applicable to the sensible world—since the sensible world seems to conform to the laws of arithmetic, which transcend the sensible world—how is it possible to know something about the world that is necessarily true, or, in other words, how can we have knowledge of the world independent of recourse to experience? In order to tackle this question, Kant distinguished between two kinds of sentences: analytically true sentences and synthetically true sentences.

An analytically true sentence is necessarily true on purely logical grounds—that is, solely in virtue of its meaning—and, in reality, it elucidates meanings already implicit in the subject. For instance, the sentence "Pediatricians are medical doctors who specialize in the medical care of infants, children, adolescents, and young adults" is an analytic statement, because it is true by definition. By contrast, the sentence "Pediatricians are rich" is not necessarily true; since it is not part of the definition of a pediatrician that a pediatrician is rich, but it is part of the definition of a pediatrician that a pediatrician is a medical doctor who specializes in the medical care of infants, children, adolescents, and young adults. The sentence "Pediatricians are rich" is a synthetic statement.

The distinction between analytic and synthetic statements is based on whether we are dealing with one concept or two concepts. If you say that "Pediatricians are rich," you are making a synthesis of two unrelated concepts—namely, the concept of being a medical doctor specialized in pediatrics and the concept of being rich. By contrast, if you say that "Pediatricians are medical doctors who specialize in the medical care of infants, children, adolescents, and young adults," you are not synthesizing two unrelated concepts, but you are analyzing a feature of one concept—namely, the concept of being a pediatrician.

Furthermore, Kant made another important epistemological distinction in order to clarify the manner in which we know things to be true—specifically, he distinguished between *a priori* philosophical methods and *a posteriori* philosophical methods. The major attribute of the *a priori* methods is that they are based on primitive hypotheses usually intuitively conceived and axiomatically accepted, which deductively give rise to series of syllogisms, which, in turn, lead to ultimate conclusions, which

are related to the preceding propositions in a logically rigorous way. For instance, we know that "pediatricians are medical doctors who specialize in the medical care of infants, children, adolescents, and young adults" *a priori*, that is, prior to any testing and any surveying. On the other hand, *a posteriori* philosophical methods are based on empirical research. For instance, the truth value of the statement that "pediatricians are rich" can only be determined *a posteriori*, that is, on the basis of doing some empirical research.

In view of the aforementioned Kantian epistemological distinctions, analytic statements are *a priori*, and synthetic statements are *a posteriori*. But mathematical knowledge exhibits the following peculiar feature: it is necessarily true, and, therefore, *a priori*, but, simultaneously, it is true of the world, and, therefore, *synthetic*. In fact, Kant observed that mathematical knowledge is a peculiar hybrid, in the sense that it is synthetic *a priori*. In other words, according to Kant, mathematical propositions, such as "$1 + 2 = 3$," are synthetic statements, abstractions from the sensing of objects, and, yet, they are *a priori*, in the sense that we do not need to do any experiments in order to verify them. Thus, Kant came up with the following question: how can we know things that are synthetic *a priori*? In order to answer this question, he developed a whole system of metaphysics that he called transcendental idealism and expounded in his *Critique of Pure Reason*.

Kant's metaphysical system is founded on the thesis that we do not know, and cannot know, the essence of things, the things-in-themselves, which he called "noumena"—meaning objects or events that exist independently of human sense and/or perception—but we can only know things as they appear to consciousness, which are called "phenomena." In Kant's philosophy, a phenomenon is a faded, dissolved declaration of the corresponding noumenon, the manner in which the corresponding noumenon (thing-in-itself) appears to an observer. According to Kant, phenomena have been put through a kind of mental filter, which is the way in which consciousness perceives the world, and mathematics is that kind of mental filter. In particular, Kant maintains that geometry is the spatial form through which consciousness perceives the world, and arithmetic—specifically, the one-dimensional sequence of numbers—is the temporal form through which consciousness perceives the world. Hence, according to Kant, we do not receive mathematics from the system of space-time itself, but we use mathematics, our spatio-temporal intuitions and intellectual glasses, in order to understand and organize the world, and this is the reason why mathematics is *a priori*. Geometry is the way in which we organize space, and arithmetic is the way in which we organize time,

and, when we combine geometry with arithmetic, we obtain the intellectual framework of the spatio-temporal world that we experience.
In his *Transcendental Aesthetic*, Kant refers to the followers of Newton's position as the "mathematical investigators" of nature, who contend that space and time "subsist" on their own; and he refers to the followers of Leibniz's position as the "metaphysicians of nature," who think that space and time "inhere" in objects and their relations. At the ontological level, Kant's position is that space and time do not exist independently of human experience, but they are "forms of intuition" (i.e., conditions of perception imposed by human consciousness). In this way, he managed to reconcile Newton's and Leibniz's arguments: he agrees with Newton that space is an irrefutable reality for objects in experience (i.e., for the elements of the phenomenal world, which are the objects of scientific inquiry), but he also agrees with Leibniz that space is not an irrefutable reality in terms of things-in-themselves. At the epistemological level, unlike David Hume, Kant argues that the axioms of Euclidean geometry are not self-evident or true in any logically necessary way. For Kant, the axioms of Euclidean geometry are logically synthetic, that is, they may be denied without contradiction, and, therefore, consistent non-Euclidean geometries are possible (as Lobachevski, Bolyai, and Riemann actually accomplished). However, Kant argues that the axioms of Euclidean geometry are known *a priori*, specifically, they depend on our intuition of space, that is, space as we can imaginatively visualize it. After the publication of Kant's philosophical works, numerous attempts have been made to articulate methods of philosophical research that synthesize idealism and positivism, or that at least combine aspects of idealism and positivism with each other. Kant has correctly highlighted and elucidated the active role of consciousness in cognition, and the distinction between cognition and the object of cognition. The distinction between cognition and the object of cognition plays a central role in the so-called analytic philosophy. However, analytic philosophy may lead to an impasse, because it urges one to repeat the distinction between cognition and the object of cognition *ad infinitum* (forever). Inherent in analytic philosophy is the risk of using Kantian philosophy in an abortive way, in the sense that the attempt to define the presuppositions of the presuppositions of philosophy can continue *ad infinitum*, annihilating epistemology. To mitigate this risk, Kant resorted to a formalist view of idealism: Kant's *Critique* is characterized by formal idealism, in the sense that it maintains that the *form* of objects is due to consciousness, but not their *matter*. Furthermore, to avoid the excesses of analytic philosophy, I would say that, at some point, a mature philosophical-scientific mind must make a final,

epistemologically responsible decision, instead of transforming philosophy into a meaningless game of words. After all, the very fact that the object of cognition, the world, exhibits a sort of resistance to cognition, and consciousness has to try hard in order to know the world and impose the intentionality of consciousness on the word, implies that—even though, under certain conditions, the world is submissive to the intentionality of consciousness—the world is not merely a projection and extension of consciousness.

The way in which Frege attempted to do justice to the objectivity of mathematics and to the reality of the world was logicism, which, as I mentioned earlier, brings together logic and arithmetic. Logicism resorts to Plato's philosophical realism regarding the objectivity of mathematics, but logicism differs from classical Platonism in two ways. Firstly, in contrast to classical Greeks, Frege and logicism in general regard arithmetic, rather than geometry, as the foundational branch of mathematics, because of the following two reasons: in the seventeenth century, Descartes's analytic geometry, adapting Viète's algebra to the study of geometric loci, showed that algebra can be used in order to model geometric objects in a systematic and rigorous way, thus establishing a correspondence between geometric curves and algebraic equations; and, in the nineteenth century, Nikolai Ivanovich Lobachevski, János Bolyai, and Bernhard Riemann invented rigorous and consistent alternatives to Euclidean geometry. Hence, for Frege and the logicists in general, the central problem in mathematical philosophy is to understand the meaning of a number. In particular, logicists endow arithmetic with the objectivity that characterizes Platonic forms, but they do so in an indirect way—through logic—trying, in a sense, to achieve a creative synthesis between Kant's transcendental idealism and Plato's philosophical realism. The role that logic plays in the "school" of logicism is the second issue with regard to which logicism differs from classical Platonism. In particular, Frege thought that we can do justice to mathematical Platonism, according to which arithmetic is about things that are forms, if we show that mathematics—particularly, arithmetic—is reducible to logic, and if we take a Platonic view of logic; hence, the name of this "school" of mathematical philosophy is logicism.

Frege fused logic and arithmetic by formulating a theory of numbers that is based on the concept of a class of objects and on structural linguistics. Hence, Frege synthesized Aristotle's work on logic and language with Plato's theory of forms. In particular, Frege thought as follows: Let us consider a variable x, meaning that x is either a symbol representing an unspecified term of a theory, or a basic object of a theory that is

manipulated without referring to its possible intuitive interpretation. Thus, given a class of sentences that have the same form, we can capture their common form by replacing their specific subjects with a variable x. For instance, given sentences such as "Plato is a philosopher," "Aristotle is a philosopher," "Kant is a philosopher," "Frege is a philosopher," etc., which have the same form, we can replace the name of the subject with a variable x, thus formulating the sentence "x is a philosopher," which captures the common form of the aforementioned sentences. In this way, we obtain a class: all the things that can satisfy the sentence "x is a philosopher," whenever we replace x with a name, belong to the class of philosophers. Hence, Plato, Aristotle, Kant, Frege, and any other person whom we could substitute for x are members of the class of philosophers.

According to Frege's terminology, whereas propositions are declarative statements that are either true or false, such as the statement "Plato is a philosopher," a statement that contains a variable x and expresses a proposition as soon as a value is assigned to x is a propositional function, such as the statement "x is a philosopher." In other words, propositions and propositional functions differ from each other by the fact that propositional functions are ambiguous, in the sense that a propositional function contains a variable of which the value is unassigned. A class is the extension of a propositional function; for instance, the collection of all philosophers constitutes the extension of the propositional function "x is a philosopher," and is a class. Frege used the so defined concept of a class in order to refer to numbers and study the foundations of arithmetic.

According to Frege, numbers are classes. In his seminal book *Basic Laws of Arithmetic* (1893, 1903), Frege explained that any number n can be used in order to count any n-membered class. For instance, the number two can be thought of as the class of all 2-membered things, namely, as the class of all pairs, independently of the nature of the objects that constitute each pair. Similarly, the number three can be thought of as the class of all triples, namely, as the class of all those things that have three members; the number four can be thought of as the class of all quadruples, namely, as the class of all those things that have four members, etc. Collect all those things that have n members, and that, according to Frege, is the number n. Notice that this way of defining numbers is substantively different from the thesis that a number is a property of a collection of objects, because, according to Frege's conception of numbers, a number is a particular kind of object, it is a class. Frege built a whole system of logic on the aforementioned concept of a class.

In order to define the concept of a natural number, in particular, Frege defined, for every 2-place relation R, the concept "x is an ancestor of y in

the R-series," and this new relation is known as the "ancestor relation on R." The underlying idea can be easily grasped if we interpret Frege's 2-place relation R as "x is the father of y in the R series." For instance, if a is the father of b, b is the father of c, and c is the father of d, then Frege's definition of "x is an ancestor of y in the fatherhood-series" ensures that a is an ancestor of b, c, and d, that b is an ancestor of c and d, and that c is an ancestor of d. More generally, given a series of facts of the form aRb, bRc, and cRd, Frege showed that we can define a relation R^* as "y follows x in the R-series." Thus, Frege formulated a rigorous definition of "precedes," and he concluded that a "natural number" is any number of the predecessor-series beginning with 0.

Chapter 1
Mathematical Logic

By the term "deductive system," we mean a calculus endowed with an interpretation of its terms. A "calculus" is a collection of symbols equipped with a set of rules for their manipulation. When a calculus is equipped with an "interpretation" of its terms, that is, with a set of rules that makes its terms meaningful, then it becomes a deductive system. A deductive system is called "pure" if the rules of the interpretation are sufficient to establish the truth or the falsity of its constituent statements. The statements of a pure deductive system are called "L-determinate," where L stands for the relevant formal language (the truth value of an L-determinate statement is determined in L by an interpretation of the symbols in L). For instance, logic (the science of correct reasoning) and mathematics are pure deductive systems. Therefore, truths derived from pure deductive systems are based on reason alone, and they are certain because they can never be empirically refuted. If a statement cannot be assigned a truth value only according to the rules of interpretation in the relevant deductive system, then it is called "non-L-determinate." A non-L-determinate statement is called true or false not only on the basis of the rules of interpretation in the relevant deductive system, but also on the basis of a rule of disposition by reference to empirical data. Non-L-determinate statements for which a rule of disposition by reference to empirical data has been established are called "factual statements," while the deductive systems in which they appear are called "applied."

A "scientific theory" is a deductive system (pure or applied) that explains generalizations (i.e., "scientific laws") or aims to criticize and change the structure of the world and/or consciousness.

In symbolic or mathematical logic, the following symbols are used:

∧ or &: conjunction ("and");
∨: disjunction ("or");
¬: negation ("not");
→ or ⇒: material implication ("if . . . then . . .");
↔ or ⇔: biconditional ("if and only if");
∀: universal quantification ("for every");
∃: "there exists";
∃!: "there exists exactly one";
∄: "there does not exist";
$P(x)$: predicate letter (meaning that x (an object) has property P);
|: "such that";

$\vdash$: turnstile ($x \vdash$ y means that x "proves" (i.e., syntactically entails) y; a sentence φ is "deducible" from a set of sentences Σ, expressed $\Sigma \vdash \varphi$, if there exists a finite chain of sentences $\psi_0, \psi_1, \psi_2, \ldots, \psi_n$ where ψ_n is φ and each previous sentence in the chain either belongs to Σ, or follows from one of the logical axioms, or can be inferred from previous sentences; $\nvdash$ denotes the negation of $\vdash$);

$\vDash$: double turnstile ($x \vDash$ y means that x "models" (i.e., semantically entails) y; a sentence φ is a "consequence" (i.e., an ordered list) of a set of sentences Σ, expressed $\Sigma \vDash \varphi$, if every model of Σ is a model of φ);

$B \subseteq A$: B is a "subset" of A, meaning that every element of a set B is an element of a set A;

$B \subset A$: B is a "proper subset" of A, meaning that $B \subseteq A$ and there is at least one element of A that is not an element of B;

$x \leq y$: x is less than or equal to y;

$x < y$: x is strictly less than y;

$x \geq y$: x is greater than or equal to y;

$x > y$: x is strictly greater than y;

x^n: this operation is called "exponentiation" (pronounced as "x raised to the power of n"), and it means that x is multiplied by itself n times, where $n = 0,1,2,3, \ldots$; $x^0 = 1$, $x^1 = x$, $x^2 = x \cdot x$, $x^3 = x \cdot x \cdot x$, etc.;

$x^{1/n}$: this operation is called the "nth root," and it is the number whose nth power equals the given number ($n \neq 0$); $x^{1/2} = \sqrt{x}$ is the square root, $x^{1/3} = \sqrt[3]{x}$ is the third root, etc.;

( ): brackets; they are used for convenience in grouping terms together (there are specific rules for removing brackets).

The English mathematician and philosopher George Boole (1815–64) realized that arguments expressed in an ordinary language (e.g., in ordinary English) can be expressed in the notation of mathematical logic and then studied in the context of "propositional calculus." For instance, consider the following argument:

- If you want to learn mathematics, then you must study methodically.
- If you must study methodically, then you must be taught an effective method of studying.
- Therefore, if you want to learn mathematics, then you must be taught an effective method of studying.

The aforementioned argument involves various propositions, which we may present by letters as follows:

P: You want to learn mathematics.
Q: You must study methodically.
R: You must be taught an effective method of studying.
These propositions can be "true" or "false." The aforementioned argument can be formalized as follows:

$P \Rightarrow Q$
$Q \Rightarrow R$

$P \Rightarrow R$

where the two propositions above the dashed line are the "premises," and the one below the dashed line is the "conclusion." The reasoning process that leads from premises to a conclusion is called a "deductive process" or just a "deduction." A "theorem" is a formula inferred by means of a rule of inference in a finite number of steps from axioms and previously inferred formulas. Those propositions where truth value is dependent on the values of the variables in them are called "predicates" (hence, we talk about "predicate calculus").
It is important to distinguish between the terms "validity" and "truth" as they are used in logic. An argument, a reasoning process, or a deduction is said to be valid (i.e., logically correct) if the truth of the conclusion follows from the truth of the premises. Notice that, if the premises are both true, then the conclusion is logically necessarily true, too. Therefore, with one or more factually incorrect premises, an argument may still be valid, although its conclusion may be false. Furthermore, a valid argument based on false premises does not necessarily lead to a false conclusion. In other words, there is a significant difference between *logical* (i.e., procedural) correctness ("validity") and *factual* correctness. If an argument is valid (i.e., logically correct), and if its premises are true (i.e., if the facts on which it is based are true), then it is said to be "sound." In logic, we focus on the validity of arguments rather than on their soundness, and this fact explains the "instrumental" role of logic in philosophy and science.
A "Boolean algebra" is the six-tuple
$\langle A, \wedge, \vee, \neg, 0, 1 \rangle$,
consisting of a set A equipped with two binary operations: $\wedge$ (called "meet" or "and") and $\vee$ (called "join" or "or"), a unary operation $\neg$ (called "complement" or "not"), and two elements 0 and 1 in A (called "bottom" and "top" respectively, and denoted by the symbols $\perp$ and $\top$ respectively), such that the truth value of a true sentence is 1, the truth value of a false sentence is 0, and, for all elements a, b, and c of A, the following axioms hold:

i. Associativity:

$a \vee (b \vee c) = (a \vee b) \vee c;\ a \wedge (b \wedge c) = (a \wedge b) \wedge c.$

ii. Commutativity:
$a \vee b = b \vee a;\ a \wedge b = b \wedge a.$

iii. Absorption:
$a \vee (a \wedge b) = a;\ a \wedge (a \vee b) = a.$

iv. Identity:
$a \vee 0 = a; a \wedge 1 = a.$

v. Distributivity:
$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c);\ a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$

vi. Complements:
$a \vee \neg a = 1$ and $a \wedge \neg a = 0$.

For instance, the 2-element Boolean algebra has only two elements, namely, 0 and 1, and it is defined by the following rules:

Table 1: Truth Tables of a 2-Element Boolean Algebra.

a	b	$a \wedge b$	$a \vee b$	a	$\neg a$
0	**0**	0	0	**0**	1
1	**0**	0	1	**1**	0
0	**1**	0	1		
1	**1**	1	1		

Formalism, Structuralism, and Mathematical Modelling

The formalist approach to mathematics maintains that, in order to analyze a mathematical text, it suffices to study its formal devices, mainly, its syntax. Hence, according to formalism, mathematical statements are statements about the consequences of the manipulation of strings (i.e., alphanumeric sequences of symbols, usually presented in the form of equations) using established rules of inference (by a "rule of inference," we mean a logical form consisting of a function that takes premises, analyzes their syntax, and returns a conclusion). In other words, according to formalism, mathematics does not consist of propositions representing an abstract sector of reality, but it is actually a game of symbols, without having more ontological commitments than, for instance, chess.

In the 1930s, the great Austrian mathematician and logician Kurt Gödel undertook to evaluate the logical rigor of formalism. Broadly speaking, Gödel considered a statement of the type

$P =$ "This statement is false,"

which leads to the following complicated situation: if $\boldsymbol{P} =$ "This statement is false" is true, then it is false, but the sentence asserts that it is false, and, if it is, indeed, false, then it must be true, and so on. The earliest study of

problems pertaining to self-reference in logic is due to the seventh-century B.C.E. Greek philosopher and logician Epimenides, who formulated the classical "liar paradox." Gödel's Incompleteness Theorem shows that such complicated situations can occur in any theory that is consistent and comprehensive enough to contain elementary arithmetic as the latter has been encoded by Peano's axioms for natural numbers (see Chapter 2). Consequently, logic is necessary and capable of organizing every mathematical and, generally, scientific theory, but logic is not sufficient to completely organize itself. According to Gödel, human consciousness, in general, and thought processes, in particular, are not merely algorithmic. Gödel established the following argument mathematically: Either the human mind (even within the realm of pure mathematics) infinitely surpasses any finite machine (algorithmic process), or else there exist absolutely undecidable arithmetic propositions (see: S. G. Shanker, ed., *Gödel's Theorem in Focus*, London: Routledge, 1991).

Formalism rightly stresses the importance of syntax and, particularly, of logical consistency, but it cannot stand as a general theory of the epistemology of mathematics or any other scientific discipline. Therefore, we have to turn from formalism to structuralism. Structuralism is concerned with the analysis of the underlying structures in a text. The structure of a mathematical text can be explained and described as follows: Let C denote the set of all basic conceptual objects (i.e., the "universe" of concepts), R the set of all basic conceptual relations, and A the set of the axioms of a structure. Then the corresponding structure is denoted by $\mathcal{S}(C, R, A)$. A segment of a structure is a set of concepts, definitions, and judgments of the given structure, it satisfies the axioms of the given structure as well as some additional conditions, and it is denoted by $\bar{\mathcal{S}}(\bar{C}, \bar{R}, \bar{A})$. Suppose that a phenomenon of the sensory-sensuous world has been described by a structure $\mathcal{S}(C, R, A)$, or by a segment of this structure. Both the phenomenon and its mathematical model can be regarded as two isomorphic models, since the original phenomenon is initially modelled by our perception of it. More precisely, it is modelled by the initial reference of our consciousness to it, and its mathematical model is $\mathcal{S}(C, R, A)$ or a segment of $\mathcal{S}(C, R, A)$.

The creation of isomorphisms between mathematics and other scientific disciplines or human activities is called mathematical modelling. Thus, mathematical modelling consists of two stages: (i) the formulation of the mathematical model of the object that one studies—that is, the transformation of the given problem into a mathematical one—and (ii) the solution of the corresponding mathematical problem, namely, the

processing of the information that is contained in the given problem by means of mathematics and mathematical informatics.

Regarding the logical-mathematical modelling of problems that belong to the realm of the social sciences, in particular, the value-system of the society in which behavior is studied must somehow find its place in the framework of action employed in the relevant analysis (see: Talcott Parsons, *The Structure of Social Action*, Glencoe, Ill.: The Free Press, 1949). In the philosophy of the social sciences, by the term "values," we mean needs that arise in consciousness and which consciousness must address. For instance, needs to know, to reap, to sustain, to socialize, to individuate, to control, to act, and so on.Then, consciousness selects some specific values-needs which it projects onto the world, thus transforming them into historical objects, and, finally, the values that have been historically objectified, specifically, have become social and institutional events, influence consciousness, shaping the subject's existential horizon. The consciousness of existence that not only functions as a witness, that is, diagnostically and ascertainably, but also functions as a judge is what we call moral consciousness. The logic of moral consciousness, that is, the logic of ethics, is called "deontic logic." Ethics is concerned with what good as a concept is and with what we should and should not do. Deontic logic is concerned with the manner in which we can represent those things that we should and should not do logically. Thus, deontic logic builds a bridge between logical rigor and ethics. For a systematic study of deontic logic, one should read the following books by Giuliano Di Bernardo: *Introduzione alla logica dei sistemi normativi*, Bologna: Il Mulino, 1972; *Le regole dell'azione sociale*, Milano: Il Saggiatore, 1983; and *Normative Structures of the Social World*, Amsterdam: Rodopi, 1988.

Chapter 2
Arithmetic and Algebra

The attempts of nineteenth-century mathematicians to found mathematical analysis in a rigorous way were based on real numbers, which also needed a rigorous foundation. Numbers are abstract objects, concepts. Simultaneously, they are intimately related to the world, since we organize the world with them (that is, we count, we measure, and we form scientific theories with numbers). In order to understand the concept of a number, we have to keep in mind that what we count are not "things," but "sets of things."

The history of set theory and of non-numerical mathematics, in general, can be traced back to the era of classical Greece, but the first systematic inquiry into the foundations of set theory was due to the German mathematician Georg Ferdinand Ludwig Philip Cantor (1845–1918). However, before Cantor, George Peacock (1791–1858), Augustus De Morgan (1806–71), and George Boole (1815–64) had already made significant contributions to the formalization of non-numerical mathematics. According to Cantor, by the term "set," we should understand a well-defined gathering together into a whole of definite, distinguishable objects of perception or of our thought that are called elements of the set. By the term "well-defined," Cantor means that, given any object and any set, the given object is either an element of the given set or not an element of the given set. By the terms "definite" and "distinguishable," Cantor means that no two elements of a set are the same.

The empty set is denoted by $\emptyset$. The empty set has no elements. If a set has only one element, then it is called a "singleton."

If every element of a set B is an element of a set A, then B is said to be a "subset" of A, and we write $B \subseteq A$. Every set is a subset of itself. If A is an arbitrary set, then $\emptyset \subseteq A$; that is, the empty set is a subset of every set. Two sets A and B are "equal" if and only if $A \subseteq B$ and $B \subseteq A$, and then we write $A = B$. If two sets A and B satisfy the condition $B \subseteq A$ and there is at least one element of A that is not an element of B, then B is said to be a "proper subset" of A, and we write $B \subset A$. If $B \subseteq A$ or $B \subset A$, then A is said to be a "superset" of B. When in a particular situation all the sets under consideration are subsets of a fixed set, this fixed set, which is the superset of every set under consideration, is called the "universal set," or the "universe of discourse."

If the elements of a set are sets themselves, then the set is called a "set of sets," "family of sets," "collection of sets," or "class of sets." For instance,

$\mathcal{C} = \{\{x\}, \{y, z\}\}$ is a class of sets (notice that x is something different from $\{x\}$).

If A and B are two arbitrary sets, then we define their

i. "union":
$A \cup B =$
$\{every\ x\ such\ that\ x\ belongs\ to\ at\ least\ one\ of\ A\ and\ B\}$;
and

ii. "intersection":
$A \cap B = \{every\ x\ such\ that\ x\ belongs\ to\ both\ A\ and\ B\}$.

Two sets are called "(relatively) disjoint" if their intersection is the empty set.

The German mathematician, logician, and philosopher Friedrich Ludwig Gottlob Frege (1848–1925) has explained that any number n can be used in order to count any n-membered set. For instance, the number two can be thought of as the set of all 2-membered sets, or as the set of all pairs, independently of the nature of the objects that constitute each pair. Similarly, the number three can be thought of as the set of all triplets, the number four can be thought of as the set of all quadruples, and so on.

In particular, in order to define the concept of a natural number $(0,1,2,3, \dots, n, n+1, \dots)$, Frege defined, for every 2-place relation R, the concept "x is an ancestor of y in the R-series," and this new relation is known as the "ancestor relation on R." The underlying idea can be easily grasped if we interpret Frege's 2-place relation R as "x is the father of y in the R-series." For instance, if a is the father of b, b is the father of c, and c is the father of d, then Frege's definition of "x is an ancestor of y in the fatherhood-series" ensures that a is an ancestor of b, c, and d, that b is an ancestor of c and d, and that c is an ancestor of d. More generally, given a series of facts of the form aRb, bRc, and cRd, Frege showed that we can define a relation R^* as "y follows x in the R-series." Thus, Frege formulated a rigorous definition of "precedes," and he concluded that a "natural number" is any number of the predecessor-series beginning with 0.

Using the concept of a "predecessor," the American mathematician John von Neumann (1903–57) has proposed an even more accurate definition of a "natural number." According to von Neumann, instead of defining a natural number n as the set of all n-membered sets, a natural number n should be defined as a particular n-membered set—more specifically, as the set of its predecessors. For instance, the number two having two predecessors, zero and one, we can think of the number two as the set $\{0,1\}$, where zero has no predecessor. Therefore, zero can be thought of as the empty set, denoted by $\emptyset$. The number one has only one predecessor,

zero. Therefore, we can think of the number one as $\{\emptyset\}$, namely, as the singleton of the empty set. Thus, von Neumann formulated the modern definition of "ordinal numbers." In particular, given the "successor operation," which is defined as

$succesor(n) = n \cup \{n\}$,

the set of von Neumann natural numbers, the ordinal numbers, denoted by ω, is defined as follows:

i. $\emptyset \in \omega$.
ii. If $n \in \omega$, then $succesor(n) \in \omega$.
iii. Nothing belongs to ω unless it can be constructed using the preceding rules.

Thus, we obtain the following definitions:

$0 = \emptyset$.

$1 = successor(0) = \emptyset \cup \{\emptyset\} = \{\emptyset\} = \{0\}$.

$2 = successor(1) = \{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\} = \{0,1\}$.

$3 = successor(2) = \{\emptyset, \{\emptyset\}\} \cup \{\{\emptyset, \{\emptyset\}\}\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = \{0,1,2\}$.

$\vdots$

Let X be a set of elements $a, b, \ldots$ Suppose that there is a binary relation expressed by $a \prec b$, defined between certain pairs (a, b) of elements of X, and satisfying the following properties:

$a \prec a$;

if $a \prec b$ and $b \prec a$, then $a = b$;

if $a \prec b$ and $b \prec c$, then $a \prec c$ (transitivity).

Then X is said to be "partially ordered" (or "semi-ordered") by the relation $\prec$.

Let X be a partially ordered set with elements $a, b, \ldots$ If $a \prec c$ and $b \prec c$, then c is said to be an "upper bound" for a and b. If, furthermore, $c \prec d$ whenever d is an upper bound for a and b, we call c the "least upper bound," or the "supremum," of a and b, and we write $sup(a, b)$. This element of X is unique if it exists. In a similar way, we define the "greatest lower bound," or the "infimum," of a and b, and we denote it by $inf(a, b)$.

A partially ordered set X is said to be "linearly ordered" (or "totally ordered") if, for every pair (a, b) in X, either $a \prec b$ or $b \prec a$ holds. A subset of a partially ordered set X is itself partially ordered by the relation that partially orders X; and the subset may even be linearly ordered by this relation. If X is a partially ordered set and A is a subset of X, then an element $m \in X$ is said to be an upper bound of A if $a \prec m$ for every $a \in A$. An element $m \in X$ is said to be "maximal" if the relations $m \in X$ and $m \prec x$ imply that $m = x$ (the maximum is the largest number of the set,

while the supremum is the smallest upper bound of the set). In a similar way, we define a "minimal element."

The major sets of numbers are the following:

The Natural Numbers

$\mathbb{N}$: the "natural numbers," namely, the positive integers $1,2,3,\ldots$, which are used to count objects, and 0. For any natural numbers m, n, and k, the following equalities hold true:

i. $m + n = n + m$,
ii. $m + (n + k) = (m + n) + k$,
iii. $mn = nm$,
iv. $m(nk) = (mn)k$,
v. $m(n + k) = mn + mk$,
vi. $m \cdot 1 = m$.

Equalities (i) and (iii) express the "commutative law" of addition and multiplication respectively; equalities (ii) and (iv) express the "associative law" of addition and multiplication respectively; and equality (v) is known as the "distributive law" of multiplication over addition. The aforementioned laws underlie all computations. If a natural number m is divisible by a natural number n, then m is said to be a "multiple" of the number n, and n, in turn, is said to be the "divisor" of the number m. If m is a multiple of the number n, then there is a natural number k such that $m = kn$. For instance, 18 is divisible by 3, and we write $18 = 6 \cdot 3$. In this case, $m = 18$ (the "dividend"), $n = 3$ (the "divisor"), and $k = 6$ (the "quotient"). If a natural number m is not exactly divisible by a natural number n, that is, if there is no natural number k such that $kn = m$, then we consider "division with a remainder." For instance, 33 divided by 2 equals 16 ("partial quotient") with a remainder of 1, and therefore $33 = 16 \cdot 2 + 1$.

For any two natural numbers a and b, there exists a unique natural number n such that $a \cdot n = b$ if and only if a is a divisor of b, and then we write $n = b \div a$. Even numbers are divisible by 2 without remainders, whereas odd numbers are not evenly divisible by 2. Notice that odd numbers end in the digit 1, 3, 5, 7, or 9.

The greatest common divisor (denoted by gcd) of two natural numbers a and b is the largest natural number that divides both a and b, and the Euclidean Algorithm for computing $gcd(a, b)$ is as follows:

i. If $a = 0$, then $gcd(a, b) = b$.
ii. If $b = 0$, then $gcd(a, b) = a$.

iii. If a and b are both non-zero natural numbers, then we write a in quotient remainder form, namely, $a = b \cdot q + r$, and, subsequently, we compute $gcd(b,r)$ using the Euclidean Algorithm since $gcd(a,b) = gcd(b,r)$. For instance, if $a = 280$ and $b = 120$, then we can compute $gcd(a,b)$ as follows: firstly, we use long division to find that $\frac{280}{120} = 2\ with\ a\ remainder of\ 40$, which can be written as $280 = 120 \times 2 + 40$; secondly, we compute $gcd(120,40) = 3\ with\ a\ remainder of\ 0$; and, therefore, $gcd(280,120) = 40$.

Let a and b be both non-zero natural numbers. Moreover, let $lcm(a,b)$ denote the least common multiple of a and b (i.e., $lcm(a,b)$ is the smallest natural number that is evenly divisible by both a and b). Then $gcd(a,b) = \frac{a \cdot b}{lcm(a,b)} \Leftrightarrow lcm(a,b) = \frac{a \cdot b}{gcd(a,b)}$.

If a natural number has only two divisors, a unity (one) and the number itself, then it is called a "prime number"; if it has more than two divisors, then it is called a "composite number." For instance, 2, 3, 5, and 7 are prime numbers, but 9 is not a prime number (9 is a composite number, because the divisors of 9 are 1, 3, and 9). Notice that 2 is the only even prime number, and that, except for 2 and 5, all prime numbers end in the digit 1, 3, 7, or 9. All numbers have prime factors. For instance, the prime factors of 10 are 2 and 5, since $10 = 2^1 \times 5^1$; the prime factors of 11 are 1 and 11, since $11 = 1^1 \times 11^1$; the prime factors of 100 are 2 and 5, since $100 = 2^2 \times 5^2$, etc.

The Italian mathematician and glottologist Giuseppe Peano (1858–1932) has organized the natural numbers as an axiomatic system on the basis of the following axioms, known as the "Peano axioms":

i. 0 is a natural number, symbolically: $0 \in \mathbb{N}$.
ii. If n is a natural number, then the successor of n (i.e., $successor(n) = n + 1$) is also a natural number.
iii. If two natural numbers have the same successor, then the two natural numbers are identical.
iv. 0 is not the successor of any natural number.
v. "Induction Axiom": If X is a set containing both 0 and the successor of every natural number belonging to X, then every natural number belongs to X.

The "Induction Axiom" gives rise to and underpins the principle of "Mathematical Induction," which is a mathematical proof technique for propositions: Suppose that P is a proposition defined on the natural numbers $\mathbb{N}$, such that:

i. $P(1)$ is true, that is, P holds true for 1;

ii. $P(n+1)$ is true whenever $P(n)$ is true.

Then P is true for every natural number. In this case, P is the "inductive hypothesis." By completing the aforementioned two steps of mathematical induction, we prove that P is true for every natural number.

Example: Let P be the proposition that the sum of the first n natural numbers is $\frac{1}{2}n(n+1)$, namely: $P(n) = 1+2+3+\cdots+n = \frac{1}{2}n(n+1)$. We can prove that P is true for every natural number $n \in \mathbb{N}$ using mathematical induction as follows:

Basis step: The proposition holds for $n = 1$, because $1 = \frac{1}{2}(1)(1+1)$. Hence, $P(1)$ is true.

Induction step: We assume that $P(n)$ is true, and we add $n+1$ to both sides of $P(n)$, obtaining

$$1+2+3+\cdots+n+(n+1) = \frac{1}{2}n(n+1)+(n+1) =$$
$$\frac{1}{2}[n(n+1)+2(n+1)] = \frac{1}{2}[(n+1)(n+2)],$$

which is $P(n+1)$. Hence, $P(n+1)$ is true whenever $P(n)$ is true. By the principle of mathematical induction, P is true for every natural number $n \in \mathbb{N}$.

The Integral Numbers

$\mathbb{Z}$: the "integral numbers," or the negative integers, zero, and the positive integers:

$$\ldots -3, -2, -1, 0, 1, 2, 3, \ldots$$

The notation $\mathbb{Z}$ for the set of integers derives from the German word "Zahlen," which means "numbers."

From the perspective of ancient mathematicians, numbers are things by means of which we count, but modern mathematical analysis, founded on Cartesianism, understands numbers mainly as positions on the number line. Let us draw a straight line l and mark on it a point 0 that will be taken as the origin. Then we choose a unit segment $0P$, where P is a natural number that lies to the right of 0, and, in this way, we specify the positive direction. In other words, the unit segment $0P$ determines the direction of the number line and corresponds to the positive unity $+1$ (or simply 1). Let us, for instance, take the number 4. Laying off the unit segment from the point 0 in the given direction four times, we obtain the point Q that corresponds to the number 4. Let us now lay off four unit segments from the zero point in the direction opposite to the specified. We then get the point Q', which is symmetric to the point Q about the origin 0. The point Q' corresponds to the number -4. Thus, the numbers 4 and -4 are said to

be "opposite." By analogy, we can define any other integer (positive or negative). In general, the numbers situated on the number line l in the specified direction are said to be "positive," whereas the numbers located on the number line in the direction opposite to the given one are said to be "negative." Hence, the natural numbers and their opposites (the opposite of the number zero being the same number) form together the set of integral numbers (integers), which is denoted by $\mathbb{Z}$.

If a point X of the line l corresponds to some number r, then this number is said to be the "coordinate of the point X," and, in this case, we write $X(r)$.

The "absolute value" of the number r is denoted by $|r|$. The absolute value of any positive number is the number itself. The absolute value of any negative number is equal to its opposite number. The absolute value of the number zero is zero.

The sum of two negative numbers is a negative number. In order to find the absolute value of a sum, it is necessary to add together the absolute values of the addends. The sum of two numbers having unlike signs is a number that has the same sign as the addend with greatest absolute value. In order to find the absolute value of their sum, it is necessary to subtract the smaller value from the larger one.

In order to subtract one number from another, it is necessary to add to the minuend a number that is the opposite of the subtrahend.

The product (resp. quotient) of two negative numbers is a positive number. The product (resp. quotient) of two numbers having unlike signs is a negative number. In order to find the absolute value of a product (resp. quotient), it is necessary to multiply (resp. divide) the absolute values of these numbers.

If the difference of two integers a and b is divisible by n, then a and b are said to be congruent with respect to the modulus n, and this is symbolically expressed as follows:

$$a \equiv b(mod n)$$

and each of the numbers a and b is said to be a residue $(mod n)$ of the other. With respect to a given modulus, every number a has an infinite number of residues which are included in the expression $a + \lambda n$ where λ is any integer. Every linear congruence with one unknown quantity can be reduced to the form

$$ax \equiv b(mod n).$$

For instance, to solve $17 \equiv x(mod 5)$, we think as follows: we have three bundles of 5 with a remainder of $17 - 5 = 2$, namely, $5 \times 5 \times 5 + 2$, so that $\frac{17}{5} = 3\ with\ a\ remainder\ of\ 2$, and, therefore, $x = 2$.

The Rational Numbers

$\mathbb{Q}$: the "rational numbers," namely, the set of all numbers of the form $\frac{p}{q}$ such that the numbers p and q are integers, $q \neq 0$, and the greatest common divisor of the integers p and q is ± 1 (that is, p and q are relatively prime integers). In other words, the integral and the fractional numbers (both positive and negative) form together the set of rational numbers, which is denoted by $\mathbb{Q}$. The notation $\mathbb{Q}$ for the set of rational numbers derives from the Italian word "quoziente," which means "quotient."

By the term "common fraction," we refer to a number of the form $\frac{m}{n}$, where m and n are integral numbers, and $n \neq 0$. The number m is called the "numerator" of the fraction, and the number n is called the "denominator" of the fraction. In particular, n may be equal to 1. In this case, we usually write m rather than $\frac{m}{1}$. In other words, any integral number can be represented in the form of a common fraction whose denominator is 1.

Two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are regarded to be equal if $ad = bc$. The "basic property of fractions" states the following: the fractions $\frac{a}{b}$ and $\frac{am}{bm}$ are equal. Therefore, if the numerator and the denominator of a given fraction are multiplied or divided by the same natural number, then an equivalent fraction is obtained (namely: $\frac{a}{b} = \frac{am}{bm}$). Taking advantage of the basic property of fractions, we may sometimes replace a given fraction with another equivalent fraction but with a smaller numerator and a smaller denominator by dividing all common factors out of the numerator and the denominator. This operation is called "reduction of a fraction to its lowest terms," or simply "reduction of a fraction." In general, reduction of a fraction is always possible if its numerator and denominator are not relatively prime numbers. If the numerator and the denominator are relatively prime numbers, then the fraction is called "irreducible."

The addition of common fractions is defined in the following way:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+b}{bd}.$$

The subtraction of common fractions is defined in the following way:

$$\frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}.$$

The multiplication of common fractions is defined in the following way:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

The division of common fractions is defined in the following way:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a/b}{c/d} = \frac{ad}{bc}.$$

A fraction $\frac{m}{n}$ is called a "proper fraction" if its numerator is less than the denominator; and it is called an "improper fraction" if its numerator is greater than the denominator.

Let us consider an improper fraction $\frac{m}{n}$. Since m is greater than n, there are two numbers p and r (with r less than n) such that $m = pn + r$, so that: $\frac{m}{n} = \frac{pn+}{n} = \frac{pn}{n} + \frac{r}{n} = p + \frac{r}{n}$. Since the remainder is always less than the divisor, $\frac{r}{n}$ is a proper fraction. Hence, we have succeeded in representing the improper fraction $\frac{m}{n}$ in the form of a sum of a natural number p and a proper fraction $\frac{r}{n}$. This operation is called the "reduction of an improper fraction to a mixed number." A number consisting of an integer and a fraction is called a "mixed number." For instance, in order to locate the mixed number $3\frac{1}{8}$ on the number line, we think as follows: laying off the unit segment $(0P = +1)$ from the point zero in the given (positive) direction three times and then $\frac{1}{8}$th part of this unit segment, we obtain the point Q that exactly corresponds to the mixed number $3\frac{1}{8}$ (the coordinate of the point Q is $3\frac{1}{8}$).

The Irrational Numbers

$\mathbb{Q}^{\sim}$: the "irrational numbers," or the set of all numbers that cannot be written as the quotient of two relatively prime integers. For instance, we can prove that $\sqrt{2} \in \mathbb{Q}^{\sim}$ by *reductio ad absurdum* as follows: For the sake of contradiction, suppose that $\sqrt{2} = \frac{p}{q}$ where $p, q \in \mathbb{Z}$, the greatest common divisor of the integers p and q is ± 1, and $q \neq 0$. Then

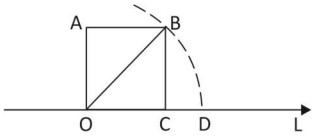
$$\sqrt{2} = \frac{p}{q} \Rightarrow 2 = \frac{p^2}{q^2} \Rightarrow p^2 = 2q^2 \Rightarrow p = 2k,$$

where k is an appropriate integer; therefore $4k^2 = 2q^2 \Rightarrow q^2 = 2k^2$; but then the greatest common divisor of the integers p and q is 2, which contradicts the hypothesis.

The German mathematician Richard Dedekind (1831–1916) observed that there exist infinitely many points on the straight number line L that correspond to no rational number. Thus, the domain of rational numbers is insufficient if we want to arithmetically follow up all phenomena on the straight line. Therefore, new numbers must be created in such a way that the domain of all numbers will gain the same "completeness" or "continuity" as the straight line. In fact, Dedekind observed that there exist infinitely many cuts that are not produced by rational numbers. For instance, as shown in Figure 1, construct a square $OABC$ on the unit

segment OC (i.e., the length of OC is equal to one) and lay off in the positive direction a line segment OD equal in length to the diagonal OB; then (according to the Pythagorean Theorem, which we shall study shortly) it is clear that D is a point that does not correspond to any rational number—it, in fact, corresponds to $\sqrt{2}$.

Figure 1: Irrational numbers.



The history of irrational numbers goes back to the Pythagorean mathematicians, who had demonstrated that there exist lengths incommensurable with a given unit of length. In the seventh century B.C.E., Thales of Miletus (a Greek mathematician, astronomer, and philosopher from Miletus, in Ionia, Asia Minor) officially initiated a new approach to mathematics. In contrast to the mathematics of other civilizations, such as the Babylonians and the Egyptians, Thales's approach to mathematics is based on the thesis that scientific propositions are not recipes for practical tasks—that is, techniques whose validity is determined by the method of trial and error—but they should be explained and proved. In other words, Thales attempted to endow mathematics with rigor—which, in this case, means logical validity.

In the context of Thales's rigorous mathematics, by the term "line segment," we mean a part of a line that is bounded by two distinct endpoints, and contains every point on the line between the endpoints. Let us consider the line segments $a_1, a_2, a_3, \dots, a_n$ and the non-zero line segments $b_1, b_2, b_3, \dots, b_n$. The line segments $a_1, a_2, a_3, \dots, a_n$ are said to be "proportional" to $b_1, b_2, b_3, \dots, b_n$, respectively, if

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \cdots = \frac{a_n}{b_n}.$$

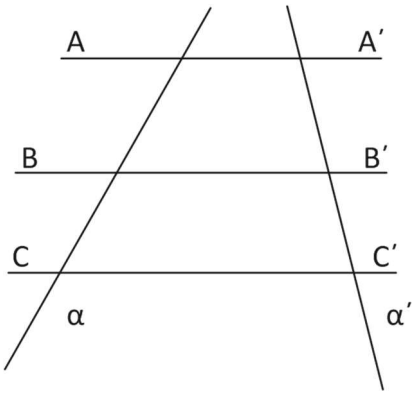
Thus, two arbitrary line segments a and c are proportional to two other arbitrary line segments b and d respectively, if and only if b and d are non-zero, and it holds that

$$\frac{a}{b} = \frac{c}{d}. \tag{1}$$

Any equality between two ratios, such as (1), is said to be a "proportion" with terms a, b, c, and d, as shown above.

Assume that AB is a non-zero straight line segment, and that P is a point on AB. Then we say that the point P "divides internally" the straight line segment AB in a ratio λ, where $\lambda \geq 0$, if it holds that
$\frac{PA}{PB} = \lambda$.
If this is the case, then
$\frac{PA}{PA+PB} = \frac{\lambda}{\lambda+1} \Leftrightarrow PA = \frac{\lambda}{\lambda+1} AB$, which implies the uniqueness of P.
Similarly, we say that a point Q "divides externally" the straight line segment AB in a ratio λ, where $\lambda \geq 0$, if the points A, B, and Q are collinear, Q is external to AB, and it holds that
$\frac{QA}{QB} = \lambda$.
If this is the case, then $\frac{QA}{|QA-Q|} = \frac{\lambda}{|\lambda-1|}$ (given that $QA \neq QB$, it holds that $\lambda \neq 1$), so that
$QA = \frac{\lambda}{|\lambda-1|}$ AB, which implies the uniqueness of Q.
Thales's Theorem: If parallel straight lines intersect two straight lines, then they define proportional straight line segments on them. For instance, if parallel straight lines l_1, l_2, and l_3 intersect straight lines a and a' at points A, B, C and A', B', C' respectively, as shown in Figure 2, then
$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$.

Figure 2: Thales's Theorem.



Corollary 1: Every straight line that is parallel to the bases of a trapezoid divides, internally or externally, the non-parallel sides of the given trapezoid in equal ratios.

Corollary 2: Every straight line that is parallel to one side of a triangle divides, internally or externally, the other two sides of the given triangle in equal ratios.
Corollary 3: If two triangles have a common angle, and if they have parallel opposite sides, then they are said to be in Thales position, and then they are similar and have proportional sides.
In the sixth century B.C.E., Pythagoras and his school (the so-called "Pythagoreans") endorsed Thales's approach to mathematics. From the Pythagorean perspective of mathematics, the relations between the objects of the world (e.g., magnitudes) correspond to the relations between natural (and, generally, integral) numbers. However, it was soon realized that things are not so simple, since it was realized that there exist magnitudes that do not have a common measure. According to the Pythagoreans, two objects (magnitudes) are "commensurable" (that is, they have a common measure) if and only if there is a magnitude of the same kind that is contained an integral number of times in both of them. In other words, two magnitudes are "commensurable" if and only if their ratio is a rational number. However, the Pythagoreans encountered "incommensurable" magnitudes: magnitudes whose ratio is an irrational number. For instance, as shown in Figure 1, the length of a diagonal of a unit square, specifically of a square whose sides have length 1, is, according to the Pythagorean Theorem, equal to $\sqrt{2}$, which is an irrational number. Similarly, a circle's circumference and its diameter are incommensurable (that is, π, the ratio of a circle's circumference to its diameter, is an irrational number). The awareness that there exist incommensurable magnitudes compelled ancient Greek mathematicians to inquire into the relations between incommensurable magnitudes. This event marked a major crisis in ancient mathematics.
According to ancient Greek mathematicians, quantities (magnitudes) are continuous and uniform objects, which are best represented by straight line segments. Their division into parts, or their measurement in terms of a "unit of measurement" (i.e., a definite magnitude of a quantity), meanwhile, represents the notion of discreteness. Ancient Greek mathematicians used the term "ratio of magnitudes" in order to refer to the relation between two magnitudes that can be measured in terms of a common unit of measurement. Thus, the ancient Greek concept of a ratio is most similar to the more abstract modern concept of a number. In the context of ancient Greek mathematics, the objects of mathematics were quantities (represented by straight line segments), and the ratio between two quantities was a meta-object, or something that was used in order to study mathematical objects without being treated as a mathematical object

itself. In other words, in the context of ancient Greek mathematics, a ratio (a number) was construed as a measuring relationship between two quantities, and such a measuring relationship could be built up (and hence proved) in finitely many steps, using a common unit of measurement. Nevertheless, the discovery of incommensurable ratios demonstrated that a ratio could not be interpreted as a measuring relationship in the aforementioned way. In fact, as a result of the discovery of incommensurable ratios, the concept of a ratio (or a number) acquired its conceptual autonomy, and, instead of being treated as a meta-object, it started being treated as an object of mathematics. Therefore, ancient Greek mathematicians had to transcend the system of mathematics that was based on commensurable ratios. Notice that a commensurable ratio could easily become an object of mathematical theory, since it is a rational number, and therefore can be constructed in finitely many steps, whereas the decimal representation of an irrational number neither terminates nor infinitely repeats but extends forever without regular repetition.

In the fourth century C.E., Theon, one of the most important Greek mathematicians and commentators of Euclid's and Ptolemy's works, attempted to solve the problems that were generated as a result of the aforementioned crisis in the foundations of ancient Greek mathematics. In particular, Theon started from an extremely small (infinitesimal) unit square such that the ratio between any of its sides and any of its diagonals is equal to 1 (given that it is infinitely small); symbolically, if a_1 is the length of each of the sides of the given infinitesimal unit square, and if δ_1 is the length of each of the diagonals of the given infinitesimal unit square, then $\frac{\delta_1}{a_1} = 1$. Subsequently, Theon formulated a recursive sequence of unit squares defined by

$a_n = \delta_{n-1} + a_{n-1}$ and $\delta_n = 2a_{n-1} + \delta_{n-1}$,

so that the ratio between a diameter and a side approaches its actual value (meaning the real relationship between a diameter and a side of these unit squares according to the Pythagorean Theorem),

$\frac{\delta_n}{a_n} \to \sqrt{2}$.

He explained that he started from the case in which $\frac{\delta_1}{a_1} = 1$ because, just as the sperm of a living organism encompasses all the subsequent properties of the given organism, any ratio (including the ratio between a diagonal and a side of a unit square) spermatically (at the infinitesimal level) encompasses the unit.

Theon's aforementioned reasoning is underpinned by Aristotle's concept of a "potential infinity." The concept of modern mathematics that is

semantically most similar to Aristotle's concept of a "potential infinity" is the convergence of a sequence of natural numbers. Thus, from the perspective of ancient Greek mathematics, infinity is not a being (i.e., it is not an actual state); it cannot be simultaneously considered in its whole extension, but it can only be considered as a becoming (i.e., a process). In this way, the concept of an infinite approach helps us to overcome the contradiction between incommensurable ratios and commensurable ratios, since we can think of an incommensurable ratio infinitely approaching a commensurable ratio (and vice versa). Similarly, the concept of an infinite approach helps us to overcome the contradiction between broken lines and curves, as well as the contradiction between continuity and discreteness. This reasoning is endorsed by Euclid; in his *Elements*, he does not consider infinitely extended straight lines, but he always works with straight line segments which, as he says, can be extended as much as one needs.

However, several intellectuals have used infinite processes in a way that is not rigorous. For instance, they have attempted to compute the length of the circumference of a circle by considering an inscribed polygon whose number of sides increases indefinitely. Therefore, the length of each side of such a polygon decreases indefinitely, so that a triangle whose base is a side of the given polygon and whose vertex (i.e., the "top" corner opposite its base) is the center of the given circle could become such that its base coincides with the given circle's circumference. To what extent is such a shape a triangle, then, and beyond which point does a straight line segment (in this case, the base of a triangle) become a chord? One may argue that these changes happen when a straight line segment becomes infinitely small, but then one may counter-argue that, by becoming infinitely small, a straight line segment is not "something" any more, and it becomes "nothing." Hence, how is it possible that an infinite series of "nothing" ("no-things") gives "something," such as a circle? The aforementioned example indicates the problems that are generated as a result of the use of infinite processes in computations.

The aforementioned crisis in the foundations of mathematics was overcome by Eudoxus's theory of proportions and by the method of exhaustion, which derives from Eudoxus's theory of proportions, and it was used by Archimedes. The method of exhaustion was originally developed in the fifth century B.C.E. by the Athenian scholar Antiphon, and it was put in a rigorous scientific setting shortly afterwards by the Greek mathematician and astronomer Eudoxus of Cnidus, who used it in order to calculate areas and volumes. The Greek mathematician and acknowledged father of "Euclidean geometry" Euclid, and the Greek

mathematician, physicist, and engineer Archimedes, made extensive use of the method of exhaustion in order to prove several mathematical propositions. For instance, Archimedes used the method of exhaustion in order to compute the area of a circle by approximating the area of a circle from above and below by circumscribing and inscribing regular polygons of an increasingly larger number of sides (so that sides become "infinitesimals," or infinitely small): each of the polygons is a union of triangles, so it is easily verified that the area of a circle of radius r and circumference C is equal to the area of a triangle whose altitude is equal to r and whose base is equal to $C = 2\pi r$. Then, given that the area of a triangle is equal to half of the product of its base and altitude, we obtain the formula for the computation of the area of a circle: $\frac{1}{2}(rC) = \frac{1}{2}(r2\pi r) = \pi r^2$. Moreover, Archimedes was able to calculate the length of various tangents to the spiral (i.e., to a curve emanating from a point moving farther away as it revolves around the point).

Archimedes was very careful in the use of infinite processes; he approximated π by using the fact that the circumference of a circle is bounded by the perimeter of an *inscribed* polygon and the perimeter of a *circumscribed* polygon. According to Eudoxus and Archimedes, there is always a ratio between any two magnitudes, and we can always make any magnitude smaller or greater than a given magnitude, so that the ratio between two magnitudes a and b is the same as the ratio between two other magnitudes c and d if and only if, for any natural numbers m and n, it holds that

$$ma \gtreqless nb \Rightarrow mc \gtreqless nd, \tag{2}$$

meaning that both of these ratios are characterized by the same placement property (i.e., ordering) with regard to other numbers. In (2), the equality sign ($=$) refers to commensurable ratios, whereas the inequality signs ($\gtreqless$) refer to incommensurable ratios. These ideas of Eudoxus and Archimedes indicate that ancient Greek mathematicians discovered not only incommensurable magnitudes but also incommensurable numbers. Eudoxus's aforementioned theory of proportions underpins Archimedes's method of exhaustion for the solution of geometric problems, and Archimedes's method of exhaustion underpins modern infinitesimal calculus.

It is important to notice that the way in which Eudoxus solved the problem of the existence of incommensurable ratios (specifically, his attempt to study the conundrum of irrationality that appears to exist in elementary geometry in a scientifically rigorous way) marks a shift away from the traditional constructivist approach to mathematics towards formalism. In

other words, Eudoxus does not explain what a ratio is (as a mathematical object), but he states only when two ratios are similar to each other. The constructivist approach to mathematics allows us to determine what an object is by being able to construct it, whereas the formalist approach to mathematics is not concerned with the substance of the mathematical object under consideration, and is concerned only with the relations of the mathematical object under consideration to other mathematical objects. Moreover, the ideas of Eudoxus and Archimedes are conceptually very similar to Dedekind's cuts.

Fusing geometry and arithmetic is an arduous task. In order to understand the difficulties that originate from fusing geometry and arithmetic, let us consider, for instance, the famous irrational number $\sqrt{2}$, which was discovered by Pythagoreans when they attempted to compute the length of a diagonal of a unit square.

The Pythagoreans realized that the diagonal of a unit square is not commensurable with the side of the given square, but, by keeping geometry and arithmetic separate from each other (that is, by refusing to identify numbers with lengths of straight line segments), ancient Greek mathematicians could argue as follows: given a straight line segment whose length is one, we can construct a straight line segment whose length is $\sqrt{2}$ (as shown in Figure 1). In general, irrational numbers are geometrically constructible (and, hence, geometrically explicable and manageable), even though, from the perspective of arithmetic, irrational numbers are ideal quantities, in the sense that the calculation of irrational numbers (such as $\sqrt{2}$) is an infinite process (namely, irrational numbers have infinitely many decimal digits).

On the other hand, having endorsed the Cartesian approach to mathematics, mathematicians in the nineteenth century realized that they had to clarify some still ambiguous fundamental concepts (such as that of a real number), to formulate new methods of doing mathematics in a logically rigorous way, and to create a rigorous theory of the arithmetic continuum—specifically, a rigorous theory of real numbers and their arithmetic.

The Real Numbers

$\mathbb{R}$: the "real numbers," or the set that is formed by the union of the set $\mathbb{Q}$ of all rational numbers and the set $\mathbb{Q}^{\sim}$ of all irrational numbers; symbolically: $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^{\sim}$.

Richard Dedekind made an in-depth study of real numbers and continuity. He began with the following three properties of rational numbers:

i. If $a > b$ and $b > c$, then $a > c$.

ii. If a and c are two distinct (rational) numbers, then there exist infinitely many distinct numbers lying between a and c.
iii. If a is any definite (rational) number, then all numbers of the system $\mathbb{Q}$ fall into two classes, A_1 and A_2, each of which contains infinitely many individuals; A_1 contains all numbers a_1 that are $< a$, while A_2 contains all numbers a_2 that are $> a$; the number a itself may be assigned at will to A_1 or A_2, being, respectively, the greatest number of A_1 or the least number of A_2.

Then Dedekind stated three properties of the points on a straight number line L:

i. If p lies to the right of q and q to the right of r, then p lies to the right of r; and q is said to lie between p and r.
ii. If p and r are two distinct points, then there always exist infinitely many points lying between p and r.
iii. If p is a definite point on L, then all points on L fall into two classes, P_1 and P_2, each of which contains infinitely many individuals; P_1 contains all the points p_1 that lie to the left of p, while P_2 contains all the points p_2 that lie to the right of p; the point p itself may be assigned at will to P_1 or P_2. In any case, every point of P_1 lies to the left of every point of P_2.

Each such division (or partition) of the set $\mathbb{Q}$ of all rational numbers defines a "cut," called the "Dedekind's cut." However, after having observed that every rational number effects a "cut" in the set of rationals, Dedekind considered the inverse question: if, by a given criterion, the set of rationals is divided into two subsets A and B so that every number in A is less than every number in B, is there always a greatest rational in A or a smallest rational in B? Dedekind immediately realized that the number line should be "continuous," or unbroken, in the intuitive sense. Like Eudoxus and Cantor before him, he developed theoretical concepts for the purpose of filling the gaps in the ordered set of rationals so that the final geometric picture is a continuous, straight number line. However, the answer to the last question is in the negative: when A has no maximum rational and B has no minimum rational, there is, indeed, a gap in the rational series (or a puncture in the number line) which must be filled. In that case, the cut (A, B) is said to define (or to be) an irrational number (as shown, for instance, in Figure 1). Hence, the set $\mathbb{R}$ of all real numbers is called the "(arithmetic or geometric) continuum" or the "straight line of real numbers."

In modern mathematical notation, the set of all real numbers x such that $a \leq x \leq b$ is said to be a "closed interval," denoted by $[a, b]$, of the real line $\mathbb{R}$, while the set of all real numbers x such that $a < x < b$ (which

does not include its endpoints) is said to be an "open interval," denoted by (a, b), of the real line $\mathbb{R}$. The intervals $[a, b) = \{x \in \mathbb{R} | a \leq x < b\}$ and $(a, b] = \{x \in \mathbb{R} | a < x \leq b\}$ are neither open nor closed, but they are sometimes called "half-open" or "half-closed." Notice that $(a, a) = \emptyset$, and $[a, a] = \{a\}$. Moreover, we define the intervals:
$(a, \infty) = \{x \in \mathbb{R} | a < x\}$,
$[a, \infty) = \{x \in \mathbb{R} | a \leq x\}$,
$(-\infty, a) = \{x \in \mathbb{R} | x < a\}$,
$(-\infty, a] = \{x \in \mathbb{R} | x \leq a\}$.
By the term "interval," we generally mean a set of points with the property that, if x and y are distinct points of the set, every point between x and y is also a point of the set (if the points x and y are included, then the interval is closed; otherwise, it is open).
Assume that ε is a positive real number—that is, $\varepsilon > 0$. Moreover, consider the open interval $N = (p - \varepsilon, p + \varepsilon)$. Hence, $p \in (a, b) \subseteq (p - \varepsilon, p + \varepsilon)$. If this is the case, then $(p - \varepsilon, p + \varepsilon)$ is called the ε-neighborhood of the point p, and it is denoted by $N_\varepsilon(p)$. In other words, the ε-neighborhood of a point p on the real line is the set of all those real numbers which are within an ε distance of p on either side of it; p is the midpoint or the center of $N_\varepsilon(p)$; ε is the radius of $N_\varepsilon(p)$. We shall use the notation $N'_\varepsilon(p)$ in order to denote the "deleted neighborhood," consisting of $N_\varepsilon(p)$ with the point p deleted. In terms of the real line $\mathbb{R}$, a deleted neighborhood is an interval on $\mathbb{R}$ with the center point removed.
Given a set S, a real number p is said to be an "interior point" of S if S is a neighborhood of p; symbolically: if $p \in (a, b) \subseteq S$. Obviously, an interior point of a set S belongs to S. The set of all interior points of a given set S is called the "interior" of S, and it is denoted by $Int(S)$. In general, a point $p \in \mathbb{R}^n$ is said to be an "interior point" of U if some neighborhood (open ball) $N_\varepsilon(p)$ with center p is contained in U. For instance, if $S = [2,5]$, then $\frac{7}{2}$ is an interior point of S, whereas neither 2 nor 5 is an interior point of S, because $[2,5]$ is not a neighborhood of 2 and 5. The interior of the closed interval $[2,5]$ is the open interval $(2,5)$.
A real number p is called a "closure point" of a set $S \subseteq \mathbb{R}$ if every neighborhood of p contains a point of S. The set of all closure points of S is called the "closure" of S, and it is denoted by $Cls(S)$. Therefore, every point of $S \subseteq \mathbb{R}$ is a closure point of S.
A real number p is called an "accumulation point," a "limit point," or a "cluster point" of S if every deleted neighborhood of p contains at least one point of S; symbolically: if $S \cap N'_\varepsilon(p) \neq \emptyset \; \forall \varepsilon > 0$ (in other words, every neighborhood of p contains at least one point of S other than p). For

instance, if $A = [a, b]$ and $B = (a, b)$, then every member of A is an accumulation point of A and of B, since, for instance, $\forall \varepsilon > 0$, the neighborhood $(a - \varepsilon, a + \varepsilon)$ of a contains infinitely many elements of A and of B. Moreover, every real number is an accumulation point of the set $\mathbb{Q}$ of all rational numbers as well as of the set $\mathbb{R}$ of all real numbers, since, for instance, given an arbitrary real number p, $\forall \varepsilon > 0$, the neighborhood $(p - \varepsilon, p + \varepsilon)$ contains infinitely many real numbers as well as infinitely many rational numbers. On the other hand, the set $\mathbb{N}$ of all natural numbers, the set $\mathbb{Z}$ of all integral numbers, and the empty set have no accumulation point. Furthermore, no finite set has any accumulation point, because, if, for instance, $A = \{a_1, a_2, a_3, \dots, a_n\}$, and if p is an arbitrary real number, we can construct a sufficiently small neighborhood N with center p such that N contains no point of A; therefore, p, which is an arbitrary real number, is not an accumulation point of A.

A "real number" is a quantity x that has a "decimal expansion":

$x = n + 0.d_1 d_2 d_3 \dots$,

where n is an integer, each d_i is a digit between 0 and 9 ($i = 1,2,3, \dots$), and no infinite sequence of 9 's appears. The aforementioned representation implies that

$$n + \frac{d_1}{10} + \frac{d_2}{100} + \cdots + \frac{d_k}{10^k} \leq x < n + \frac{d_1}{10} + \frac{d_2}{100} + \cdots + \frac{d_k}{10^k} + \frac{1}{10^k},$$

for all positive integers k.

Let a be a real number. Then the product $a \cdot a \cdot a \dots$ (n times) is denoted by a^n, where n is called the "exponent," and a is called the "base." Therefore, the following results hold $\forall a, b \in \mathbb{R}$:

i. $a^n a^m = a^{n+m}$,
ii. $(a^n)^m = a^{nm}$,
iii. $\frac{a^n}{a^m} = a^{n-m}$,
iv. $a^0 = 1$, and
v. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$.

A "factorial" is a function in mathematics denoted with the symbol ! that multiplies a positive integer n by every number that precedes it:

$n! = n \cdot (n - 1) \cdot (n - 2) \cdot \dots \cdot 2 \cdot 1$.

For instance, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. Notice that $0! = 1$, and $1! = 1$. In fact, $n!$ is the number of "permutations" of n elements. The number of "arrangements" of n elements taken m at a time is $A_m^n = \frac{n!}{(n-m)!}$, and the number of "combinations" of n elements taken m at a time is $C_m^n = \frac{n!}{(n-m)!m!} = \frac{A_m^n}{m!}$.

Intimately related to the concepts of an exponent and an index is the concept of a logarithm, which is the inverse function to exponentiation. The "logarithm" of an arbitrary real number a is the exponent to which another fixed real number, the base b, must be raised to produce the real number a; symbolically:
$log_b a = x \Leftrightarrow b^x = a$.
For instance, $log_{10} 1{,}000 = 3$, since $10^3 = 1{,}000$, and $log_3 81 = 4$, since $3^4 = 81$. The method of logarithms was originally developed by the Scottish mathematician, physicist, and astronomer John Napier (1550–1617), who published his book *Mirifici Logarithmorum Canonis Descriptio* (*Description of the Wonderful Rule of Logarithms*) in 1614.
In case the base $b = e = \sum_{n=0}^{\infty} \frac{1}{n!} \approx 2.718$, which is known as Euler's number (in honor of the Swiss mathematician Leonhard Euler), then $log_e a$ is written as lna, and it is said to be the "natural logarithm" of a. Euler's number e is irrational, and it was originally derived from the study of compound interest: if one places $1USD$ into a deposit account at a banking institution with 100% interest, and the compounding period is n, as a fraction of a year, then the formula of the compound interest $(1 + \frac{r}{n})^n$, where, in our case, $r = 1$ (annual interest rate as a decimal: $100\% = \frac{100}{100} = 1$), tends to e as n tends to infinity. However, the problem of compound interest was systematically investigated by the Swiss mathematician Jacob Bernoulli (1655–1705), who studied the following question: if an account starts with $1USD$ and pays 100% interest per year, and if the interest is credited once at the end of the year, then the value of the account at the year-end will be $2USD$, but what will happen if the interest is computed and credited more frequently during the year? In fact, Bernoulli noticed that, if there are n compounding intervals, the interest for each interval will be $\frac{100\%}{n}$, and the value of the aforementioned account (which started with $1USD$) at the end of the year will be $1\ USD \times \left(1 + \frac{1}{n}\right)^n$. Furthermore, Bernoulli noticed that this sequence approaches a limit (the "force of interest"). More specifically, it approaches the number e, as n increases—that is, as compounding intervals become smaller. For instance, compounding monthly (i.e., $n = 12$) yields approximately 2.613 USD, while compounding daily (i.e., $n = 365$) yields approximately 2.7146 USD. The limit as n tends to infinity is the number $e = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \approx 2.718$, meaning that, with continuous compounding, the value of the aforementioned account will reach approximately 2.718 USD. Leonhard Euler proved that the number e

is irrational by showing that its simple continued fraction expansion is infinite (by a "continued fraction," we mean an expression obtained through an iterative process of representing a number as the sum of its integral part and the reciprocal of another number, then writing this other number as the sum of its integral part and another reciprocal, etc.).
The following properties of the logarithm can be easily verified:

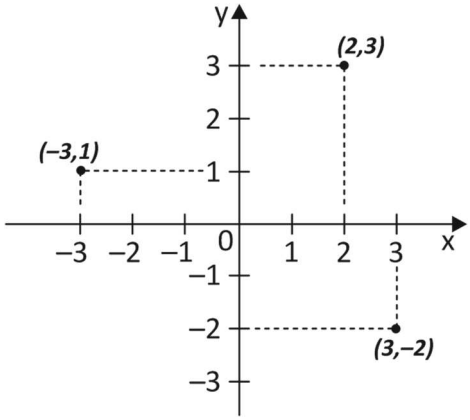
i. $log_b(xy) = log_b x + log_b y$,
ii. $log_b\left(\frac{x}{y}\right) = log_b x - log_b y$,
iii. $log_b x^k = k log_b x$,
iv. $log_b 1 = 0$,
v. $log_b b^x = x = b^{log_b x}$,
vi. $log_b x = \frac{log_a x}{log_a b}$ (change of base rule).
vii. If x, y, and b are positive real numbers with $b \neq 1$, then
$x = y \Rightarrow log_b x = log_b y$, and, conversely,
$log_b x = log_b y \Rightarrow x = y$. Hence, we can solve exponential equations (i.e., equations in which the unknown is in the exponent) by taking the logarithm of both sides of the equation. For instance, let us solve the exponential equation $5^{2x} = 21$ using log base of e:
$5^{2x} = 21 \Rightarrow ln(5^{2x}) = ln21 \Rightarrow 2x \cdot ln5 = ln21 \Rightarrow 2x = \frac{ln21}{ln5} \Rightarrow$
$x = \frac{\frac{ln21}{ln}}{2} \approx 0.9458.$

Ordered Pairs and the Cartesian Product

The Fundamental Property of Ordered Pairs: For any ordered pairs $(w, x) and\ (y, z)$, it holds that:
$(w, x) = (y, z) \Leftrightarrow w = y\ \&\ x = z$,
and, in this case, the two ordered pairs are called "equal."
The "Cartesian product" (also known as the "direct product") $A \times B$ of two sets A and B is the set of all ordered pairs (a, b) such that $a \in A\ and\ b \in B$; symbolically:
$A \times B = \{(a, b) | a \in A\ \&\ b \in B\}$.
For instance, if $A = \{1,2\}$ and $B = \{1,3\}$, then the Cartesian product $A \times B$ is the set $\{(1,1), (1,3), (2,1), (2,3)\}$. In general, the Cartesian product of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots \times A_n$ is the set of all ordered n-tuples of the form $(a_1, a_2, \ldots, a_n)$, where a_i is an element of $A_i (i = 1,2, \ldots, n)$.
Remark: It is easily checked that, for any sets A, B, and C, we have:
$A \times (B \cup C) = (A \times B) \cup (A \times C)$,

$A \times (B \cap C) = (A \times B) \cap (A \times C)$.
If $A = \emptyset$ or $B = \emptyset$, then $A \times B = \emptyset$.
$A \times B = B \times A \Leftrightarrow A = B$.
Let $A \times B = \{(a, b) | a \ \& \ b \ are \ real \ numbers\}$. Then $A \times B$ is the set of all points in a plane whose coordinates are (a, b). Thus, $A \times B$ is the Cartesian plane
$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$,
as shown, for instance, in Figure 3. In this case, each point P in the plane represents an ordered pair (a, b) of real numbers, and vice versa. In other words, the vertical line through P meets the x-axis at a, and the horizontal line through P meets the y-axis at b. Thus, we can understand the relationship between set theory, mathematical analysis, and geometry. In other words, a two-dimensional coordinate system consists of the horizontal axis (namely, the x-axis) and the vertical axis (namely, the y-axis), and the intersection of the two axes is the origin $O(0,0)$ of the coordinate system (by the term "axis," we mean a straight line with respect to which a body or structure is symmetrical). By analogy, we can define an n-dimensional coordinate system for $n \geq 3$ ($n = 3,4,5, ...$), using n axes of reference at right angles to each other.

Figure 3: The Cartesian Plane.



As noted above, the set $\mathbb{R}$ of all real numbers is called the real line, or the continuum. A set of pairs of real numbers is called a "number plane," and it is denoted by $\mathbb{R}^2$. As already mentioned, the set $\mathbb{R}$ can be represented geometrically as a horizontal number line. A geometric representation of the set $\mathbb{R}^2$ is the coordinate plane xOy, defined as two perpendicular

number lines with a common origin O and the same scale (the number of units represented by a unit length along an axis is called the "scale"). The point $O(0,0)$ is called the "origin of coordinates." If P_0 is a point in the coordinate plane, then, by projecting it on the coordinate lines Ox and Oy, we find the coordinates of the projections x_0 and y_0 (notice: if you drop a perpendicular from a point to a line or plane, then the point you reach on that line or plane is called the projection of the point onto the line or plane). The coordinates are called respectively the "abscissa" (i.e., the x-coordinate) and the "ordinate" (i.e., the y-coordinate) of the point P_0, and the straight lines Ox and Oy are respectively called the "axis of abscissas" and the "axis of ordinates". Hence, to the point P_0 there corresponds one pair of numbers (x_0, y_0); conversely, given a pair of numbers (x_0, y_0), we mark the points x_0 and y_0 on the coordinate lines (axes) Ox and Oy respectively, and, drawing through these points straight lines parallel to the coordinate lines (axes) Ox and Oy, we find the point of their intersection P_0. By analogy we work in $\mathbb{R}^n$.

In general, the use of coordinate systems implies that space itself is encoded by n-tuples (i.e., by sequences, ordered lists, of n numbers), and, specifically, that the two-dimensional space, the "plane," is encoded by pairs of numbers, so that the conception of space becomes subordinate to the conception of arithmetic.

Relations and Functions between Sets

Let A and B be two arbitrary sets. Then a "relation" between A and B, denoted by R, is defined to be a subset of the Cartesian product $A \times B$; symbolically: $R \subseteq A \times B$. The "domain" of relation R is defined by $D_R = \{a|(a,b) \in R\}$, and the "range" of relation R is defined by $R_R = \{b|(a,b) \in R\}$. If R is a relation from A to B, then the relation from B to A is called the "inverse" of R, and it is defined by $R^{-1} = \{(b,a)|(a,b) \in R\}$. A relational proposition is often denoted by aRb, where R relates a term a to a term b. Hence, a relation of two terms proceeds, somehow, from one to the other.

If R_1 is a relation from a set A to a set B, and if R_2 is a relation from B to a set C, then their "composition," denoted by $R_2 \circ R_1$, is a relation from A to C, symbolically:

$R_2 \circ R_1 = \{(a,c)|for\ some\ b \in B, (a,b) \in R_1\ \&\ (b,c) \in R_2\ with\ a \in A, c \in C\}$.

If R_1 and R_2 are relations such that $R_1 \subseteq R_2$, then R_2 is said to be an "extension" of R_1, and R_1 is said to be a "restriction" of R_2.

A relation R on a set A is "reflexive" if (a,a) is an element of R for every $a \in A$; it is "symmetric" if (a,b) is an element of R whenever (b,a) is an

element of R; and it is "transitive" if (a,c) is an element of R whenever (a,b) and (b,c) are elements of R. A relation R on a set A is "antisymmetric" if, whenever a and b are distinct, then (a,b) is an element of R only if (b,a) is not an element of R. For instance, if $A = \{u,v,w\}$ and R is a relation on A, then:
$R = \{(u,v),(v,u),(u,u),(v,v),(v,w),(w,w)\}$ is a reflexive relation on A;
$R = \{(u,v),(v,u),(w,w)\}$ is a symmetric relation on A;
$R = \{(u,v),(v,w)(u,w),(v,v)\}$ is a transitive relation on A;
$R = \{(u,w),(v,v),(u,v),(u,u)\}$ is an antisymmetric relation on A.
Let A and B be two arbitrary sets. A relation $f \subseteq A \times B$ is called a "function," "mapping," or "transformation," denoted by $f: A \to B$, if it assigns to each element $a \in A$ exactly one element $b \in B$. The set A is called the "domain" of the function f and is denoted by D_f, while the set B is called the "codomain" of the function f. The set of all elements of B that are related to the elements of A via f is called the "range" of the function f, and it is denoted by R_f, meaning that the range of f is the image of A by f:
$f(A) = \{f(a) | a \in A\}$.
By the term "graph" of a function $f: A \to B$, we mean the set $\{x, f(x)\}$, where $x \in A$. If c is a positive constant, then:

i. The graph of $y = f(x) + c$ is the graph of f raised by c units.
ii. The graph of $y = f(x) - c$ is the graph of f lowered by c units.
iii. The graph of $y = f(x + c)$ is the graph of f shifted c units to the left. In fact, if we analyze the x-values, we can see a pattern, and we realize that the new x that we need in order to obtain $f(0)$ is the one that makes $f(x + c) = f(0)$, namely, $-c$. We can generalize this result as follows:
$f(x_{new} + c) = f(x) \Rightarrow x_{new} + c = x \Rightarrow x_{new} = x - c$,
meaning that the new x-values are the old x-values translated $-c$ units (that is, c units to the left, since that direction is the negative direction).
iv. The graph of $y = f(x - c)$ is the graph of f shifted c units to the right.

The graph of $y = -f(x)$ is the graph of f reflected about the x-axis.
If $c > 1$, then the graph of $y = cf(x)$ is the graph of f stretched by a factor of c. If $0 < c < 1$, then the graph of $y = cf(x)$ is the graph of f flattened out by a factor of c.
Vertical line test: Imagine a vertical line sweeping across a graph. Assume that the vertical line at any position intersects the graph in more than one point. Then the graph is not the graph of a function.

Two functions $f: A \to B$ and $g: A \to B$ are called "equal" if $f(x) = g(x), \forall x \in A$, and they are called "different" if there is at least one $x_0 \in A$ such that $f(x_0) \neq g(x_0)$.

A function f is said to be "odd" if $f(-x) = -f(x)$ for every x in the domain of f. The graph of an odd function has symmetry about the origin.

A function f is said to be "even" if $f(-x) = f(x)$ for every x in the domain of f. The graph of an even function has symmetry about the y-axis.

A function $f: X \to Y$ is called "one-to-one" (or "injective," or an "injection," or a "monomorphism") if

$f(x_1) = f(x_2) \Rightarrow x_1 = x_2, \forall x_1, x_2 \in X$;

that is, a function is "one-to-one" if each x value in the domain is assigned a different y value, so that no two ordered pairs have the same second component. If more than one element of X has the same f-image in Y, then the function $f: X \to Y$ is said to be "many-to-one."

Horizontal line test: Imagine a horizontal line sweeping down the graph of a function. Assume that the horizontal line at any position intersects the graph in more than one point. Then, the function is not one-to-one, and its inverse is not a function.

A function $f: X \to Y$ is called "into" if there exists at least one element of Y that is not the f-image of an element of X. In other words, for any into function $f: X \to Y$, the range set $f(X)$ is a proper subset of Y; symbolically, $f(X) \subset Y$.

If the range of a function f is the whole codomain of f, then f is said to be "onto" (or "surjective," or a "surjection," or an "epimorphism"). In other words, for any onto function $f: X \to Y$, $f(X) = Y$.

If a function is both one-to-one and onto, then it is called "bijective," or a "bijection," or an "one-to-one correspondence."

For instance:

i. If A is a subset of X, then the restriction to A of the identity mapping id_x, defined by $A \ni x \to x \in A$, is an injection j_A, called the "natural injection."
ii. The identity mapping of any set is bijective.
iii. The function $f: X \times Y \to Y \times X$ defined by $(x, y) \to (y, x)$, where $x \in X \; and \; y \in Y$, is bijective.
iv. The function $f(x) = x^2$, where $x \in \mathbb{R}$, is not injective, since $f(-x) = f(x) = x^2$, but the restriction to $\mathbb{R}^+$ (the set of all positive real numbers) of f is injective.
v. $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is an one-to-one and onto mapping, that is, a bijection from $\mathbb{R}$ to $\mathbb{R}$.

Sequences and Series

A "sequence" is a function whose domain is the set of positive integers (i.e., 1,2,3, ...). The functional values (i.e., the range elements) are called the terms of the sequence. In other words, a sequence is a set of numbers arranged in a definite order.

An "arithmetic progression" is a sequence of numbers in which each term after the first is found by adding a constant to the preceding term. This constant is called the "common difference" and is symbolized by d. Thus, the formula for the nth term in an arithmetic progression with first term a_1 and common difference d is:

$a_n = a_1 + (n-1)d.$

A "geometric progression" is a sequence of numbers in which each term after the first is found by multiplying the preceding term by a constant. This constant is called the "common ratio" and is symbolized by r. Thus, the formula for the nth term in a geometric progression with first term a_1 and common ratio r is:

$a_n = a_1 r^{n-1}.$

Associated with any sequence $a_1, a_2, a_3, \ldots$ is a "series"

$$a_1 + a_2 + a_3 + \cdots$$

which is the sum of all the terms in the sequence. A series that is associated with an arithmetic progression is called an "arithmetic series." A series that is associated with a geometric progression is called a "geometric series."

The sum of the first n terms of an arithmetic series is given as the following formula:

$S_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}[2a_1 + (n-1)d].$

The sum of the first n terms of a geometric series is given as the following formula:

$S_n = \frac{a_1 - a_1 r^n}{1-r} = \frac{a_1 - a_n r}{1-r}.$

Real Equations and Algebra

By the term "equation," we mean a statement that two quantities are equal. For instance, $1{,}000m = 1km$. More often, an equation contains an unknown quantity that is represented by a symbol, and we try to find the value of this unknown quantity. By the term "algebra," we refer to methods and techniques for solving equations. In fact, the core of the study of structures in mathematics consists of taking numbers and putting them into equations in the form of "variables"; and the rules for manipulating these equations are contained in algebra. Moreover, in the context of

algebra, we study multidimensional numbers, such as matrices and vectors (see chapters 3 and 7).

The word "algebra" derives from the Arabic word "al-Jabr," meaning "transformation." It refers to a methodology developed by the Persian mathematician Al-Khwarizmi, who lived in Baghdad early in the Islamic era. Al-Khwarizmi was interested in solving algebraic equations, and his method consists in applying a transformation to the given equation in order to put it into a standard form for which the solution method is known.

Equations requiring multiplication and division:

i. We can solve the equation $\frac{x}{12} = 4$ as follows: multiplying each side by 12, we get $\frac{x}{12} \times 12 = 4 \times 12 \Rightarrow x = 48$. Check: when $x = 48$, the left-hand side of the given equation becomes $\frac{48}{12} = 4$. The right-hand side of the given equation is equal to 4. Therefore, the solution is correct.

ii. We can solve the equation $6x = 3$ as follows: dividing each side by 6, we get $\frac{6x}{6} = \frac{3}{6} \Rightarrow x = \frac{1}{2}$. Check: when $x = \frac{1}{2}$, the left-hand side of the given equation becomes $6 \times \frac{1}{2} = 3$. The right-hand side of the given equation is equal to 3. Therefore, the solution is correct.

Equations requiring addition and subtraction:

i. We can solve the equation $x - 2 = 4$ as follows: adding 2 to each side, we get $x - 2 + 2 = 4 + 2 \Rightarrow x = 6$. The operation of adding 2 to each side is the same as transferring -2 to the right-hand side, but, in so doing, the sign is changed from a minus to a plus. Hence, $x - 2 = 4 \Leftrightarrow x = 4 + 2 \Leftrightarrow x = 6$. Check: when $x = 6$, the left-hand side of the given equation becomes $6 - 2 = 4$. The right-hand side of the given equation is equal to 4. Therefore, the solution is correct.

ii. We can solve the equation $x + 18 = 30$ as follows: subtracting 18 from each side, we get $x + 18 - 18 = 30 - 18 \Rightarrow x = 12$. Alternatively, moving $+18$ to the right-hand side (changing the sign from a plus to a minus), we get $x = 30 - 18 \Leftrightarrow x = 12$. Check: when $x = 12$, the left-hand side of the given equation becomes $12 + 18 = 30$. The right-hand side of the given equation is 30. Therefore, the solution is correct.

Equations containing the unknown quantity on both sides: In equations of this kind, we group all the terms containing the unknown quantity on one side of the equation and the remaining terms on the other side.

i. We can solve the equation $4x + 3 = 6x + 11$ as follows: transferring $6x$ to the left-hand side and $+3$ to the right-hand side, we get $4x - 6x = 11 - 3 \Rightarrow -2x = 8 \Rightarrow x = -\frac{8}{2} = -4$. Check: when $x = -4$, the left-hand side becomes $4(-4) + 3 = -13$, and the right-hand side becomes $6(-4) + 11 = -13$. Therefore, the solution is correct.

ii. We can solve the equation $7x - 2 = 5x + 8$ as follows: $7x - 5x = 8 + 2 \Rightarrow 2x = 10 \Rightarrow x = 5$. Check: when $x = 5$, the left-hand side becomes $7 \times 5 - 2 = 33$, and the right-hand side becomes $5 \times 5 + 8 = 33$. Therefore, the solution is correct.

Equations containing brackets: When an equation contains brackets, we remove these first, and then we solve according to the aforementioned methods. For instance, $3(2x - 1) = 9 \Rightarrow 6x - 3 = 9 \Rightarrow 6x = 12 \Rightarrow x = 2$. Check: when $x = 2$, the left-hand side is $3(2 \times 2 - 1) = 9$, and the right-hand side is 9. Therefore, the solution is correct.

Equations containing fractions: When an equation contains fractions, we multiply each term of the equation by the least common multiple of the denominators. For instance, we can solve the equation $\frac{x}{3} + \frac{2}{5} = \frac{5x}{2} - 1$ as follows: The least common multiple of the denominators 3, 5, and 2 is 30. Multiplying each term by 30 gives $\frac{x}{3} \times 30 + \frac{2}{5} \times 30 = \frac{5x}{2} \times 30 - 1 \times 30 \Rightarrow 10x + 12 = 75x - 30 \Rightarrow -65x = -42 \Rightarrow x = \frac{42}{65}$. The solution may be verified by the check method shown in the previous examples.

Simultaneous equations: Consider the two following equations:

$$\left.\begin{matrix} ax + by = c \\ px + qy = r \end{matrix}\right\}.$$

Each equation contains the unknown quantities x and y. The solutions of the equations are the values of x and y that satisfy both equations. Equations such as these are called "simultaneous equations" (or a "system of equations").

i. We can solve the simultaneous equations

$$4x + 5y = 14 \qquad (*)$$

$$x + 2y = 11 \qquad (**)$$

as follows: If we multiply equation $(**)$ by 4, we shall have the same coefficient of x in both equations:

$$4x + 8y = 44 \qquad (***)$$

We can now eliminate x by subtracting equation $(*)$ from equation $(***)$:

$$\begin{matrix} 4x + 8y = 44 \\ 4x + 5y = 14 \end{matrix}$$

$$3y = 30$$

Hence, $y = 10$. In order to find x, we substitute $y = 10$ in either of the original equations. Therefore, substituting for $y = 10$ in equation $(*)$, we get $4x + 5 \times 10 = 14 \Rightarrow x = -9$. In order to check these values, it suffices to substitute them in equation $(**)$.

ii. We can solve the simultaneous equations

$$5x + 7y = 15 \qquad (*)$$

$$4x + \frac{8}{5}y = 24 \qquad (**)$$

as follows: the same coefficient of x can be obtained in both equations if equation $(*)$ is multiplied by 4 (the coefficient of x in equation $(**)$) and equation $(**)$ is multiplied by 5 (the coefficient of x in equation $(*)$). Multiplying equation $(*)$ by 4, we get

$$20x + 28y = 60 \qquad (***)$$

Multiplying equation $(**)$ by 5, we get

$$20x + 8y = 120 \qquad (****)$$

Subtracting equation $(***)$ from equation $(****)$, we get

$$-20y = 60 \Rightarrow y = -3.$$

Substituting for $y = -3$ in equation $(*)$, we get $x = \frac{36}{5}$. In order to check these values, it suffices to substitute them in equation $(**)$.

iii. We can solve the simultaneous equations

$$7x + 4y = 20 \qquad (*)$$

$$3x - 2y = 3 \qquad (**)$$

as follows: in this system of equations, it is easier to eliminate y, since the same coefficient of y can be obtained in both equations by multiplying equation $(**)$ by 2. In fact, multiplying equation $(**)$ by 2, we get

$$6x - 4y = 6 \qquad (***)$$

Adding equations $(*)$ and $(***)$, we get $13x = 26 \Rightarrow x = 2$. Substituting for $x = 2$ in equation $(*)$, we get $y = \frac{3}{2}$. In order to check these values, it suffices to substitute them in equation $(**)$.

iv. We can solve the simultaneous equations

$$\frac{x}{5} - \frac{y}{3} = \frac{1}{10} \qquad (*)$$

$$\frac{3x}{4} - \frac{2y}{3} = \frac{2}{3} \qquad (**)$$

as follows: first, we shall clear each equation of fractions. In equation $(*)$, the least common multiple of the denominators is 30. Hence, by multiplying equation $(*)$ by 30, we get

$6x - 10y = 3$ $(***)$

In equation $(**)$, the least common multiple of the denominators is12. Hence, by multiplying equation $(**)$ by 12, we get

$9x - 8y = 8$ $(****)$

We now proceed in the usual way. Multiplying equation $(**)$ by 6, we get

$36x - 60y = 18$ (A)

Multiplying equation $(****)$ by 4, we get

$36x - 32y = 32$ (B)

Subtracting equation (B) from equation (A), we get $-28y = -14 \Rightarrow y = \frac{1}{2}$. Substituting for $y = \frac{1}{2}$ in equation $(***)$, we get $x = \frac{8}{6} = \frac{4}{3}$. Therefore, the solutions are $y = \frac{1}{2}$ and $x = \frac{4}{3}$. Since equation $(***)$ came from equation $(*)$, we must do the check in equation $(**)$. Indeed, $\frac{3(4/3)}{4} - \frac{2(1/2)}{3} = \frac{2}{3}$.

Factoring Models

Common factor: $ax + ay = a(x + y)$.
Difference of squares: $x^2 - y^2 = (x + y)(x - y)$.
Trinomial (leading coefficient 1): $x^2 + (a + b)x + ab = (x + a)(x + b)$.
Perfect square trinomial: $x^2 + 2xy + y^2 = (x + y)^2$.
General trinomial: $(ac)x^2 + (ad + bc)x + bd = (ax + b)(cx + d)$.
Sum of cubes: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.
Difference of cubes: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

Real Polynomials

A function of a single variable x is said to be a "polynomial" on its domain if it can be put in the following form:

$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$,

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are constants. Hence, every polynomial can be expressed as a finite sum of monomial terms of the form $a_k x^k$, in which the variable is raised to a non-negative integral power. Notice that $x^0 = 1$, and so $a_0 x^0 = a_0$. For the aforementioned polynomial with $a_n \neq 0$:

the numbers a_i (where $0 \leq i \leq n$) are called "coefficients";
a_n is the "leading coefficient";
$a_n x^n$ is the "leading term";

a_0 is the "constant term" or the "constant coefficient";
a_1 is the "linear coefficient";
a_1x is the "linear term";
when the leading coefficient, a_n, is equal to 1, the polynomial is said to be "monic";
the non-negative integer n is the "degree" of the polynomial, and we write $\deg(p) = n$.
A "constant polynomial" has only one term, specifically, a_0. A non-zero constant polynomial has degree 0, and, by convention, the "zero polynomial" (with all coefficients vanishing) has degree $-\infty$.
A "zero" of a polynomial $p(x)$ is any number r for which $p(r)$ takes the value 0. Hence, when $p(r) = 0$, we say that r is a "root," or a "solution" of the equation $p(x) = 0$.
Let
$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ and
$q(x) = b_0 + b_1x + b_2x^2 + \cdots + b_mx^m$
be two arbitrary polynomials. Then we can operate with them as follows:
Sum: $(p + q)(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots$
Difference: $(p - q)(x) = (a_0 - b_0) + (a_1 - b_1)x + (a_2 - b_2)x^2 + \cdots$
Product of a constant and a polynomial: $(cp)(x) = ca_0 + ca_1x + ca_2x^2 + \cdots$
Product of two polynomials: $(p \cdot q)(x) = a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \cdots + (a_0b_k + a_1b_{k-1} + \cdots + a_ib_{k-i} + \cdots + a_kb_0)x^k + \cdots + (a_nb_m)x^{m+n}$.
Composition of two polynomials: $(p \circ q)(x) = p\big(q(x)\big)$, so that we replace each occurrence of x in the expression for $p(x)$ with $q(x)$.
Notice that we divide one polynomial by another in a manner similar to the division of two integers. Firstly, we arrange the terms of the dividend and the divisor in descending powers of x. If a term is missing, then we write 0 as its coefficient. Then, we divide the first term of the dividend by the first term of the divisor to obtain the first term of the quotient. Next, we multiply the entire divisor by the first term of the quotient, and we subtract this product from the dividend. We use the remainder as the new dividend, and we repeat the same process until the remainder is of lower degree than the divisor. As with the division of numbers,
$dividend = (divisor)(quotient) + remainder$.
Remainder Theorem: If a polynomial $p(x)$ is divided by $x - b$, then the remainder is $p(b)$.
Proof: Let $q(x)$ and r be, respectively, the quotient and the remainder when $p(x)$ is divided by $x - b$. Then, given that
$dividend = (divisor)(quotient) + remainder$,

it holds that, for any x,
$p(x) = (x - b)q(x) + r$.
If $x = b$, then $p(b) = r$.■
Factor Theorem: Given an arbitrary polynomial function $y = p(x)$, b is a zero of $y = p(x)$ if and only if $x - b$ is a factor of $p(x)$.
Proof. It can be easily verified using the Remainder Theorem. ■
Remark: The real number zeros of $y = p(x)$ are also the x-intercepts in the graph of $y = p(x)$. If b is a real number zero with multiplicity n of $y = p(x)$, then the graph of $y = p(x)$ crosses the x-axis at $x = b$ if n is odd, whereas the graph turns around and stays on the same side of the x-axis at $x = b$ if n is even. Hence, the x-intercepts can be obtained from the Factor Theorem, and the behavior of the graph at an x-intercept, say $(b, 0)$, is determined by the multiplicity of zero$b$ or, equivalently, by the highest power of $(x - b)$ that is a factor of $p(x)$. For instance, if $p(x) = (x + 1)(x - 2)^2$, then, by setting $x = 0$, we realize that the y-intercept is $(0,4)$. Because $(x + 1)$ is a factor with an odd exponent, it holds that $(-1,0)$ is an x-intercept at which the graph crosses the x-axis. Because $(x - 2)^2$ is a factor with an even exponent, it holds that $(2,0)$ is an x-intercept at which the graph touches the x-axis and then turns around.
In fact, the fundamental problem in algebra consists in finding ways of solving polynomial equations; specifically, we seek formulas for zeros/roots in terms of the coefficients of the corresponding polynomial. A well-known example is the "quadratic formula." If we have the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, then we have the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where the expression $b^2 - 4ac$ is known as the "discriminant," meaning that, if we have a number r such that $r^2 = b^2 - 4ac \Leftrightarrow r = \sqrt{b^2 - 4ac}$, then
$x_1 = \frac{-b+r}{2a}$ and $x_2 = \frac{-b-r}{2a}$
are the solutions of $ax^2 + bx + c = 0$.
If a function $y = f(x)$ satisfies an equation of the form
$p_0(x)y^n + p_1(x)y^{n-1} + \cdots + p_{n-1}(x)y + p_n(x) = 0$,
where $p_0(x), \ldots, p_n(x)$ are polynomials in x, then it is said to be an "algebraic function." In other words, an algebraic function is a function that can be defined as the root of a polynomial equation. If a function can be expressed as the quotient of two polynomials,
$f(x) = \frac{p(x)}{q(x)}$,
then it is called a "rational algebraic function."

A Few Applications:

I. Applications of Arithmetic in Biology: Let A be a prey and B be a predator. Suppose that the hibernation period of B is X. Then what hibernation period should A choose for itself in order to minimize the chance of getting hunted? This question can be answered as follows: Suppose that the maximum hibernation period for A is M. Then A must choose some value Y between 0 and M. Let X and Y have some greatest common divisor (gcd). Hence, the maximum time A can stay alive is equal to $\frac{X \cdot Y}{gcd}$, and A has to maximize this over 0 to M.

II. Applications of Arithmetic in Cryptography: In the context of cryptography, there is a plaintext (i.e., an intelligible message) that is converted into a ciphertext (i.e., an unintelligible message) according to an encryption algorithm, and this ciphertext is transmitted on the internet and is received by a receiver who will use the decryption algorithm (which is the opposite to the encryption algorithm) in order to convert the ciphertext into the original plaintext. Thus, in cryptography, the computer converts information into a signle number (representing one's message), say m. In order to be computationally secure, many encryption algorithms are based on prime numbers because of the following reason: generally, multiplying two large prime numbers can be very fast, but it is very difficult to do the reverse (it is extremely computer-intensive to find the prime factors of large numbers).

Chapter 3
Matrices and Applications in Input-Output Analysis and Linear Programming

Matrices are often used in physics, statistics, and economics, and they are particularly useful when they are used in connection with systems of linear equations. For instance, let us considers the following linear simultaneous equations:

$$4x + 5y = 14$$
$$x + 2y = 11$$

By arranging the coefficients of x and y in the way in which they occur in the equations, we obtain the array

$$\begin{pmatrix} 4 & 5 \\ 1 & 2 \end{pmatrix},$$

which is an example of a matrix.

In general, consider the following rectangular array

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix},$$

consisting of m rows (i.e., horizontal n-tuples) and n columns (i.e., vertical n-tuples). This is called an "$m \times n$ matrix," usually denoted by $A = (a_{ij})$. If the number of rows in the matrix is m and the number of columns is n, then the matrix is said to be of order $m \times n$. The term "matrix" was introduced by the nineteenth-century English mathematician James Sylvester, but it was his friend the mathematician Arthur Cayley who developed the algebra of matrices in the 1850s.

Types of matrices:

i. *Row matrix.* This is a matrix having only one row; for instance, the following is a row matrix:
$$(4 \quad 5).$$
ii. *Column matrix.* This is a matrix having only one column; for instance, the following is a column matrix:
$$\begin{pmatrix} 5 \\ 2 \end{pmatrix}.$$
iii. *Null matrix.* this is a matrix with all its elements zero.
iv. *Square matrix.* This is a matrix having the same number of rows and columns.
v. *Diagonal matrix.* This is a square matrix in which all the elements are zero except the main diagonal elements (the main diagonal in a matrix always runs from upper left to lower right, so that the

main diagonal of a matrix $A = (a_{ij})$ is the list of entries a_{ij} where $i = j$); for instance, the following is a diagonal matrix:
$\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$.

vi. *Identity matrix.* This is a diagonal matrix in which the main diagonal elements are equal to 1 (an identity matrix is usually denoted by *I*); for instance, the following is an identity matrix:
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Addition and Subtraction of Matrixes: Two matrices may be added or subtracted provided that they are of the same order. Addition of matrices is done by adding together the corresponding elements of each of the two matrices. For instance:

$$\begin{pmatrix} 4 & 5 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 4+3 & 5+6 \\ 1+2 & 2+4 \end{pmatrix} = \begin{pmatrix} 7 & 11 \\ 3 & 6 \end{pmatrix}.$$

Subtraction of matrices is done in a similar way except the corresponding elements are subtracted. For instance:

$$\begin{pmatrix} 4 & 5 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 4-3 & 5-6 \\ 1-2 & 2-4 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & -2 \end{pmatrix}.$$

Multiplication of Matrices:

i. Scalar multiplication: A matrix may be multiplied by a number as follows:

$$4\begin{pmatrix} 5 & -2 \\ 1 & 8 \end{pmatrix} = \begin{pmatrix} 4\times 5 & 4\times(-2) \\ 4\times 1 & 4\times 8 \end{pmatrix} = \begin{pmatrix} 20 & -8 \\ 4 & 32 \end{pmatrix}.$$

ii. General Matrix Multiplication: Two matrices can only be multiplied by each other if the number of columns in the one is equal to the number of rows in the other. Multiplication of matrices is done by multiplying a row by a column as follows:

$$\begin{pmatrix} 4 & 5 \\ 1 & 2 \end{pmatrix} \times \begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 4\times 3 + 5\times 2 & 4\times 6 + 5\times 4 \\ 1\times 3 + 2\times 2 & 1\times 6 + 2\times 4 \end{pmatrix} = \begin{pmatrix} 22 & 44 \\ 7 & 14 \end{pmatrix}.$$

The product of an $m \times n$ matrix $A = (a_{ij})$ and an $n \times p$ matrix $B = (b_{ij})$ is a matrix $C = AB = (c_{ij})$ whose (i, j) entry is

$c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}$, where $1 \le i \le m$ and $1 \le j \le p$.

Inverting a Matrix: An n-square matrix A is said to be "invertible" or "non-singular" if there exists an n-square matrix B with the following property:

$AB = BA = I_n$,

where I_n is the n-square identity matrix, namely, the $n \times n$ matrix with ones along the main diagonal and zeros elsewhere. If this is the case, then

the matrix B is called the inverse of A, and the notation A^{-1} is used to designate B. If no such B exists, then A is said to be "singular." If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Transposition of Matrices: The "transpose" of a matrix A is denoted by A^T, and it is the matrix obtained by writing the rows of A, in order, as columns; if $A = (a_{ij})$ is an $m \times n$ matrix, then $A^T = (a_{ij}^T)$ is the $n \times m$ matrix where $a_{ij}^T = a_{ji}$, for all i and j. For instance, if

$$A = \begin{pmatrix} 1 & 7 \\ 4 & 3 \end{pmatrix}, \text{ then } A^T = \begin{pmatrix} 1 & 4 \\ 7 & 3 \end{pmatrix}.$$

Determinants: The determinant of a matrix A is a scalar assigned to A, and it is denoted by $det(A)$. Given a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

its determinant is

$$det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

Solution of simultaneous equations using matrices: Let us consider a system of two linear equations with two unknowns:

$$\begin{Bmatrix} a_{11}x_1 + a_{12}x_2 = c_1 \\ a_{21}x_1 + a_{22}x_2 = c_2 \end{Bmatrix},$$

which gives rise to the following three matrices:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, B = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \text{ and } X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Thus, the original system of linear equations can be reformulated as follows:

$$AX = B \Leftrightarrow X = A^{-1}B,$$

where A is the matrix of the system's coefficients, X is the matrix of the system's unknowns, and B is the matrix of the system's constant terms. The system has a unique solution if and only if the determinant $\det(A) = a_{11}a_{22} - a_{12}a_{21} \neq 0$, and that solution is:

$$x_1 = \frac{B_{x_1}}{\det(A)} = \frac{\begin{vmatrix} c_1 & a_{12} \\ c_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{c_1a_{22} - a_{12}c_2}{a_{11}a_{22} - a_{12}a_{21}},$$

and

$$x_2 = \frac{B_{x_2}}{\det(A)} = \frac{\begin{vmatrix} a_{11} & c_1 \\ a_{21} & c_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{a_{11}c_2 - c_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}},$$

where the numerators B_{x_1} and B_{x_2} are obtained by substituting the column of constant terms in place of the column of coefficients of the

corresponding unknown in the matrix of coefficients. If $\det(A) = 0$, then the system has either no solution or an infinite number of solutions.
Consider the 3-square matrix

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}.$$

The determinant of A is

$$\det(A) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_1c_2b_3 - b_1a_2c_3 - c_1b_2a_3.$$

Moreover, it can be easily shown that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

Let us consider a system of 3 linear equations with 3 unknowns:

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2. \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

The aforementioned system has a unique solution if and only if the determinant of the matrix of coefficients is not zero:

$$\det(A) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0.$$

In this case, the unique solution of the given system can be expressed as quotients of determinants as follows:

$$x = \frac{B_x}{\det(A)},$$

$$y = \frac{B_y}{\det(A)},$$

$$z = \frac{B_z}{\det(A)},$$

where the numerators B_x, B_y, and B_z are obtained by substituting the column of constant terms for the column of coefficients of the corresponding unknown in the matrix of coefficients, so that:

$$B_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, B_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \text{ and } B_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}.$$

If $\det(A) = 0$, then the system has either no solution or an infinite number of solutions.

Advances in computing power have contributed significantly to the application of matrix algebra in several scientific disciplines, such as physics and mathematical economics.

The application of matrices in input-output analysis

The major economic tasks that every society must accomplish pertain to decision-making about an economy's inputs and outputs. In economics, the term "input" refers to commodities or services used by firms in their production processes. Thus, by means of its technology, an economy combines inputs to produce outputs. In economics, the term "output" refers to the various useful goods or services that are either employed in further production or consumed.

The acknowledged founder of "input-output analysis" is the Russian-American economist Wassily Leontief, who won the Nobel Prize in Economics in 1973. An input-output matrix is a square matrix, say $A = (a_{ij})$, whose entries a_{ij} represent the amount of input i required per unit of output j. A column of such a matrix depicts the inputs needed for the achievement of a specific output. Therefore, from the perspective of economics, it can be considered as a "production technique." Hence, an input-output matrix is a "constellation" of production techniques. If the list of inputs is complete, including factor inputs, then the input-output matrix contains techniques for the production of the factor services as well. Input-output is an integral part of general equilibrium analysis. As the American economist Campbell R. McConnell has pointed out, the economy is "an interlocking network of prices wherein changes in one market are likely to elicit numerous and significant changes in other markets," so economists need to study "the price system as a whole" and focus on "general equilibrium analysis" (Campbell R. McConnell, *Economics*, fifth edition, New York: McGraw-Hill, 1972, p. 579).

For instance, let us consider a small economic network that consists of two interdependent industries A and B (e.g., A may represent the final goods industry, and B may represent the energy industry). This method can obviously be generalized to any number of industries. We assume that, for each dollar's worth of goods/services produced by A, A needs to consume a quantity of A's output and a quantity of B's output, and, for each dollar's worth of goods/services produced by B, B needs to consume a quantity of B's output and a quantity of A's output. In particular, the production of each dollar's worth of A requires $\$q_{11}$ worth of A and $\$q_{21}$ worth of B; and the production of each dollar's worth of B requires $\$q_{12}$ worth of A and $\$q_{22}$ worth of B. Therefore, both industries sell to each other and buy from each other. In addition, assume that there is an external demand for A

and B; specifically, let the final demand from the outside sector of the economy be $\$d_1$ million for A and $\$d_2$ million for B. Let x_1 and x_2 represent the total output from A and B respectively.
Then we formulate the following equation:
$X = QX + D \Rightarrow X - QX = D \Rightarrow IX - QX = D \Rightarrow (I - Q)X = D,$
where: $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$, and $D = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$,
and X is the "output matrix" (i.e., X is a column matrix representing the equilibrium output levels in industry A and industry B), Q is the "technology matrix," D is the "final demand matrix," and I is the identity matrix.
If $I - Q$ is invertible, then the solution for X is given by
$X = (I - Q)^{-1}D,$
which is the optimum level of production for the given economic network, meaning that the given economic network must produce x_1 million dollars of A (e.g., final goods) and x_2 million dollars of B (e.g., energy) in order to meet both the internal demand and the external demand for A and B (and thus avoid both oversupplying and undersupplying the market with the corresponding commodities).

The application of matrices in linear programming

By the term "linear programming," we mean a method to achieve the best outcome (e.g., to maximize profit, minimize cost, etc.) in a mathematical model whose requirements are represented by linear functions. The first contributions to linear programming are due to the Soviet mathematician and economist Leonid Vitaliyevich Kantorovich (1912–86), who won the Nobel Prize in Economics in 1975. Moreover, one of the acknowledged founders of linear programming is the American mathematician George Bernard Dantzig (1914–2005), who managed to make significant contributions to industrial engineering, operations research, economics, statistics, and computer science. In fact, input-output analysis is a special and very important case of linear programming.
The "canonical form" of linear programming is the following: given a system of m linear constraints (or linear inequalities) with n variables, we wish to find non-negative values (i.e., ≥ 0) of these variables that will satisfy the constraints and will maximize a function of these variables; symbolically: given m linear inequalities and/or equalities
$\sum_j a_{ij}\, x_j \leq b_i, i = 1,2, \dots, m, and\ j = 1,2, \dots, n,$ $(*)$
we wish to find those values of x_j which satisfy the constraints $(*)$ and the condition that $x_j \geq 0$ (for $j = 1,2, \dots, n$) and simultaneously maximize the linear function

$$z = \sum c_j x_j \, , j = 1,2, \dots , n. \qquad (**)$$

For instance, consider a problem where we wish to maximize the gross profit of an industry (or of a firm offering several product lines) that produces n commodities, and thus has n sectors of production. In this case, $(*)$ and $(**)$ can be interpreted as follows: z denotes an overall performance measure (specifically, total gross profit); x_j denotes the level of activity j ($j = 1,2, \dots , n$), specifically, the output of the jth sector of production (i.e., the produced quantity of the jth commodity); c_j denotes the performance measure coefficient for activity j, specifically, the gross profit per unit of output in the jth sector of production (so that the total gross profit in the jth sector of production is $c_j x_j$); b_i denotes the available quantity of resource i ($i = 1,2, \dots , m$); and a_{ij} denotes the quantity of resource i consumed by each unit of activity j.

In matrix form, the constrained maximization problem $(**)$ can be rewritten as follows:

$$z_{max} = \begin{pmatrix} c_1 & c_2 & \cdots & c_n \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

under the constraints

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \leq \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix},$$

and

$x_j \geq 0$ for $j = 1,2, \dots , n$. More simply, given the above concepts, we can write:

$$\left. \begin{matrix} maxz = cx \\ under\ the\ constraints \\ Ax \leq b \\ x_j \geq 0 \end{matrix} \right\}. \qquad (***)$$

Regarding the geometric significance of $(***)$, notice that the constraints $Ax \leq b$ and $x_j \geq 0$ define a convex polyhedron P_n in $\mathbb{R}^n$, and such P_n is called the "feasible region" of the corresponding model, meaning the region of all the feasible solutions of the corresponding problem. In general, a polyhedron P_n in $\mathbb{R}^n$ is the set of all points $x \in \mathbb{R}^n$ that satisfy a finite set of linear inequalities. Moreover, a set Q in $\mathbb{R}^n$ is called "convex" if, for any two points x and y in Q, the line segment joining them is also in Q; symbolically: $\forall x, y \in Q$, the "convex combination" $kx + (1 - k)y \in Q$ for any k such that $0 \leq k \leq 1$. The goal of constrained maximization in the context of linear programming is to choose that feasible combination

$(x_1, x_2, \dots, x_n)$ of actions that maximize a given function $z = cx$. This occurs at the maximum (most extreme) point $(x_1^*, x_2^*, \dots, x_n^*)$ of the feasible region.

The constrained maximization problem $(***)$ is known as the "primal problem," while the so-called "dual problem" is the corresponding constrained minimization problem where, given a system of m linear constraints (linear inequalities) with n variables, we wish to find non-negative values (i.e., ≥ 0) of these variables that will satisfy the constraints and will minimize a function (e.g., a cost function) of these variables; symbolically:

$$\left.\begin{array}{c} minz = cx \\ under\ the\ constraints \\ Ax \geq b \\ x_j \geq 0 \end{array}\right\}. \qquad (****)$$

For instance, using the "dual problem," we can create models of constrained cost minimization in economics and business management. Firms seek to minimize cost subject to the constraint that they produce at least b units of output, so that the firm's cost minimization problem is given by $(****)$. In general, linear programming is useful for guiding quantitative decisions in business planning, industrial engineering, and the social and physical sciences.

Chapter 4
The Basic Equations of Money

By the term "economy," we refer to a system for making decisions about the use of limited resources so that goods and services can be produced and cosumed. By the term "market," we refer to a system in which two or more parties participate in order to engage in economic transactions.

According to the standard functional definition of "money," four functions have been ascribed to money—namely: medium of exchange, unit of account, store of value, and standard of deferred payment. The stock of money held in an economy is held for various reasons: firstly, money is held in order to facilitate exchange (i.e., it is to be spent rather than saved), and, secondly, it may be held as an asset (i.e., to be saved rather than spent).

If the supply of money falls below the level that is necessary to support the growth of the economy, then the growth of the economy will be held below its potential. On the other hand, if the supply of money is above the level that is necessary to support the potential growth of the economy in real terms, then the growth of the economy in money terms will be greater than the growth in real terms, and this, other things equal, will manifest itself in inflation. The "central bank" is a public institution that is responsible for implementing and managing the monetary policy of a country, or of a group of countries, and it controls the money supply.

In an economy, there will always exist two groups of economic agents: (i) surplus units, namely, those whose revenue exceeds their current expenditure during a given period of time, and (ii) deficit units, namely, those whose expenditure exceeds their current revenue in a given period of time. Therefore, some mechanism is required to ensure that the surplus funds are channeled to the deficit units.

The surplus units can lend their excess funds directly to the deficit units. For instance, a person can buy company or government securities through a public issue. However, it is very often the case that a surplus unit will lend its excess funds to a financial institution ("financial intermediary"), which will then on-lend these funds by itself, buying company stocks, government bonds, or other assets in which it invests. Thus, instead of a direct contractual relationship between the provider and the user of the funds, there are two contractual relationships: (i) the surplus unit lends to or acquires a financial claim on the financial intermediary, and (ii) the financial intermediary lends to or acquires a financial claim on the ultimate borrower, the user of the funds. Financial intermediation facilitates the reconciliation of the differing needs of lender and borrower

by means of: (i) maturity transformation (since a financial intermediary can borrow short and lend long), (ii) aggregation (i.e., by collecting together a large number of relatively small amounts), and (iii) risk transformation. The most important financial intermediaries are commercial banks, investment banks, insurance companies, mutual funds, hedge funds, pension funds, venture capitals, savings and loans associations, credit unions, mutual savings banks, and consumer finance companies.

In economics, by the term "interest," we refer to the profit return on investment. The money that is invested is called the "principal." The percentage return per annum is called the "rate per cent." Thus, if P stands for the principal, T stands for the time in years, R stands for the rate per cent per annum, and I stands for the interest, then

$$I = \frac{PRT}{100}$$

where P and I must be in the same monetary units. This formula can be transposed to give P, R, and T in terms of the other letters:

$T = \frac{100I}{PR}$,

$R = \frac{100I}{PT}$, and

$P = \frac{100I}{RT}$.

Compound interest is different from simple interest in that the interest which is added also attracts interest. If a sum of P monetary units is invested at $r\%$ per annum for n years, then the value or amount after n years is

$P\left(1 + \frac{r}{100}\right)^n$.

For instance, the value of $\$2{,}500$ invested at 5% compound interest after eight years (i.e., $P = \$2{,}500$, $r = 5$, and $n = 8$) will be

$P\left(1 + \frac{r}{100}\right)^n = \$2{,}500\left(1 + \frac{5}{100}\right)^8 = \$3{,}693.$

The mathematical formula of compound interest is the following: assume that you borrow an amount P of money (the "principal") at an (annual) interest rate of $r > 0$, and that, at the end of each year, you have to pay back a fixed amount (a "deposit") d. Let A_n be the total amount of money owed after n years. The formula for computing A_n in terms of P (the principal of the loan), r (the interest rate of the loan), and d (the loan deposits) is the following (where $0 < r \leq 1$):

$$\begin{aligned} A_n &= A_{n-1}(1 + r) - d \\ &= P(1 + r)^n - d(1 + r)^{n-1} - d(1 + r)^{n-2} - \cdots \\ &\quad - d \end{aligned}$$

$= P(1+r)^n - d\frac{(1+r)^n-1}{(1+r)-1} \Leftrightarrow A_n = P(1+r)^n - \frac{d}{r}[(1+r)^n - 1], r \neq 0;$
so that the initial condition is $A_0 = P$; at the end of the first year, you owe P (the principal) plus an interest equal to rP minus the deposit you have agreed to pay each year. Therefore, $A_1 = P + rP - d = P(1+r) - d$; by analogy, at the end of the second year, you owe $A_2 = A_1(1+r) - d = P(1+r)^2 - d(1+r) - d$, etc. By allowing the owners of large sums of money to lend (that is, trade) money on interest, we give them power to immunize themselves against loss (in fact, this is the ultimate purpose of charging interest on loans: to immunize the lender of money against loss), while socializing loss and risks. Thus, we create an exceptionally privileged financial oligarchy.

The net present value (NPV) of an investment project consists in calculating the amount by which the value of that investment project exceeds its cost. If i is the interest rate (which, for convenience, is assumed to be fixed for the project under consideration), then the NPV is defined as follows:

$$NPV = \frac{X_1}{1+i} + \frac{X_2}{(1+i)^2} + \cdots + \frac{X_n}{(1+i)^n} - C_0$$

where X_t $(t = 1,2,\dots,n)$ denotes the cash flow that corresponds to year t, C_0 is the capital cost of the investment project in year 0, and n is the lifetime (in years) of the investment project. Hence, according to the Nobel Prize-winning Italian-American economist Franco Modigliani and the American economist Merton Miller, under certain conditions (in particular, if we assume that there is total information transparency and total rationality), the intrinsic or real value of a company can be considered to be the net present value of all the investment projects of that company. Furthermore, if we divide the intrinsic or real value of a company by the total number of outstanding shares issued by that company, we can find the real or intrinsic value per share (for the given company).

Whereas the term "stock" means a share in the ownership of a company, the term "bond" means debt. In fact, a bond is a debt instrument issued for a period of more than one year with the purpose of raising capital by borrowing. By the term "maturity," we mean the date on which a debt becomes due for payment. The "face value" (also known as the "par value" or "principal") is the amount of money a holder of a fixed income security will receive back once the given security matures. The "coupon" is the amount that a holder of a fixed income security will receive as interest payments. The coupon is expressed as a percentage of the par value. "Yield" is a figure that shows the return one gets on a bond.

$$Current\ Yield = \frac{coupon\ amount}{market\ price}$$

(when we buy a bond at par, yield is equal to the coupon, and, when price changes, so does the yield). For instance, suppose that a bond has a par value of $\$1{,}000$ and that its coupon rate is equal to 6%. Since the market price of a bond changes, an investor may purchase a bond at a discount (i.e., less than par value) or a premium (i.e., more than par value). In particular, if an investor buys this 6% coupon rate bond for a discount of $\$900$, then the investor earns an annual interest income of $(\$1{,}000 \times 6\%) = \$60$, and the current yield is $\$60/\$900 = 6.67\%$. Notice that the annual cash flow of $\$60$ is fixed, regardless of the price paid for the bond. On the other hand, if an investor buys this 6% coupon rate bond at a premium of $\$1{,}100$, then the investor earns again an annual interest income of $(\$1{,}000 \times 6\%) = \$60$, but, in this case, the current yield is $\$60/\$1{,}100 = 5.45\%$.

A "zero-coupon bond" is a type of bond that makes no coupon payments but, instead, is issued at a considerable discount to par value. For instance, a zero-coupon bond with a $\$1{,}000$ par value and ten years to maturity might be trading at $\$600$. In case of a zero-coupon bond,

$$Y = \left(\frac{M}{P}\right)^{1/N} - 1$$

where Y denotes the yield to maturity, M denotes the value of the given zero-coupon bond at the time of maturity (i.e., the par value), P denotes the price of this bond, and N denotes the years to maturity.

In general, as we have seen, when price goes up, yield goes down, and vice versa. The factor that influences a bond more than any other is the level of prevailing interest rates in the economy. When interest rates rise, the prices of bonds in the market fall, and, thus, we see an increase in the yield of the older bonds, which are brought into line with the newer bonds being issued with a higher coupon. On the other hand, when interest rates fall, the prices of bonds in the market rise, thereby lowering the yield of the older bonds and bringing them into line with the newer bonds being issued with a lower coupon. Moreover, another important factor that influences a bond is the issuer's default risk. In fact, investors try to determine if the bond rating agencies are going to change the issuer's rating. Rating changes may be prompted by changes in such factors as: financial ratios, Gross National Product, inflation, etc.

In 1911, the American economist Irving Fisher expressed the "quantity theory of money" in what is known as the equation (actually, identity) of exchange:

$$MV = PQ$$

where M is the quantity of money in the economy, V is the velocity of the circulation of money (i.e., the amount of nominal Gross National Product each year divided by the money stock), P is the general price level (i.e., the average value of each transaction), and Q is aggregate output (i.e., the physical volume of transactions during the given time period, so that $Gross\ National\ Product = PQ$). Thus, according to Fisher, if we assume that, at least in the short-run, both V and Q are constant (given that the velocity of circulation is determined by institutional factors, such as the payments interval for wages, and Q is determined by the productive capacity of the economy), then a change in the money supply, M, results in an equal percentage change in the price level P.

The previous equation implies that

$$M = \frac{PQ}{V}$$

and, since V is (assumed to be) constant, $1/V$ can be replaced by a constant k. Additionally, when the money market is in equilibrium, the demand for money, M_d, is equal to M. Hence,

$$M_d = kPQ$$

which means that, according to Fisher's model, the demand for money is a function of income and does not depend on interest rates.

However, in practice, the velocity of the circulation of money, V, is not constant, even in the short-run, and especially during periods of recession. Therefore, the English economist John Maynard Keynes extended Fisher's equation of exchange by pointing out that there are three motives for holding money: (i) Transactions motive: money is a medium of exchange, and, as income rises, people conduct more transactions and hold more money. (ii) Precautionary motive: people hold money for emergencies, and money demand is again expected to rise with income. (iii) Speculative motive: money is also a way for people to store wealth, and, under the speculative motive, the demand for money is negatively related to the interest rate. Moreover, Keynes modeled the demand for money as the demand for the real (as opposed to the nominal) quantity of money (real balances), M/P . According to Keynes, the demand for real money balances is a function of both income and interest rates:

$$\frac{M}{P} = f(Q, i)$$

where Q denotes output or income and i denotes the interest rate (and, hence, the velocity of the circulation of money fluctuates with the interest rate).

The level of interest rates can indeed be treated as a monetary target, but it is important to determine the extent to which interest rates are a major

factor in decisions of either businesses, consumers, or governments. For instance, if an economy is characterized by important structural inefficiencies, then an increase in the supply of money (other things equal), instead of boosting economic growth, may lead to an increase in inflation and money incomes.

Moreover, it is worth mentioning that central banks have at their disposal a number of policy instruments that can affect certain intermediate targets, such as the money supply, interest rates, etc. The three major instruments of monetary policy are:

(i) *Open market operations:* this is the activity of a central bank in buying or selling government bonds to influence the money supply, interest rates, and bank reserves. In fact, if securities are bought (by the central bank), the money paid out by the central bank increases commercial-bank reserves, and the money supply increases. On the other hand, if securities are sold (by the central bank), then money supply decreases.

(ii) *Discount-rate policy:* given that the discount rate is the interest rate charged by the central bank on a loan that it makes to a commercial bank, it follows that the central bank can increase the discount rate to reduce the money supply, whereas the central bank can reduce the discount rate to increase the money supply.

(iii) *Reserve-requirements policy:* by the term "required reserves," we mean that portion of deposits that a bank sets aside in the form of vault cash or non-interest-earning deposits with the central bank. In fact, if the central bank wants to tighten money overnight, then it can raise reserve requirements, whereas, if the central bank wants to ease credit conditions (and, thus, increase the money supply), then it can cut reserve requirements.

Chapter 5
Probability and Statistics

First of all, it should be clarified that, by the term "quantitative analysis," we mean the study of phenomena by means and on the basis of any type of quantitative information. Such an inquiry takes place by applying suitable methods that determine the nature of the available information and the phenomena under consideration. Quantitative methods mainly include methods that derive from mathematical analysis, mathematical programming, probability theory, and statistics.

In fact, statistics emerged from the constant efforts of humankind to deal with situations of uncertainty in which they lived. In these situations, the element of luck always appeared as a key determining factor which prevented the identification of the existence of systematicness in the manifestations of various phenomena and in the formulation of relations between them. Aristotle was the first philosopher to offer a systematic account of "luck" and to include it as a significant topic in both physics and ethics (Aristotle, *Physics*, 2:4–6, and *Metaphysics*, 7:7–9). A method is called statistical if it relates facts and hypotheses of some kind. Hence, statistics investigates and develops methods for evaluating hypotheses in reference to empirical facts.

In general, luck is involved in all things where actors do not hold full control over the outcome of action. One of the basic attributes of the statistical method is the fact that it refers to properties of populations instead of individual cases. Statistics examines a unit only in its capacity as a member of a population. The statistical method can be applied to any problem related to the definition of overall behavior, based on individual observations expressed numerically. The concept of luck is commonly used in statistics in order to display all the possible outcomes given a very large sample and the probability of each outcome. In science, "probabilities," often called chances or stochastic processes, are relative frequencies in series of events, or tendencies or propensities in the systems that give rise to those events. The "frequency" is the number of times each measurement occurs.

Probability theory is primarily concerned with the issue of uncertainty. In fact, "probability," usually denoted by p, is a quantitative measure of uncertainty. It is a number between 0 and 1, where 0 indicates impossibility and 1 indicates certainty. Assume that we take any very large number, N, out of a series of cases in which an event, A, is in question, and that A happens on pN occasions (where $0 \leq p \leq 1$). The probability of the event A is said to be p (the certainty of the corresponding proposition

increases as the number N of specimen cases selected increases). Furthermore, the following corollaries and extensions may be added to the aforementioned definition of a probability: (i) If the probability of an event is p, then, out of N cases in which it is in question, it will happen pN times, where N is any very large number (where $0 \leq p \leq 1$). (ii) If the probability of an event is p, then the probability of its failing is $1 - p$.

Probability theory is based on set theory. By the term "experiment," we mean a process that leads to one of several possible outcomes. By the term "outcome," we mean an observation or measurement. The "sample space" is the set of all possible outcomes of an experiment. An "event" is a subset of a sample space—or, in other words, a set of basic outcomes. Thus, we say that the event "occurs" if the corresponding experiment gives rise to a basic outcome belonging to the event. Therefore, we obtain the following formula:

$$Probability\ of\ event\ A = \frac{n(A)}{n(S)},$$

where $n(A)$ is the number of elements in the set of the event A, and $n(S)$ is the number of elements in the sample space S. For instance, roulette as it is played in Las Vegas or Atlantic City consists of a wheel that has 36 numbers, numbered 1 through 36, and the numbers 0 and 00 (double zero). Therefore, in this case, the sample space, S, consists of 38 numbers, and the probability of winning a single number that you bet is $P = 1/38$.

When the sets corresponding to two events are disjoint (their intersection is the empty set), then these events are called "mutually exclusive."

The axiomatic definition of probability is the following: Let E be a space of elementary events (i.e., the space of outcomes of experiments, or the space of states of a system, since the state of a system can be construed as the outcome of an experiment). The "probability of an event" $A \subseteq E$ is denoted by $p(A)$, and it is defined as a single number that corresponds to A and has the following properties:

(P1) $p(A) \geq 0$;

(P2) for each pair of mutually exclusive events, $A, B \subseteq E$, it holds that
$p(A \cup B) = p(A) + p(B)$;

(P3) $p(E) = 1$ (i.e., the total probability, after adding all possibilities, is equal to one).

Remark: For each $A, B \subseteq E$, $p(A \cup B) = p(A) + p(B) - p(A \cap B)$; but, in case A and B are mutually exclusive, it holds that $p(A \cap B) = 0$, so we obtain (P2).

By the term "conditional probability," we mean the probability of event A conditional upon the occurrence of event B. Assume that we investigate

the probability of an event A given that we know that an event B has occurred, and that event B influences the probability of event A. The "conditional probability" of event A given the occurrence of event B is defined as the quotient of the probability of the intersection of A and B over the probability of event B; symbolically:

$P(A|B) = \frac{P(A \cap B)}{P(B)}$,

assuming that $P(B) \neq 0$. The aforementioned formula for the computation of conditional probability is known as Bayes's Law, since it was originally formulated by the eighteenth-century English statistician and philosopher Thomas Bayes. Notice that A is independent of B if $P(A|B) = P(A)$; that is, knowing that B occurred does not change the probability that A occurred. Thus, according to Bayes's Law, two events A and B are independent of each other if and only if

$P(A \cap B) = P(A)P(B)$.

Bayes's Law provides a method of revising existing predictions or theories (specifically, updating probabilities) given new additional evidence. In fact, Bayes's Law implies that the interpretation of any risk assessment depends on an estimate of the base rate, and the corresponding base rate, which is never known with complete certainty at the time of the assessment, is a Bayesian "prior probability."

Probability theory has several significant applications in the natural sciences and in the social sciences. For instance, in genetics, probability is a measurement tool that helps us to predict the chances of an offspring being inherited with a particular trait of interest (assuming Mendel's laws of inheritance). The sum law helps us to find the probability of two or more events occurring as long as they are mutually exclusive. The product law helps us to find the probability of two or more events occurring as long as they are independent of each other. Moreover, probability theory helps us to estimate the chances of success or failure of a business project, an investment, or product launch.

One of the most important methods that is used to discover, describe, and explain "typical" behavior of mass data is the "arithmetic mean." The formula is

$$\bar{X} = \frac{\sum_{i=1}^{N} X_i}{N}$$

where $\bar{X}$ denotes the arithmetic mean, $\sum_{i=1}^{N} X_i$ denotes the summation of the values of the individual observations X_i under consideration ($i = 1,2, \dots, N$), and N is the total number of items in the series that have been summated. It is worth noticing that arithmetic means are often "weighted" averages, in the sense that, when averaging values, it is sometimes

logically necessary to assign more importance to some than to others (by multiplying each value with a suitable statistical weight), so that particular values may be more influential in determining the "typical" value than others. Formally, the weighted arithmetic mean of a non-empty finite set of data $\{X_1, X_2, \ldots, X_N\}$ with corresponding non-negative weights $\{w_1, w_2, \ldots, w_N\}$ is

$$\bar{X} = \frac{\sum_{i=1}^{N} w_i X_i}{\sum_{i=1}^{N} w_i} = \frac{w_1 X_1 + w_2 X_2 + \cdots + w_N X_N}{w_1 + w_2 + \cdots + w_N}$$

(the weights can be in the form of decimals, whole numbers, percentages, etc.). For instance, if $x_1, x_2, x_3, \ldots$ are the measured observations and $f_1, f_2, f_3, \ldots$ are the corresponding frequencies, then the arithmetic mean is

$$\bar{x} = \frac{f_1 x_1 + f_2 x_2 + f_3 x_3 + \cdots}{f_1 + f_2 + f_3 + \cdots}$$

(this is the arithmetic mean of a frequency distribution). Moreover, notice that a consumer price index (CPI) is typically calculated as a weighted average of the price change of the goods and services covered by the index (in this case, the weights are meant to reflect the relative importance of the goods and services as measured by their shares in the total consumption of households).

Whereas the mean is the average value of a set of data, the median is the middle value in a set of data (so that we find the median by dividing the observations by two, and, if the number of observations is odd, we round that number up). The mode is the value that appears most frequently in a set of data.

By the term "probability distribution," we mean a statistical function that describes all the possible values and likelihoods that a random variable can take within a given range. A probability distribution is called a "normal distribution," or a "Gaussian distribution," if it is symmetric about the mean, showing that data near the mean are more frequent in occurrence than data far from the mean. In the normal distribution, its mean (average), median (midpoint), and mode (most frequent observation) are all equal to each other; and these values all represent the peak, or highest point, of the distribution. In graphical form, the normal distribution appears as a "bell curve":

In other words, the "normal curve" is bell-shaped and perfectly symmetric (centered on the mean).

One of the most important methods that are used to discover, describe, and explain "risk" or "uncertainty" is the "standard deviation," which is a quantity expressing by how much the members of a database (i.e., the data

under consideration) differ from the arithmetic mean of the given database. The formula is:

$$\sigma = \sqrt{\frac{\sum_{i=1}^{N} x_i^2}{N}}$$

where: firstly, we calculate the arithmetic mean $\bar{X}$ of the values X_i $(i = 1,2, \dots, N)$ under consideration; secondly, we record the deviation of each value X_i from the arithmetic mean, namely, $x_i = X_i - \bar{X}$; thirdly, we square these deviations (we compute x_i^2); fourthly, we summate the squared deviations and divide by N (thus finding the "variance" of our data); fifthly, we extract the square root to obtain σ. However, the aforementioned formula for the standard deviation is used when N is the entire population of the species or kind under consideration; if we do not have the entire population, we use the following formula for the standard deviation:

$$s = \sqrt{\frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}}$$

where n is the size of the sample (i.e., the number of the point data that are contained in the database that we use), X_i is the ith point of the sample $(i = 1,2, \dots, n)$, and $\bar{X}$ is the arithmetic mean of the sample (namely, of the database that we use).

The normal curve's standard deviation tells us what percentage of observations falls within a specific distance from the mean: When we have a normal curve, the area below the curve contains 100% of all observations; 68% of all observations fall within one standard deviation from the mean; 95% of all observations fall within about two standard deviations from the mean; and 99% of all observations fall within about three standard deviations from the mean.

When we have two sets of data and we want to find how strong a relationship is between them, we use Pearson's correlation coefficient (PCC), also known as Pearson's r. In other words, PCC calculates the level of change in one variable due to the change in the other. When applied to a sample of the variables x and y, PCC is commonly represented by r_{xy}.

Given paired data $\{(x_1, y_1), \dots, (x_n, y_n)\}$, consisting of n pairs, r_{xy} is defined as follows:

$$r_{xy} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}}$$

where:

n is the sample size,
x_i are the values of the x-variable in the sample,
$\bar{x}$ is the mean of the values of the x-variable,
y_i are the values of the y-variable in the sample, and
$\bar{y}$ is the mean of the values of the y-variable.
PCC returns values between -1 and 1, symbolically,
$-1 \leq r_{xy} \leq 1$,
where:
1 indicates a strong (actually, perfect) positive relationship,
-1 indicates a strong (actually, perfect) negative relationship, and
a result of zero indicates no relationship at all. In general, a positive correlation between two variables means that both the variables move in the same direction, whereas a negative correlation between two variables means that both the variables move in opposite directions.
For instance, in biology, the relation between independent or the predictor variables and outcome or the dependent variable is explored using correlation analysis. In this way, one can explain how the risk factors or the predictor variables account for the possibility of the occurrence of a disease or presence of a phenotype. The disease outcome or the dependent variable is associated with biological factors (e.g., age and gender), lifestyle variables, psychological variables, and genetic factors (genetic mutations), and correlation tests help us to understand such "risk factors–disease" relationships. Moreover, correlation is an important part of statistical analysis in economics and social policy, and it helps us to understand economic and social phenomena and trends.

Chapter 6
Classical Euclidean Geometry, Analytic Geometry, and Trigonometry

Geometry is the scientific study of the quantitative and the qualitative properties of spatial forms and relations (the criteria for equality of triangles provide instances of qualitative geometric knowledge, and the computation of lengths, areas, and volumes exemplifies quantitative geometric knowledge).

Around 300 B.C.E., Euclid published the definitive treatment of Greek geometry and number theory in his thirteen-volume *Elements*, building on the experience and the achievements of previous Greek mathematicians: on the Pythagoreans for Books I–IV, VII, and IX, on Archytas for Book VIII, on Eudoxus for Books V, VI, and XII, and on Theaetetus for Books X and XIII. The axiomatic method used by Euclid is the prototype for the entire field of "pure mathematics," which is "pure" in the sense that we need only pure thought, no physical experiments, in order to verify that the statements are correct—that is, we need only to check the reasoning in the demonstrations. All mathematical theorems are conditional statements—namely, statements of the form

If (hypothesis) *then* (conclusion).

Put simply, one condition (hypothesis) implies another (conclusion). In particular, in a given mathematical system, the only statements that are called "theorems" are those statements for which a proof has been supplied. By a "proof," we mean a list of statements that is endowed with a justification for each statement, and it ends up with the conclusion desired. The following are the six types of justifications allowed for statements in proofs: (i) "by hypothesis . . ."; (ii) "by axiom . . ."; (iii) "by theorem . . ."; (iv) "by definition . . ."; (v) "by step . . ."; (vi) "by rule . . . of logic"; and a justification may involve several of the aforementioned types.

In particular, Euclid articulated:

i. *A set of definitions, such as the following:*
   - A point is that which has no part or magnitude (i.e., it does not have a concrete size).
   - A line is length without breadth.
   - The ends of a line are points. A straight line is a line that lies evenly with the points on itself.
   - A surface is that which has length and breadth only.
   - The edges of a surface are lines.

- A plane surface is a surface that lies evenly with the straight lines on itself.

ii. *A set of fundamental rules (axioms):*

- Things that are equal to the same thing are equal to each other.
- If equals are added to equals, then the wholes are equal.
- If equals are subtracted from equals, then the remainders are equal.
- Things that coincide with each other are equal to each other.
- The whole is greater than the part.
- Things that are double of the same things are equal to each other.
- Things that are halves of the same things are equal to each other.

iii. *A set of fundamental propositions (postulates):*

- Postulate 1: a straight line may be drawn from one point to any other point. Given two distinct points, there is a unique straight line that passes through them.
- Postulate 2: a terminated straight line can be produced indefinitely.
- Postulate 3: a circle can be drawn with any center and any radius.
- Postulate 4: all right angles are equal to each other.
- Postulate 5 (known as the Parallel Postulate): if a line segment intersects two straight lines forming two interior angles on the same side that sum to less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles.

According to Euclidean geometry, space is three-dimensional and isotropic (i.e., it has the same value when measured in different directions). This scientific conception of space clashes with several mythical and folk perceptions of space, according to which space is connected with a form of temporality, and it is unisotropic (for instance, the "upward" and the "forward" directions are evaluated as superior to the "downward" and the "backward" directions). The Euclidean perception of space, combined with the concept of gravity, found its fullest expression in Isaac Newton's calculus and mechanics.

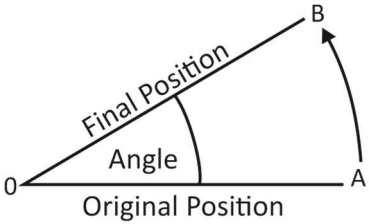
In view of Euclid's geometric treatises and the subsequent development of geometry as a scientific discipline, geometry is "an axiomatic in which we ignore all representation, and in which the word 'space' designates a structure, i.e., a system of axioms and deductions" (Saddo Ag Almouloud, "Demonstration in Geometry: Historical and Philosophical Perspectives,"

Quantitative Research Journal, vol. 8, Special Edition: Philosophy of Mathematics, 2020, p. 562). In other words, in mathematics, by the term "space," we mean a non-empty set endowed with some mathematical structure. In general, in mathematics, the term "structure" refers to a class of mathematical objects described by axioms. Moreover, sometimes mathematicians use the term "structure" in order to refer to the description of the way in which an object could be reconstructed from simpler objects of the same kind.

Euclidean Geometry

The two most basic geometric concepts are those of an angle and of a straight line. An angle may be considered to be an amount of a rotation or turning. In Figure 4, the line OA has been rotated about O in an anti-clockwise direction, until it takes up the position OB. The angle through which the line has turned is the amount of opening between the lines OA and OB. If the line OA is rotated until it returns to its original position, then it will have described one revolution. Angles are usually measured in degrees, minutes, and seconds as follows: $60\,seconds = 1\,minute$, $60\,minutes = 1\,degree$, and $360\,degrees = 1\,revolution$. For instance, an angle of 32 degrees 18 minutes and 3 seconds is written as follows: $32^o18'3''$. A "right angle" is the $\frac{1}{4}$th of a revolution, and, therefore, it contains 90^o. An "acute angle" is less than 90^o. An "obtuse angle" lies between 90^o and 180^o. A "reflex angle" is greater than 180^o. "Complementary angles" are angles whose sum is 90^o. "Supplementary angles" are angles whose sum is 180^o.

Figure 4: An angle.

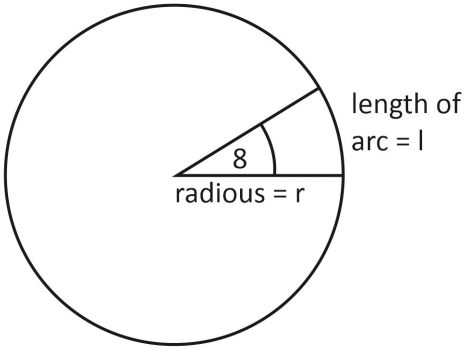


While we usually measure angles in degrees, we can also measure angles in radians. Referring to Figure 5,

$$angle\ in\ radians = \frac{length\ of\ arc}{radius\ of\ circle}$$

so that $\theta\ radians = \frac{l}{r} \Leftrightarrow l = r\theta$.

Figure 5: Measuring angles in radians.

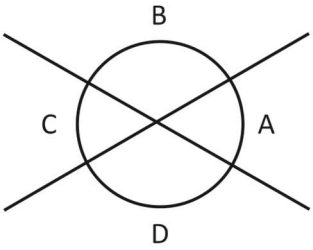


In geometry, the abstraction of a straight line can be attributed to mathematical intuition. According to the ancient Greek mathematician Euclid, an arbitrary straight line can be construed as a "length without breadth" that is perceived as a whole. Furthermore, there are points on every straight line, each point on the straight line corresponds to a real number, and the straight line is complete. For this reason, it is known as the arithmetic or geometric continuum. In fact, the ancient Greek mathematicians' awareness of the existence of real numbers was developed with reference to geometric processes, in the sense that they construed a real number either as a completed process of combining units or monads (that is, as a rational number) or as an incomplete process of measuring non-commensurable quantities (that is, as an irrational number).

Properties of angles and straight lines:

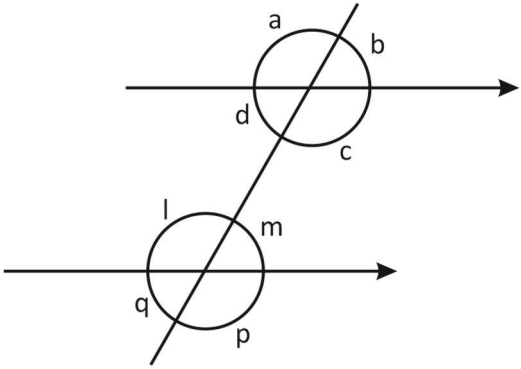
i. The total angle of a straight line is 180^o.
ii. When two straight lines intersect, the opposite angles are equal, as shown in Figure 6, where $\angle A = \angle C$ and $\angle B = \angle D$.

Figure 6: Opposite angles formed by intersecting straight lines.



iii. If two parallel lines are cut by a transversal, then, as shown in Figure 7: the corresponding angles are equal (i.e., $a = l$, $b = m$, $c = p$, and $d = q$); the alternate angles are equal (i.e., $d = m$ and $c = l$); and the interior angles are supplementary (i.e., $d + l = 180^o$ and $c + m = 180^o$). Conversely, if two straight lines are cut by a transversal, the lines are parallel if one of the following is true: two corresponding angles are equal; two alternate angles are equal; two interior angles are supplementary.

Figure 7: Angles formed by two parallel lines cut by a transversal.



Types of triangles on the basis of their angles and their sides:

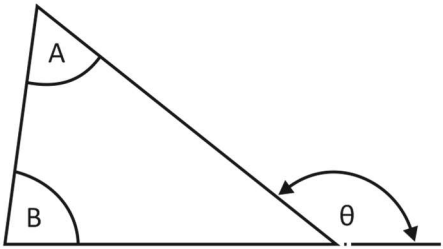
i. An "acute-angled" triangle has all its angles less than 90^o.
ii. A "right-angled" triangle has one of its angles equal to 90^o. The side opposite to the right angle is the longest side, and it is called the "hypotenuse."
iii. An "obtuse-angled" triangle has one angle greater than 90^o.

iv. A "scalene" triangle has all three sides of different length.
v. An "isosceles" triangle has two sides and two angles equal. The equal angles lie opposite to the equal sides.
vi. An "equilateral" triangle has all its sides and angles equal. Each angle of an equilateral triangle is equal to 60^o.

Angle properties of triangles:

i. The sum of the angles of a triangle is equal to 180^o
ii. In every triangle, the greatest angle is opposite to the longest side, and the smallest angle is opposite to the shortest side. Moreover, in every triangle, the sum of the lengths of any two sides is always greater than the length of the third side.
iii. When the side of a triangle is produced, the exterior angle so formed is equal to the sum of the opposite interior angles. For instance, in Figure 8, $\angle\theta = \angle A + \angle B$.

Figure 8: Exterior angle.



iv. In an isosceles triangle, the perpendicular (drawn from the point where the two equal sides meet) to the base bisects the angle between the two equal sides. Moreover, it bisects the base of the triangle.

Two triangles are said to be "congruent" if they are equal in every respect, both with regard to their corresponding angles and to their corresponding sides. If one side and two angles in one triangle are equal to one side and two similarly located angles in another triangle, then these triangles are congruent. Moreover, if two sides and the angle between them in one triangle are, respectively, equal to two sides and the angle between them in another triangle, then these triangles are congruent. Given two right-angled triangles, if their hypotenuses are equal to each other and one other side in each triangle are also equal to each other, then these right-angled triangles are congruent.

Two triangles are said to be "similar" if they are equi-angular. Two triangles are equi-angular if and only if their corresponding sides are

proportional. For instance, a triangle $\triangle ABC$ and a triangle $\triangle XYZ$ are equi-angular if and only if
$\frac{AB}{XY} = \frac{AC}{XZ} = \frac{BC}{YZ}$.
Areas of triangles: The area of any triangle is:
$area = \frac{1}{2} \times base \times height$.
Triangles having equal bases and equal heights are equal in area. Moreover, the areas of congruent triangles are equal.
One of the most important geometric theorems is the Pythagorean Theorem, which states that, in every right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides. As mentioned earlier, the Pythagorean Theorem led Greek mathematicians to prove the existence of irrational numbers. The Pythagorean Theorem can be proved in an algebraic way, using the concept of a locus, as follows.
Pythagorean Theorem: Consider a right-angled triangle $\triangle ABC$, whose hypotenuse is c, and whose other two sides are a and b, as shown in Figure 9. Then
$a^2 + b^2 = c^2$.
Proof: Given the triangle shown in Figure 9, we create four triangles identical to it, and we use them in order to form a square with side lengths $a + b$ as shown in Figure 10. The area of this square is
$A = (a + b)(a + b)$.

Figure 9: A right-angled triangle.

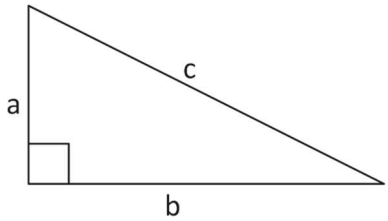
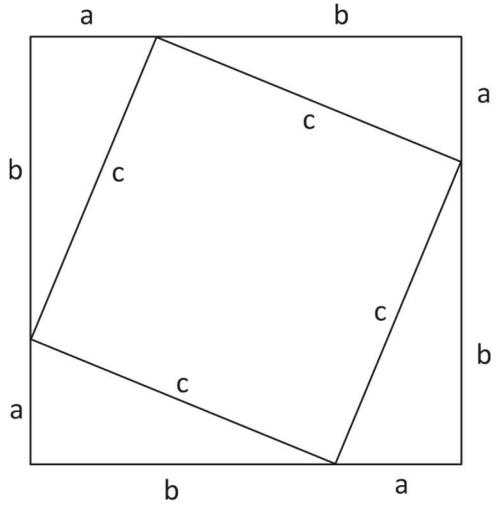


Figure 10: Proof of the Pythagorean Theorem.



In Figure 10, inside the big square, the hypotenuses of the four identical triangles form another smaller square, whose area is equal to c^2. Each of the four triangles has an area of $\frac{ab}{2}$. In general, notice that, given an arbitrary rectangle $ABCD$ whose height is h, and whose base is b, its area is equal to hb. Therefore, if we draw a diagonal from one vertex, say diagonal AC, it will break the rectangle into two congruent, or equal, triangles, and the area of each of these triangles is half the area of the rectangle, that is, $\frac{hb}{2}$. The area of all four of the triangles that are shown in Figure 10 is equal to $4\frac{ab}{2} = 2ab$. Adding up the areas of the smaller square and of the four triangles, we obtain
$A = c^2 + 2ab$.
Hence, given that, as we have shown, $A = (a + b)(a + b)$, it holds that
$(a + b)(a + b) = c^2 + 2ab \Leftrightarrow a^2 + b^2 = c^2$.■

Quadrilaterals and Polygons

A "quadrilateral" is any four-sided figure. Given that a quadrilateral can be split up into two triangles, the sum of its angles is 360^o.
A "parallelogram" has both pairs of opposite sides parallel. If the base of a parallelogram is equal to b and its height is equal to h, then its area is given by the following formula: $A = bh$. Parallelograms having equal bases and equal heights are equal in area. A parallelogram has the

following properties: (i) the sides that are opposite to each other are equal in length; (ii) the angles that are opposite to each other are equal; (iii) the diagonals bisect each other; (iv) the diagonals each bisect the parallelogram.

A "rectangle" is a parallelogram with all its angles equal to 90^o. If the length of a rectangle is equal to l and its width is equal to w, then its area is equal to lw, and its perimeter is equal to $2l + 2w$. A rectangle has all the properties of a parallelogram, but the diagonals are equal in length.

A "rhombus" is a parallelogram with all its sides equal in length. It has all the properties of a parallelogram, but in addition it has the following properties: (i) the diagonals bisect at right angles; (ii) the diagonal bisects the angle through which it passes.

A "square" is a rectangle with all its sides equal in length. If the length of each side of a square is equal to a, then its area is equal to a^2, and its perimeter is equal to $4a$. A square has all the properties of a parallelogram, a rectangle, and a rhombus.

A "trapezoid" is a quadrilateral having only one pair of parallel sides (as opposed to a parallelogram, which has both pairs of opposite sides parallel). The parallel sides are called the "bases" of the trapezoid, while the other two sides are called the "legs" of the trapezoid. If the bases (parallel sides) of a trapezoid are equal to a and b, respectively, and if its height is equal to h, then its area is equal to $\frac{1}{2}h(a + b)$.

Analytic Geometry and Trigonometric Functions

Analytic geometry signifies the introduction of coordinates into geometry in a systematic way—specifically, by unifying aspects of algebra and aspects of geometry. The development of analytic geometry through the algebraization of geometry set the stage for the development of infinitesimal calculus. The first pioneers of analytic geometry were the second-century B.C.E. Greek astronomer and mathematician Hipparchus of Nicaea, who introduced coordinates for the sphere (in the context of his studies of the night sky), and the third-century B.C.E. Greek geometer Apollonius of Perga, who introduced coordinates for the study of conic sections.

Ancient Greek mathematicians, such as Apollonius of Perga, were the first to observe that circles, ellipses, hyperbolas, and parabolas result from the intersection of a cone by an adequate plane. A cone is defined to be a three-dimensional geometric shape that tapers smoothly from a flat circular base to a point called the vertex (or apex). A circle is produced when the cone is cut by a plane that is parallel to the base of the cone. An ellipse is produced when the cone is cut by a plane that is not parallel to

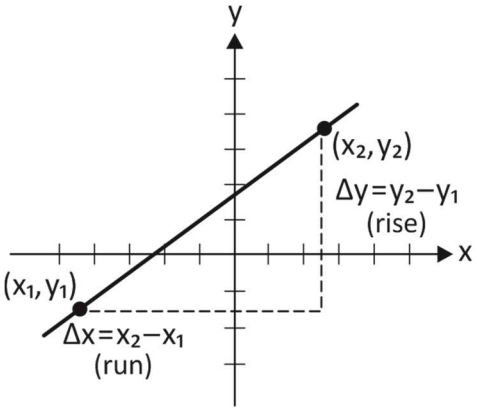
the base of the cone or the side of the cone, and it cuts only one nappe of the cone. A hyperbola is produced when the intersecting plane cuts both nappes of the cone. A parabola is produced when the oblique section of the cone is parallel to the slant height (the height of a cone from the vertex to the periphery, rather than the center, of the base). In the Middle Ages, the use of coordinates in mathematics and analytic geometry was further analyzed and developed by the fourteenth-century French philosopher and mathematician Nicolas d'Oresme.

By the term "locus," we mean a set of all the points that satisfy a specific rule. Moreover, the path drawn by a point moving according to a given rule is called the "locus of the point." Thus, using the concept of a locus, we can study geometric problems through algebra. In analytic geometry, we put traditional (Euclidean) geometry on the Cartesian plane. René Descartes has pointed out that "any problem in geometry can easily be reduced to such terms that knowledge of lengths of certain straight lines is sufficient for its construction" (René Descartes, "On Analytic Geometry," translated by David E. Smith and Marcia L. Latham, in *A Source Book in Mathematics*, edited by David E. Smith, New York: Dover, 1959, p. 397). In particular, according to Descartes, "just as arithmetic consists of only four or five operations, namely, addition, subtraction, multiplication, division, and the extraction of roots, which may be considered a kind of division, so in geometry," we can find required lines by merely adding or subtracting other lines; or else, by working as follows (ibid, pp. 397–98):

> . . . taking one line which I shall call unity in order to relate it as closely as possible to numbers, and which can in general be chosen arbitrarily, and having given two other lines, to find a fourth line which shall be to one of the given lines as the other is to unity (which is the same as multiplication); or, again, to find a fourth line which is to one of the given lines as unity is to the other (which is equivalent to division); or, finally, to find one, two, or several mean proportionals between unity and some other line (which is the same as extracting the square root, cube root, etc., of the given line).

Consider two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ on the xy-plane and connect them with a straight line segment as shown in Figure 11.

Figure 11: Slope and Distance.



The x-coordinate of point P is x_1, the x-coordinate of point Q is x_2, and the distance between x_1 and x_2 is $x_2 - x_1$; in order to avoid the use of plus and minus signs, we can use the absolute value $|x_2 - x_1|$. The y-coordinate of point P is y_1, the y-coordinate of point Q is y_2, and the distance between y_2 and y_1 is $y_2 - y_1$; in order to avoid the use of plus and minus signs, we can use the absolute value $|y_2 - y_1|$. Therefore, the horizontal distance between points P and Q is $x_2 - x_1$, and the vertical distance between points P and Q is $y_2 - y_1$. Now, consider the right-angled triangle that is defined by the points $P(x_1, y_1)$, $Q(x_2, y_2)$, and the point R (the intersection between the horizontal side and the vertical side): the three sides of this right-angled triangle are the hypotenuse PQ, the horizontal side, which is $x_2 - x_1$, and the vertical side, which is $y_2 - y_1$. The "slope," or "gradient," of the straight line segment PQ, denoted by m_{PQ}, is the quotient of the "rise" over the "run," comparing how much one travels vertically ("up and down") versus how much one travels horizontally. Thus, it relates the steepness or inclination of the straight line segment PQ to the coordinates; symbolically:

$$slope = m_{PQ} = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

(see Figure 11; the Greek letter Δ is used to indicate change).

In Figure 11, the distance between points P and Q, denoted by d_{PQ}, is given by (and, indeed, is a version of) the Pythagorean Theorem. Therefore, in Figure 11,

$$\left(d_{PQ}\right)^2 = (run)^2 + (rise)^2$$

$\Leftrightarrow d_{PQ} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

It can be easily verified that the midpoint of the straight line segment joining points (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

All points (x, y) in $\mathbb{R}^2$ satisfying the equation $y = mx + b$ form a straight line, and m is the slope of the straight line. For the slope m of the straight line passing through the points (x_1, y_1) and (x_2, y_2), we have:

i. If $x_1 = x_2$, m is undefined (the line is vertical).
ii. If $x_1 \neq x_2$, then $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$.

Two non-vertical straight lines y_1 and y_2, with slopes m_1 and m_2 respectively, are parallel if and only if $m_1 = m_2$ (i.e., their slopes are equal), and they are perpendicular if and only if $m_1 m_2 = -1$ (i.e., the product of their slopes is -1).

In order to find the equation of a non-vertical straight line, we work as follows:

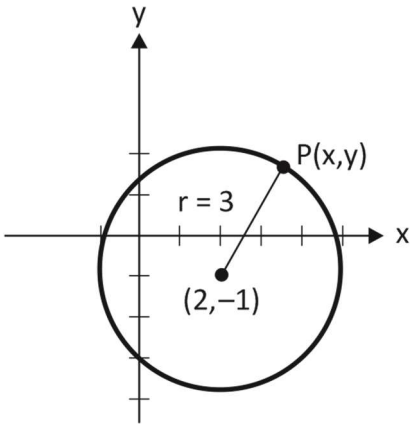
i. we find a point (x_1, y_1) on the line;
ii. we find the slope m of the line;
iii. we write the equation of the line as follows:
$y - y_1 = m(x - x_1)$; this equation is called the "point-slope" form of the equation of a line.

For instance, let us find the equation of the straight line passing through the points $(5, -0.5)$ and $(10, 9.5)$. Firstly, we define the point $(x_1, y_1) = (5, -0.5)$. Secondly, we find the slope of the required line: $m = \frac{9.5-(-0.5)}{10-5} = 2$. Thirdly, we find the equation of the required line: $y - y_1 = m(x - x_1) \Rightarrow y - (-0.5) = 2(x - 5) \Rightarrow y = 2x - 10.5$.

Circle

As we can see in Figure 12, a circle with center $O(v, w)$ and radius r is the set of all points in the xy-plane whose distance from O is r (in Figure 12, $O(v, w) = O(2, -1)$, and $r = 3$).

Figure 12: Circle.



If (x, y) is a point on the circle with center $O(v, w)$ and radius r, then the distance formula implies that

$r = \sqrt{(x-v)^2 + (y-w)^2} \Leftrightarrow r^2 = (x-v)^2 + (y-w)^2$,

which is the standard form of the equation of a circle with center (v, w) and radius r. The circumference of a circle of radius r is $C = 2\pi r$, and the area of a circle of radius r is $A = \pi r^2$, where $\pi \approx 3.14$ is Archimedes's constant (the ratio of the circle's circumference to its diameter). Archimedes approximated π by using the fact that the circumference of a circle is bounded by the perimeter of an inscribed polygon and the perimeter of a circumscribed polygon. In particular, he used a 96-sided inscribed polygon and a 96-sided circumscribed polygon to find the following approximation:

$3 + \frac{10}{71} < \pi < 3 + \frac{10}{70}$.

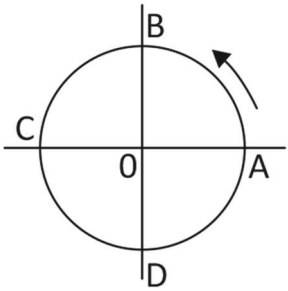
It is worth mentioning that the degenerate possibilities for a circle are the following: a point or no graph at all.

The study of the circle underpins trigonometry. The term "trigonometry" appeared for the first time in the book *Trigonometria* by Bartholomaeus Pitiscus (1561–1613) in 1595, and it literally means measuring (and, more broadly, studying) "trigons" ("trigon" being the Latin word for "triangle"). The acknowledged founder of trigonometry is the ancient Greek astronomer and mathematician Hipparchus of Nicaea (ca. 190–ca. 120 B.C.E.). Moreover, around 100 C.E., another Greek mathematician, Menelaus of Alexandria, published a series of treatises on chords.

Trigonometric Functions

In the context of analytic geometry, we can also study the basic trigonometric functions on the unit circle (specifically, on a circle whose center is $(0,0)$ and whose radius $r = 1$).

Figure 13: The Number Circle.



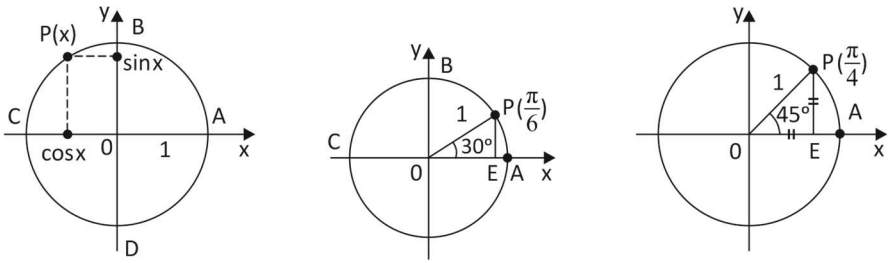
Consider a circle of unit radius, as shown in Figure 13, and let point A (the right-hand endpoint of the horizontal diameter) be a reference point. Let an anti-clockwise motion round the circle be a positive direction, and a clockwise motion be a negative direction. A circle of unit radius with a reference point and the direction of tracing specified is called the "number circle." Given an arbitrary point P of the number circle, there are infinitely many arcs beginning at the point A and terminating at the point P. One of these arcs is the shortest arc connecting the points A and P, and all the other arcs are obtained from the shortest arc by adding or subtracting an integral number of complete revolutions. Hence, every point P of the number circle is associated with an infinite set of numbers that consists of the values of all the arcs beginning at the point A and terminating at the point P (the lengths of the arcs are taken with the plus or minus sign according as the motion from the point A to the point P is anti-clockwise or clockwise, respectively).

The circumference of the circle of unit radius is equal to 2π. Therefore, the lengths of all the arcs terminating at the given point P differ from one another by an integral number multiple of 2π, so that the general form of these quantities is $x + 2\pi a$, where $a \in \mathbb{Z}$, and x is the length of the shortest arc connecting the points A and P. Thus, for every real number x, there is a point $P(x)$ of the number circle such that the length of the arc AP is x, and every point P of the circle corresponds to an infinite set of

numbers of the form $x + 2\pi a$, where $a \in \mathbb{Z}$, and x is the length of one of the arcs connecting the points A and P.

Assume that the center of the number circle coincides with the origin $O(0,0)$ of the rectangular coordinate system XOY, as shown in Figure 14. Let x be an arbitrary real number. Then, on the number circle, we find the point $P(x)$ that corresponds to x. The ordinate of the point $P(x)$ is called the "sine" of the number x (denoted by $sinx$), the abscissa of the point $P(x)$ is called the "cosine" of the number x (denoted by $cosx$), the ratio $\frac{sinx}{cosx}$ is called the "tangent" of the number x (denoted by $tanx$), and the ratio $\frac{cosx}{sinx}$ is called the "cotangent" of the number x (denoted by $cotx$).

Figure 14: Trigonometric Functions.



Notice that the reference point A on the number circle corresponds to the number 0, that is, $A = A(0)$. Since the abscissa and the ordinate of this point are 1 and 0, respectively, we have $cos0 = 1$, $sin0 = 0$, and $tan0 = \frac{sin0}{cos0} = 0$. The point B of intersection of the circle and the positive ray of the axis OY corresponds to the number $\pi/2$. Since the abscissa and the ordinate of the point B are 0 and 1 respectively, we have $\cos(\frac{\pi}{2}) = 0$ and $\sin(\frac{\pi}{2}) = 1$, whereas $\tan(\frac{\pi}{2})$ is not defined. Similarly, as shown in Figure 14, given the coordinates of the points C and D, we realize that $cos\pi = -1$, $sin\pi = 0$, $tan\pi = 0$, $\cos(\frac{3\pi}{2}) = 0$, $sin(\frac{3\pi}{2}) = -1$, and $tan(\frac{3\pi}{2})$ is not defined. The parametrization of the unit circle can be written as follows:

$$(cos\theta, sin\theta)$$

where $0 \leq \theta \leq 2\pi$.

We can summarize the basic definitions and the basic formulas of trigonometry as follows:

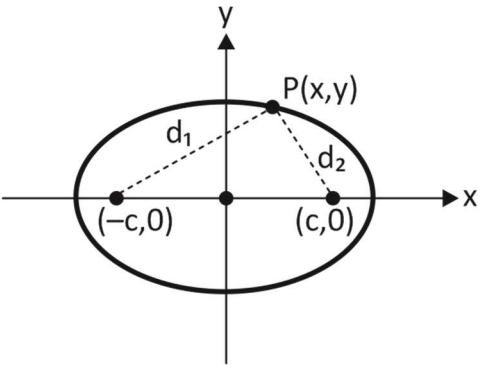
$Sine$: $\sin\theta = \frac{opposite\ side}{hypotenuse}$,

$Cosine: cos\theta = \frac{adjacent\ side}{hypotenuse}$,
$Tangent: tan\theta = \frac{opposite\ side}{adjacent\ side}$,
$Cosecant: csc\theta = \frac{hypotenus}{opposite\ side} = \frac{1}{sin\theta}$,
$Secant: sec\theta = \frac{hypotenuse}{adjacent\ side} = \frac{1}{cos\theta}$,
$Cotangent: cot\theta = \frac{adjacent\ side}{opposite\ side} = \frac{1}{tan\theta}$,
and the basic trigonometric identities:
$sin^2a + cos^2a = 1$,
$sin(-a) = -sina$,
$cos(-a) = cosa$,
$\sin(a \pm b) = sina \cdot cosb \pm cosa \cdot sinb$,
$\cos(a \pm b) = cosa \cdot cosb \mp sina \cdot sinb$,
$sina + sinb = 2sin\frac{1}{2}(a+b) \cdot cos\frac{1}{2}(a-b)$,
$cosa + cosb = 2cos\frac{1}{2}(a+b) \cdot cos\frac{1}{2}(a-b)$,
$sin2a = 2sina \cdot cosa$,
$cos2a = cos^2a - sin^2a$,
$sin\frac{1}{2}a = \sqrt{\frac{1-cos}{2}}$,
$cos\frac{1}{2}a = \sqrt{\frac{1+cos}{2}}$,
$\sin(a \pm \pi/2) = \pm cosa$,
$\cos(a \pm \pi/2) = \mp sina$.
The inverse trigonometric functions are denoted as follows: $arcsinx \equiv sin^{-1}x$ $(\ y = arcsinx \Leftrightarrow x = siny)$, $arccosx \equiv cos^{-1}x$ $(\ y = arccosx \Leftrightarrow x = cosy)$, and $arctanx \equiv tan^{-1}x$ $(\ y = arctanx \Leftrightarrow x = tany)$.

Ellipse

As we can see in Figure 15, an "ellipse" is the set of all points in a plane the sum of whose distances from two fixed points ("foci") is constant. Foci: $(-c, 0)$ and $(c, 0)$. Notice that, if the two foci coincide, then we receive a circle. The Greek word ellipse, literally meaning "omission," was first applied by Apollonius of Perga, because, in the case of an ellipse, the conic section of the cutting plane makes a smaller angle with the base than does the side of the cone.

Figure 15: Ellipse.



The standard form of the equation of an ellipse with center at the origin and foci on the x-axis is

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

By setting $y=0$, we find that the x-intercepts are $(-a,0)$ and $(a,0)$. By the setting $x=0$, we find that the y-intercepts are $(0,-b)$ and $(0,b)$. The larger segment from $(-a,0)$ to $(a,0)$ is called the "major axis," while the "minor axis" is the segment from $(0,-b)$ to $(0,b)$. The endpoints of the major axis are called the "vertices of the ellipse"; vertices: $(-a,0)$ and $(a,0)$.

If the foci are placed on the y-axis at $(0,-c)$ and $(0,c)$, then the standard form of the equation of an ellipse is

$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$.

In this case, the major axis is along the y-axis, the foci are $(0,c)$ and $(0,-c)$, and the vertices are $(0,a)$ and $(0,-a)$.

Given the definition of an ellipse, the degenerate possibilities for an ellipse are the following: a point or no graph at all.

In our solar system, many bodies revolve in elliptical orbits around a larger body that is located at one focus. In the seventeenth century, Johannes Kepler, based on Apollonius's mathematical study of the ellipse, articulated a rigorous explanation of planetary motions.

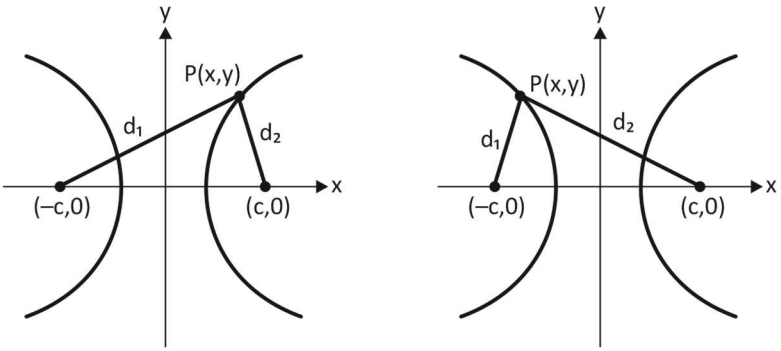
Moreover, regarding the ellipse, it should be mentioned that it has a reflection property that causes any ray or wave that originates at one focus to strike the ellipse and pass through the other focus. In terms of acoustics, the aforementioned property implies that, in a room with an elliptical

ceiling, even a slight noise made at one focus can be heard at the other focus, but, if people are standing between the foci, then they hear nothing. Such rooms are known as whispering galleries.
As regards architecture, it should be mentioned that ornamental arches are often elliptical in shape; in other words, arches whose main purpose is beauty and not strength are often elliptical in shape.

Hyperbola

As we can see in Figure 16, a "hyperbola" is the set of all points in a plane the difference of whose distances from two fixed points ("foci") is a positive constant (the Greek word hyperbola literally means "extravagance"). Hence, the distances between the foci and a point on the figure maintain a *constant difference* for a hyperbola and a *constant sum* for an ellipse.

Figure 16: Hyperbola.



Given the definition of a hyperbola, the degenerate possibilities for a hyperbola are two intersecting straight lines.
The standard form of a hyperbola with center at the origin and foci on the x-axis is
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$
By setting $y = 0$, we find that the x-intercepts are $(-a, 0)$ and $(a, 0)$. The line segment joining these two points is called the "transverse axis." The endpoints of the transverse axis are called the "vertices of the hyperbola." By setting $x = 0$, we find that there are no y-intercepts. The line segment from $(0, b)$ to $(0, -b)$ is called the "conjugate axis." In order to determine the significance of b, we write

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ as $y=\frac{\pm bx}{a}\sqrt{1-\frac{a^2}{x^2}}$.

As $|x|$ tends to infinity, $1-\frac{a^2}{x^2}$ tends to 1, and, therefore, the graph of the hyperbola approaches the lines

$y=\pm\frac{b}{a}x$.

These lines are called the "asymptotes of the hyperbola" (they are the diagonals of a rectangle of dimensions $2a$ by $2b$).

If the foci are placed on the y-axis at $(0,-c)$ and $(0,c)$, then the standard form of the equation of a hyperbola is

$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$,

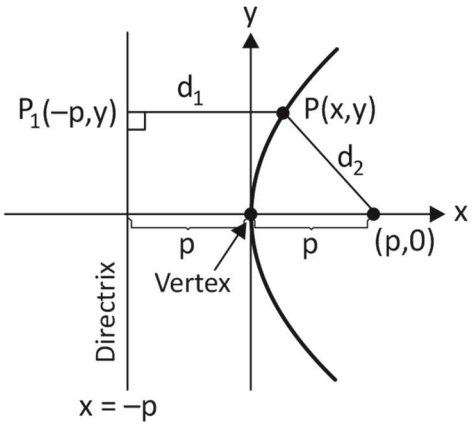
and, in this case, the asymptotes are given by

$y=\pm\frac{a}{b}x$.

Parabola

As we can see in Figure 17, a "parabola" is the set of all points in a plane that are equidistant from a fixed line ("directrix") and a fixed point ("focus") not on the line (the word "parabola" derives from the Greek terms "parā," meaning "beside," and "bolē," meaning "a throw," and, therefore, "parabola" literally means "para-beside"—that is, placing side by side).

Figure 17: Parabola.



The standard form of the equation of a parabola with directrix $x=-p$ and focus at $(p,0)$ is

$4px = y^2$.
The line through the focus that is perpendicular to the directrix is called the "axis of symmetry." In this case, the axis of symmetry is the x-axis, and the parabola opens to the right. The point on the axis of symmetry that is midway between the focus and the directrix is called the "vertex," and the vertex is the turning point of the parabola. The standard form of the equation of a parabola with directrix $x = p$ and focus at $(-p, 0)$ is
$-4px = y^2$,
and, in this case, the parabola opens to the left.
Obviously, the axis of symmetry of a parabola may be the y-axis. If the directrix is $y = -p$ and the focus is at $(0, p)$, then the standard form of the equation of a parabola is
$x^2 = 4py$,
and the parabola opens upward. If the directrix is $y = p$ and the focus is at $(0, -p)$, then the standard form of the equation of a parabola is
$x^2 = -4py$,
and the parabola opens downward.
As regards the parabola in general, it should be mentioned that it has a reflection property that causes any ray or wave that originates at the focus and strikes the parabola to be reflected parallel to the axis of symmetry. Thus, for instance, flashlights and searchlights use a parabolic reflector with the bulb located at the focus. Additionally, due to the reflection property of a parabola, any ray or wave that comes into a parabolic reflector parallel to the axis of symmetry is directed to the focus point. For this reason, radars, radio antennas, and reflecting telescopes operate according to this principle. In astronomy, the parabola features in both the construction of telescopes and in the motion of comets around the Sun. Finally, due to their great strength, parabolic arches are used extensively in bridges, cathedrals, and elsewhere in architecture and engineering, especially in case we have equally spaced load.

Volumes and Surface Areas

By the term "volume," we mean the amount of three-dimensional space enclosed by a closed surface. The volume of any solid having a uniform cross-section is equal to:
cross-sectional area×length of solid.
The surface area of any solid having a uniform cross-section is equal to:
curved surface+ends; namely:
perimeter of cross-sections×length of solid+total area of ends.
The volume of a sphere with radius r is equal to
$\frac{4}{3}\pi r^3$,

and its surface area is equal to
$4\pi r^2$.
The volume of a cylinder whose height is h and whose base is a circle with radius r is equal to
$\pi r^2 h$,
and its surface area is equal to
$2\pi rh + 2\pi r^2 = 2\pi r(h + r)$.
The volume of a cone whose vertical height is h and whose base is a circle with radius r is equal to
$\frac{1}{3}\pi r^2 h$,
and, if l is its slant height, then its surface area is equal to
$\pi rl + \pi r^2$.
The volume of a pyramid whose height is h and whose base's area is equal to A is given by the following formula:
$V = \frac{1}{3}Ah$.
The surface area of a pyramid is equal to the sum of the areas of the corresponding triangles plus the area of the base.

Chapter 7
Vectors, Vector Spaces, and Normed Vector Spaces

The discipline of mathematics that deals with matrices (covered in Chapter 3) and vectors (and, more generally, with vector spaces and linear transformations) is called Linear Algebra.

A "scalar" is a quantity that can be specified by determining only its magnitude. However, the quantities that are specified by determining both magnitude and direction are called "vectors." In other words, a "vector" is a quantity that has both a direction and a magnitude of length; therefore, it is graphically denoted by an oriented line segment ("arrow"). In physics, vectors are very useful, because they can visually represent position, displacement, velocity, and acceleration. Moreover, vector graphics are used in computers, since they can be scaled to a larger size without losing any image quality.

If the coordinates of a point P in the coordinate plane are (x, y), and if we denote the origin of the coordinate system by $O(0,0)$, then a vector OP is denoted by $\overrightarrow{OP}$, since the length OP represents the magnitude, and the arrow represents the direction. The column vector (matrix) corresponding to $\overrightarrow{0P}$ is

$\begin{pmatrix} x \\ y \end{pmatrix}$.

Since the coordinates of point P are (x, y), the length from $O(0,0)$ to P is $\sqrt{x^2 + y^2}$, according to the Pythagorean Theorem. Notice that, frequently, we do not need to use arrows in order to indicate that letters represent vectors (in particular where there is no likelihood of confusion).

The operations between vectors are based on matrix algebra. For instance, given two vectors $\overrightarrow{OA} = \begin{pmatrix} p \\ q \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} r \\ s \end{pmatrix}$,

their sum is a vector $\overrightarrow{OC}$ such that

$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB} = \begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} p + r \\ q + s \end{pmatrix}$.

In general, we can define the following vector operations:

Vector addition: $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n)$;

Scalar multiplication: $k\vec{u} = (ku_1, ku_2, \dots, ku_n)$;

Negation: $-\vec{u} = (-1)\vec{u} = (-u_1, -u_2, \dots, -u_n)$;

Dot Product (or Scalar Product or Inner Product):

$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = \sum_{i=1}^{n} u_i v_i$;

Norm (Length): $\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$

(specifically, the norm of a vector is the distance of the vector from the origin);

where: $\vec{u} = (u_1, u_2, \dots, u_n)$ and $\vec{v} = (v_1, v_2, \dots, v_n)$ are vectors in $\mathbb{R}^n$, and k is a real number (scalar).

Cross Product of two vectors in a 3-dimensional space: Consider two vectors $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$, and let $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$ be the unit vectors of the three coordinate axes respectively. Then the cross product of $\vec{u}$ and $\vec{v}$ is a vector given by the following determinant:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \vec{\imath} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \vec{\jmath} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \vec{k} =$$

$$(u_2 v_3 - u_3 v_2)\vec{\imath} - (u_1 v_3 - u_3 v_1)\vec{\jmath} + (u_1 v_2 - u_2 v_1)\vec{k}.$$

The geometric significance of this operation is that, if θ is the angle between $\vec{u}$ and $\vec{v}$ with $0 \leq \theta \leq \pi$, then

$$\vec{u} \times \vec{v} = \|\vec{u}\|\|\vec{v}\|(sin\theta)\vec{n},$$

where $\vec{n}$ is a unit vector perpendicular to the plane containing $\vec{u}$ and $\vec{v}$ (with your right hand, point your index finger along vector $\vec{u}$, and point your middle finger along vector $\vec{v}$; then $\vec{n}$ goes in the direction of your extended thumb). The magnitude of the cross product ($|\vec{u} \times \vec{v}|$) can be interpreted as the positive area of the parallelogram having $\vec{u}$ and $\vec{v}$ as its sides. Whilst the resultant of the dot product of two vectors $\vec{u}$ and $\vec{v}$ is a scalar quantity, the cross product of two vectors $\vec{u}$ and $\vec{v}$ is a third vector whose direction is perpendicular to both $\vec{u}$ and $\vec{v}$ (the direction is given by the aforementioned right-hand rule). Two vectors are parallel to each other if and only if they are scalar multiples of each other.

The most abstract definition of a vector is that a vector is an element of a "vector (or linear) space," which, in turn, can be defined as follows: let U be a set endowed with two operations: addition and scalar multiplication, defined in the following way:

$+: U \times U \to U$ defined by $(u, v) \in U \times U \to u + v \in U$ for all $u, v \in U$, that is, U is "closed under addition";

$\cdot: k \times U \to U$ defined by $(k, u) \in K \times U \to k \cdot u \in U$ for every $k \in K$ (where K is a field, such as $\mathbb{R}$) and for every $u \in U$, that is, U is "closed under scalar multiplication." Of course, $0 \in U$, since, for every $u \in U$, $(-1)u \in U$, and, therefore, $u - u \in U \Rightarrow 0 \in U$. As a result of the aforementioned definition, we say that U under the operations of $+$ (addition) and $\cdot$ (scalar multiplication) forms a "vector space" (or "linear space") over the field K; and, therefore, a "vector" can be defined as an element of such a U.

For instance, we can prove that, if

$V = \{ax^2 + bx + c | a, b, c \in \mathbb{R}\}$,

then V is a vector space over $\mathbb{R}$ as follows:

Step 1: $0 = 0x^2 + 0x + 0 \in V$.
In other words, $0 \in V$.
Step 2: Let
$\begin{cases} v_1 = a_1x^2 + b_1x + c_1 \\ v_2 = a_2x^2 + b_2x + c_2 \end{cases}$.
Then $v_1 + v_2 = (a_1 + a_2)x^2 + (b_1 + b_2)x + (c_1 + c_2) \in V$.
In other words, V is closed under addition.
Step 3: Let $v = ax^2 + bx + c$ with $a, b, c \in \mathbb{R}$.
Then $kv = (ka)x^2 + (kb)x + (kc) \in V$.
In other words, V is closed under scalar multiplication.
Therefore, $V = \{ax^2 + bx + c | a, b, c \in \mathbb{R}\}$ is a vector space over $\mathbb{R}$. In other words, the set of all real quadratic polynomials forms a vector space over $\mathbb{R}$.
On the other hand, we can prove that a sphere S is not a vector space as follows: let v be a vector belonging to the sphere S. If we multiply v by an adequate number k, then kv does not belong to S any more (it "pierces" the sphere). Hence, a sphere is not a vector space (it is not closed under scalar multiplication). This example helps us to understand why no bounded set, in general, is a vector space.
Linearly Independent Vectors: Let V be a vector space over K. The vectors $v_1, v_2, \dots, v_n$ of V are "linearly independent" if and only if every time $k_1v_1 + k_2v_2 + \cdots + k_nv_n = 0 \Rightarrow k_1 = k_2 = \cdots = k_n = 0$.
For instance, the vectors $v_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $v_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are linearly independent, since

$$k_1v_1 + k_2v_2 + \cdots + k_nv_n = 0$$
$$\Rightarrow \begin{pmatrix} k_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & k_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ k_3 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & k_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow k_1 = k_2 = k_3 = k_4 = 0.$$

Linearly Dependent Vectors: Let V be a vector space over K. The vectors $v_1, v_2, \dots, v_n$ of V are "linearly dependent" if and only if $k_1v_1 + k_2v_2 + \cdots + k_nv_n = 0$ for some $k_i \neq 0$, where $i = 1,2, \dots, n$.
For instance, the vectors $v_1 = (0,1)$, $v_2 = (1,0)$, and $v_3 = (1,1)$ are linearly dependent.
Basis: Let V be a vector space over K. The vectors $v_1, v_2, \dots, v_n$ form a "basis" of V if and only if these vectors are linearly independent and generate (or span) V; that is, every vector of V must be expressed in terms of $v_1, v_2, \dots, v_n$. For instance, if $V = \{a + bx + cx^2 | a, b, c \in \mathbb{R}\}$, then $v_1 = 1$, $v_2 = x$, and $v_3 = x^2$ form a basis of V, because: (i) v_1, v_2, and v_3

are linearly independent, since no vector from $\{1, x, x^2\}$ can be written in terms of the other vectors; and (ii) $\{1, x, x^2\}$ generate V, since, for any $v \in V$, it holds that $v = k + lx + mx^2 = k \cdot 1 + lx + mx^2$. Every (non-zero) vector space over a field K has at least one basis (actually, it has many different bases). However, every vector space V has an invariant property: the number of vectors in every basis of V remains the same, and the "dimension" of a vector space V is the number of elements of any of its bases.

When we study vector spaces, and abstract mathematical spaces in general, we must keep in mind that the term "space" signifies a collection of vectors that interact in a certain way, which is determined by the corresponding structure (e.g., by a set of operations, by a norm, etc.). We can define a norm in an abstract way as follows: given a vector (or linear) space X over $\mathbb{R}$, a "norm" $\|\cdot\|$ for X is a function on X that assigns to each element a real number (symbolically: $\|\cdot\|: X \to \mathbb{R}$) satisfying the following properties:

for every $x \in X$:

i. $\|x\| \geq 0$,
ii. $\|x\| = 0$ if and only if $x = 0$,
iii. $\|kx\| = |k|\|x\|$ for any scalar k, and,
for every $x, y \in X$,
iv. $\|x + y\| \leq \|x\| + \|y\|$ (the triangle inequality).

A vector (or linear) space that is equipped with a norm $\|\cdot\|$ is denoted by $(X, \|\cdot\|)$ and is called a "normed vector space" (or "normed linear space"). Different norms can be defined on the same vector space, thus giving rise to different normed vector spaces.

Example 1: $(\mathbb{R}, |\cdot|)$. The set of real numbers $(\mathbb{R})$ is a normed vector space with norm given by the absolute value (or modulus), that is,

$\|x\| = |x|$,

and we call this the "usual norm" for $\mathbb{R}$.

Example 2: $(\mathbb{R}^n, \|\cdot\|_2)$. The set of ordered n-tuples of real numbers $(\mathbb{R}^n)$ is a normed vector space with norm $\|\cdot\|_2$ defined as follows:

for any real vector $x = (k_1, k_2, \dots, k_n)$,

$\|x\|_2 = \sqrt{|k_1|^2 + |k_2|^2 + \cdots + |k_n|^n}$,

and we call this the "Euclidean norm."

Example 3: $(\mathbb{R}^n, \|\cdot\|_1)$. The set of ordered n-tuples of real numbers $(\mathbb{R}^n)$ is a normed vector space with norm $\|\cdot\|_1$ defined as follows:

for any real vector $x = (k_1, k_2, \dots, k_n)$,

$\|x\|_1 = |k_1| + |k_2| + \cdots + |k_n|$.

Example 4: $(\mathbb{R}^n, \|\cdot\|_\infty)$. The set of ordered n-tuples of real numbers $(\mathbb{R}^n)$ is a normed vector space with norm $\|\cdot\|_\infty$ defined as follows:

for any real vector $x = (k_1, k_2, \dots, k_n)$,
$\|x\|_\infty = max\{|k_i|, where\ i = 1,2, \dots, n\}$,
and we call this the "supremum (or uniform) norm" for $\mathbb{R}^n$.
Example 5: $(\mathcal{B}(X), \|\cdot\|_\infty)$. For any non-empty set X, we denote by $\mathcal{B}(X)$ the set of bounded real functions on X. Notice that a function f on some set X with real values is said to be "bounded" if the set of its values is bounded—that is, if there exists a real number M such that, for every $x \in X$, it holds that $|f(x)| \leq M$.
$\mathcal{B}(X)$ is a real vector space under the pointwise definitions of addition and scalar multiplication. Moreover, $\mathcal{B}(X)$ is a normed vector space with norm $\|\cdot\|_\infty$ defined by
$\|f\|_\infty = sup\{|f(x)|, where\ x \in X\}$,
and we call this the "supremum (or uniform) norm" for $\mathcal{B}(X)$. Notice that Example 4 is the special case when $X = \{1,2, \dots, n\}$.
Example 6: l_2-space, also known as the "Hilbert (sequence) space." This is a generalization of the Euclidean n-space. The set l_2 whose elements are sequences of scalars (real numbers) $x = \{k_1, k_2, \dots, k_n, \dots\}$ such that $\sum|k_n|^2$ is convergent, is a real vector space under the pointwise definitions of addition and scalar multiplication, and it is a normed vector space with norm $\|\cdot\|_2$ defined by
$\|x\|_2 = \sqrt{\sum_{i=1}^{\infty}|k_i|^2}$.
In an arbitrary normed vector space $(X, \|\cdot\|)$, the set

$$S(0;1) = \{x \in X\ such\ that\ \|x\| = 1\}$$

is called the "unit sphere"; the set

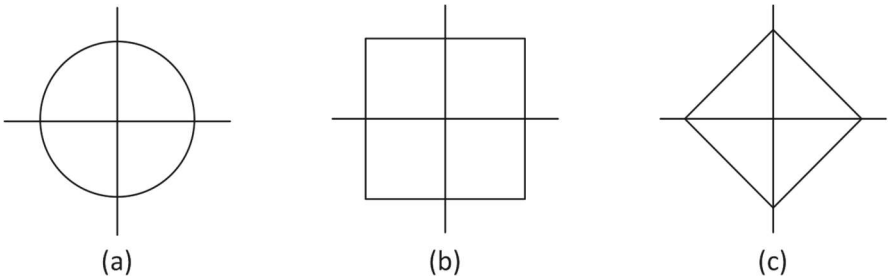
$$B[0;1] = \{x \in X\ such\ that\ \|x\| \leq 1\}$$

is called the "closed unit ball"; and the set

$$B(0;1) = \{x \in X\ such\ that\ \|x\| < 1\}$$

is called the "open unit ball." In Figure 18, we consider the shape of the unit sphere in several coordinate space examples: (a) in $(\mathbb{R}^2, \|\cdot\|_2)$, where $S\big((0,0);1\big) = \{(k,l)\ such\ that\ k^2 + l^2 = 1\}$; (b) in $(\mathbb{R}^2, \|\cdot\|_\infty)$, where $S\big((0,0);1\big) = \{(k,l)\ such\ that\ max\{|k|,|l|\} = 1\}$; and (c) in $(\mathbb{R}^2, \|\cdot\|_1)$, where $S\big((0,0);1\big) = \{(k,l)\ such\ that\ |k| + |l| = 1\}$.

Figure 18: The shape of the unit sphere in several coordinate space examples.



Linear Transformations: Linear transformations are transformations (functions) that preserve the operations of vector addition and scalar multiplication. Thus, a transformation T is linear if:

i. $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$

ii. $T(c\vec{u}) = cT(\vec{u})$, where c is a scalar quantity.

Remark: If T is a linear transformation, then $T(\vec{0}) = \vec{0}$.

Example 1: If A is any $m \times n$ matrix, then the mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ which is matrix-vector multiplication

$$T(\vec{x}) = A\vec{x}$$

is a linear transformation.

Example 2: Projection is a linear transformation. In particular, in $\mathbb{R}^2$, a projection is a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$, which takes every vector in the plane into a vector in the plane. The "vector projection" of $\vec{v}$ onto $\vec{u}$ is denoted by $proj_{\vec{u}}\vec{v}$, and it is defined as follows:

$$proj_{\vec{u}}\vec{v} = \left(\frac{\vec{v}\cdot\vec{u}}{\|\vec{u}\|^2}\right)\vec{u}$$

where the operator $\cdot$ denotes the dot product, and $\|\vec{u}\|$ is the length of $\vec{u}$. This formula indicates that the new vector is going in the direction of $\vec{u}$ (notice that the vector projection is the vector produced when one vector is resolved into two component vectors, one that is parallel to the second vector and one that is perpendicular to the second vector). The "scalar projection" of of $\vec{v}$ onto $\vec{u}$ is equal to

$$v_1 = \|\vec{v}\,\|cos\theta$$

where θ is the angle between $\vec{v}$ and $\vec{u}$ (notice that the scalar projection is the length of the vector projection).

Example 3: Rotation is a linear transformation. In particular, in $\mathbb{R}^2$, we write $Rot_\theta: \mathbb{R}^2 \to \mathbb{R}^2$ for the linear transformation that rotates vectors in $\mathbb{R}^2$ counter-clockwise through the angle θ. Its matrix is

$$\begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$$

and, to perform the rotation on a plain point with standard coordinates $\vec{v} = (x, y)$, it should be written as a column vector and multiplied by $Rot_\theta: \mathbb{R}^2 \to \mathbb{R}^2$, namely:

$$Rot_\theta \vec{v} = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xcos\theta - ysin\theta \\ xsin\theta + ycos\theta \end{pmatrix}.$$

The kernel (or null space) of a linear transformation is the subset of the domain that is transformed into the zero vector. In formal notation, the kernel of a linear transformation $T: V \to W$ is denoted by $ker(T)$, and it is the set of all input vectors $\vec{v} \in V$ such that $T(\vec{v}) = \vec{0}$. The kernel is a measure of injectivity. In fact, since the kernel consists of the elements sent to $\vec{0}$, the dimension of the kernel tells us how much the corresponding linear transformation shrinks the source space into the target space. Hence, a linear transformation is injective if and only if its kernel is trivial.

In linear algebra, we often need to know which vectors have their directions unchanged by a linear transformation. An "eigenvector" (or "characteristic vector") is such a vector. Hence, an eigenvector $\vec{v}$ of a linear transformation T is merely scaled by a constant factor λ when the linear transformation is applied to it; symbolically, $T(\vec{v}) = \lambda\vec{v}$. The corresponding "eigenvalue" (or "characteristic value") is the multiplying factor λ. In other words, if T is a linear transformation from a vector space V over a field F into itself and $\vec{v}$ is a non-zero vector in V, then $\vec{v}$ is an eigenvector of T if $T(\vec{v})$ is a scalar multiple of $\vec{v}$, that is, $T(\vec{v}) = \lambda\vec{v}$ where λ is a scalar in F, and then λ is said to be the eigenvalue associated with $\vec{v}$. Let A be an $n \times n$ matrix, and let $X \in \mathbb{R}^n$ be a non-zero vector for which

$$AX = \lambda X$$

for some scalar λ. Then λ is said to be the eigenvalue of the matrix A, and X is said to be an eigenvector of A associated with λ. If this is the case, then

$$AX - \lambda X = 0 \Leftrightarrow (A - \lambda I)X = 0 \Leftrightarrow (\lambda I - A)X = 0$$

where I is the corresponding identity matrix. Therefore, when we have to find eigenvectors, we have to find the non-trivial solutions of this homogeneous system of equations. The expression (determinant) $det(\lambda I - A)$ is a polynomial (in the variable x) called the "characteristic polynomial" of A.

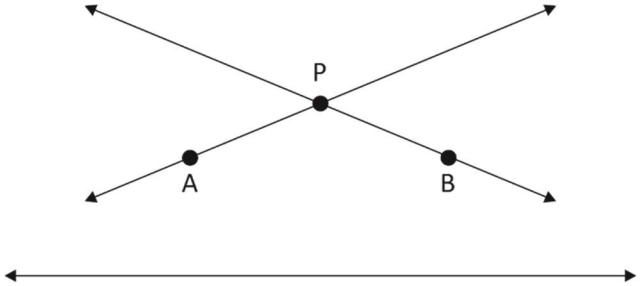
Chapter 8
Non-Euclidean Geometries

Ancient geometry culminated in Euclid's *Elements*. However, in the fifth century C.E., the Greek philosopher Proclus criticized Euclid's parallel postulate ("if a line segment intersects two straight lines forming two interior angles on the same side that sum to less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles") by arguing that it should be struck out of the axioms of geometry altogether, because it is actually a theorem involving many difficulties. Proclus offered the example of a hyperbola that approaches its asymptotes as closely as one likes without ever meeting them, thus indicating that the opposite of Euclid's conclusion is at least conceivable. Consequently, according to Proclus, Euclid's parallel postulate should be treated as a theorem, which should be proved from the other axioms.

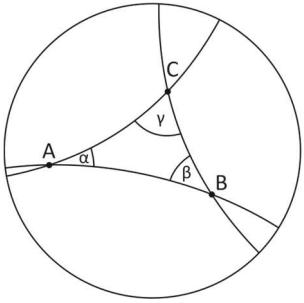
Euclid's parallel postulate was so obscure and so intimately related to the other axioms that, for many centuries, mathematicians vainly attempted to prove it by proceeding from the others. Leo Gersonides (1288–1344) was arguably the first mathematician in Western Europe who tried to prove the parallel postulate. Euclid's parallel postulate became the focus of keen attention by famous mathematicians, such as: Ch. Clavius (1574), P. Cataldi (1603), G. A. Borelli (1658), G. Vitale (1680), J. Wallis (1663), G. G. Saccheri (1733), J. H. Lambert (1766), A.-M. Legendre (1800), F. K. Schweikart (1818), F. Taurinus (1825), and C. F. Gauss. Although their attempts were unsuccessful, they played a very important role in the development of modern mathematics, because they laid the foundations for a new, non-Euclidean geometry. This geometry was based on the rejection of Euclid's parallel postulate, and was invented by the Russian mathematician Nikolai Ivanovich Lobachevski (1792–1856), who initially called this geometry "imaginary" and, later, "pangeometry." In 1826, Lobachevski delivered his first communication on non-Euclidean geometry. Lobachevski's work opened up a new era in the development of geometry. In 1832, a similar research paper of the Hungarian mathematician J. Bolyai was published. In the 1830s, Lobachevski argued that, in order to establish the validity of his non-Euclidean geometry, he needed the aid of experiments, such as astronomical observations, as in the case of other natural laws (see: Eric Temple Bell, *The Search for Truth*, New York: Reynal and Hitchcock, 1934). Similar ideas were put forward by C. F. Gauss, who did not, however, publish his work.

In Gaussian–Lobachevskian geometry, known as hyperbolic geometry, Euclid's parallel postulate is replaced by the so-called "hyperbolic axiom": for any given line L and point P not on L, in the plane containing both line L and point P, there exist at least two distinct lines through P that do not intersect L, as shown in Figure 19(a). In Euclidean geometry, the sum of the three interior angles of a triangle is always equal to π radians (i.e., 180^o, a straight line). In hyperbolic geometry, however, the sum of the three interior angles of a triangle is always strictly less than π radians, as shown in Figure 19(b). The difference is referred to as the "defect."

Figure 19: Hyperbolic Axiom and Hyperbolic Triangle.



(a)



(b)

The renowned German mathematician Bernhard Riemann (1826–66), who was a student of Gauss, had the most profound insight in non-Euclidean geometry. In his investigations of the function theory, Riemann

developed new methods based on geometric representation. Remember that $\mathbb{R}^n$ is the space of all ordered sets (called points or vectors) $x = (x_1, x_2, \dots, x_n)$ of n real numbers; the numbers $x_1, x_2, \dots, x_n$ are the coordinates of a point (or vector) x. We say that $\mathbb{R}^n$ is "standardly embedded" in $\mathbb{R}^{n+k}$ if a point $(x_1, x_2, \dots, x_n)$ from $\mathbb{R}^n$ is identified with the point $(x_1, \dots, x_n, 0, \dots, 0)$ from $\mathbb{R}^{n+k}$. In the 1850s, Riemann invented the concept of an abstract geometric surface that need not be embeddable in Euclidean three-dimensional space. On this surface, the "lines" can be interpreted as geodesics, and the intrinsic curvature of the surface can be precisely defined, as shown in Figure 20(a): a "geodesic" is the shortest path between two points on a curved surface (i.e., the non-Euclidean equivalent of a Euclidean straight line); like, for instance, on the surface of the Earth (e.g., airplanes, wishing to minimize the time that they spend on the air, do not follow Euclidean straight lines, but they follow shortest curves known as geodesics). In spherical geometry, "great circles," or "geodesics," are intersections with planes through the center of the sphere. Thus, it is not quite true that, given any two points, there is a unique line through them, because, if one chooses two points on the sphere that are opposite, or "antipodal," then there is a whole family of great circles that go through them.

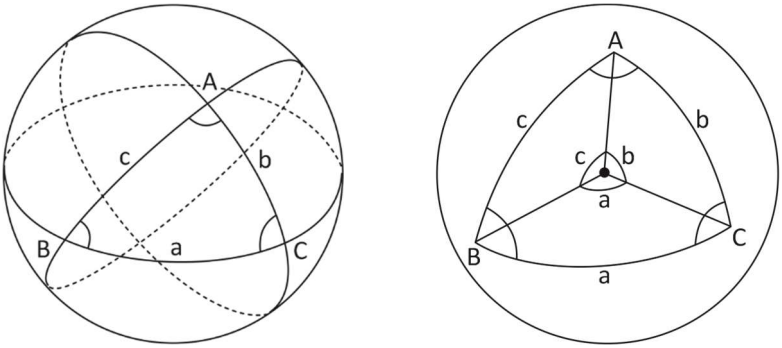
In general, Riemannian geometry is geometry on the ellipsoid or on the sphere; thus, it exists on surfaces that have constant positive curvature (Fig. 20). Gaussian–Lobachevskian geometry exists on surfaces that have constant negative curvature (Fig. 19). Euclidean geometry exists on surfaces that have constant zero curvature. This is the way in which modern geometers construe the reality of non-Euclidean planes. Therefore, whereas hyperbolic triangles are "thin" triangles (i.e., their angle sum is strictly less than 180º, as shown in Figure 19(b)), Riemannian triangles (i.e., triangles on the ellipsoid or on the sphere) are "fat" triangles (i.e., their angle sum is strictly greater than 180º, as shown in Figure 20(b)).

Notice that, if $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ are points on a sphere of radius $r > 0$ centered at the origin of Euclidean 3-space, then the distance from a to b along the surface of the sphere is

$$d(a, b) = r \cdot arccos\left(\frac{a \cdot b}{r^2}\right) = r \cdot arccos\left(\frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{r^2}\right)$$

as can be easily seen by considering the plane through a, b, and the origin. If θ is the angle between the vectors a and b, then $a \cdot b = r^2 cos\theta$, and the short arc joining a and b has length $r\theta$.

Figure 20: Riemannian Geometry on the Sphere (where "Lines" are Geodesics) and a Spherical Triangle.



When the great French mathematician and philosopher Henri Poincaré (1854–1912) was asked which geometry is true, he answered as follows:

> If geometry were an experimental science, it would not be an exact science. It would be subjected to continual revision . . . The geometric axioms are therefore neither synthetic *a priori* intuitions [as Kant has contended] nor experimental facts [as Newton has assumed]. They are conventions. Our choice among all possible conventions is guided by experimental facts; but it remains free, and is only limited by the necessity of avoiding every contradiction, and thus it is that postulates may remain rigorously true even when the experimental laws which have determined their adoption are only approximate . . . One geometry cannot be more true than another: it can only be more convenient (Henri Poincaré, *Science and Hypothesis*, translated by Mélanie Frappier, Andrea Smith, and David J. Stump, London: Bloomsbury, 2017, p. 50).

Guided by Max Planck's research work in quantum physics, by Bernhard Riemann's research work in non-Euclidean geometry, and by Constantin Carathéodory's research work in mathematical analysis and the axiomatization of thermodynamics, Albert Einstein concluded that space and time are functions of each other, so that, by referring to space, we actually refer to a temporal correspondence of space, and vice versa. Einstein's theory of relativity implies that, in contrast to Newton's perception of a three-dimensional space, we should perceive a four-dimensional space whose fourth dimension is time. Thus, time is part of the substance of space; conversely, time is underpinned by the three classical dimensions of space.

The experiments on which Einstein was based in order to articulate his general theory of relativity—according to which mass and energy are, in essence, mutually transformable forms of the same reality—presupposed the existence of a four-dimensional continuum (space-time) whose curvature is determined by gravity.

Riemannian geometry was used by Albert Einstein in order to formulate the general theory of relativity. According to Newtonian mechanics, which is formulated in the context of Euclidean geometry, assuming zero curvature, the natural trajectory of a physical body that is not acted upon by any external force is a straight line. According to the general theory of relativity, gravity manifests itself as space-time curvature. Therefore, what Newton has called natural straight-line trajectories should be generalized into curved paths known as geodesics, or great circle arcs.

The general theory of relativity explains the operation of gravity. A very simple way in which one can present Einstein's general theory of relativity is the following metaphor: imagine a big rubber sheet stretched nice and taut before your eyes. If you watch a little marble as it rolls across the surface of this rubber sheet, then you will realize that it follows a simple straight-line trajectory. But if you watch the movement of a heavy rock on this rubber sheet, then you will realize that now the rubber sheet is deformed, warped, curved. In contrast to the previous marble, this rock does not follow a straight-line trajectory, but it follows a curved trajectory along the curved surface of the rubber sheet. Einstein took this idea and applied it to the fabric of space. Originally, the fabric of space may look nice and flat, like the rubber sheet in the previous example. However, if the Sun appears, the fabric of space curves. Similarly, in the vicinity of the Earth, the fabric of space curves, and the Moon is kept in orbit around the Earth because it rolls along a valley in the curved environment that is created by the Earth's mass. This is the manner in which, according to Einstein, gravity is communicated from place to place: through warps and curves in the fabric of the space, more specifically through warps and curves in space-time. For instance, the Earth is kept in orbit around the Sun because it rolls along a valley in the curved environment that is created by the Sun's mas, and, similarly, as I mentioned before, the Moon is kept in orbit around the Earth because it rolls along a valley in the curved environment that is created by the Earth's mass. For this reason, the general theory of relativity is necessarily founded on Riemannian geometry.

It is worth mentioning that the general theory of relativity makes the following predictions: rays of light passing close to a star should be bent towards it, and physical processes should take place more slowly in

regions of low gravitational potential than in regions of high gravitational potential.

Moreover, the German physicist Arnold Sommerfeld in 1909 and the Serbian-Croatian mathematician Vladimir Veričak in 1912 proved that the special theory of relativity, which explains how speed affects mass, time, and space, is intimately related to hyperbolic geometry. In particular, according to the special theory of relativity, a light wave always travels at the same speed. In other words, the speed of light is always constant but time is relative, depending on one's state of motion. Observers in relative motion experience time differently.

Appendix: Measuring distance between two points on the Earth's surface

Lines of longitude (i.e., lines running North-South that measure angular distance from the Prime Meridian) and lines of latitude (i.e., lines running East-West that measure distance from the Equator) are used as reference points. Meridians coincide with points of the same longitude, and parallels coincide with points of the same latitude. By the term "great circle," we mean a circle that circumnavigates the Earth and passes through the center of the Earth. A great circle divides the Earth in half, and, thus, the Equator is a great circle, but no other latitudes. All lines of latitude, except for the Equator, are "small circles." All lines of longitude are "great circles." The shortest distance between any two points on the Earth's surface lies along a great circle.

First of all, we know that the circumference of a circle is given by the formula $C = 2\pi r$, and an arc length is a fraction of a circle; and such a fraction is equal to $\frac{\theta}{360^o}$. Hence, the formula for the computation of an arc length is

$$l = \frac{\theta}{360^o} \times 2\pi r. \quad (1)$$

When we have to find the distance between two points on the Earth's surface, we use formula (1) with the angle θ being the angular distance from the center of the Earth. The radius of the Earth is approximately $6{,}371km$. Therefore: the formula for finding the distance between two points with the same longitude is

$$d(x, y) = \frac{Angular\ distance}{360^o} \times 2\pi \times 6{,}371km$$

where the angular distance is the angle between the two points relative to the center of the Earth; and the formula for finding the distance (along a parallel) between two points with the same latitude is

$$d(x,y) = \frac{Angular\ distance}{360^o} \times 2\pi \times 6{,}371km \times cos\theta$$

where θ is the latitude, and the angular distance is the angle between the two points relative to the center of the small circle of the parallel on which they are located.

Chapter 9
Infinitesimal Calculus: Limits, Continuity, Differentiation, Integration, Partial Differentiation, and Multiple Integration

"Infinitesimal calculus" is a branch of mathematical analysis that concerns itself with the systematic study of the concept of an "infinitely small function," a function of a variable x whose absolute value, $|f(x)|$, becomes and remains smaller than any given number as a result of variation of x. The method of the "infinitesimals" ("infinitely small" quantities) was originally used by ancient Greek mathematicians, who determined areas and volumes by the so-called "method of exhaustion," in which infinitesimal quantities are used in order to prove that two given magnitudes (or two ratios between given magnitudes) are equal.

The method of exhaustion was originally developed in the fifth century B.C.E. by the Athenian scholar Antiphon, and it was put in a rigorous scientific setting shortly afterwards by the Greek mathematician and astronomer Eudoxus of Cnidus, who used it in order to calculate areas and volumes. The Greek mathematician and acknowledged father of "Euclidean geometry" Euclid, and the Greek mathematician, physicist, and engineer Archimedes, made extensive use of the method of exhaustion in order to prove several mathematical propositions. For instance, as already mentioned, Archimedes used the method of exhaustion in order to compute the area of a circle by approximating the area of a circle from above and below, by circumscribing and inscribing regular polygons of an increasingly larger number of sides (so that sides become "infinitesimals," namely, infinitely small). Moreover, Archimedes was able to calculate the length of various tangents to the spiral (i.e., to a curve emanating from a point moving farther away as it revolves around the point).

In few words, infinitesimal calculus, or simply calculus, is concerned with two kinds of problems: problems of tangents to curves, and problems of areas or volumes of regions. Thus, having studied both of these kinds of problem in a rigorous and systematic way, Archimedes can be considered to be the first pioneer of calculus. Some other great pioneers of calculus are the Flemish Jesuit and mathematician Gregory of Saint Vincent (1584–1667), the Dutch-French philosopher and mathematician René Descartes (1596–1650), the Italian mathematician and Jesuate Bonaventura Francesco Cavalieri (1598–1647), the French lawyer and amateur mathematician Pierre de Fermat (ca. 1607–65), the English clergyman and mathematician John Wallis (1616–1703), the English Christian theologian

and mathematician Isaac Barrow (1630–77), and the Scottish mathematician and astronomer James Gregory (1638–75).

Infinitesimal calculus is primarily aimed at solving problems concerning "change." Thus, infinitesimal calculus is used in many fields, including physics, engineering, biology, economics, statistics, and the mathematical modelling of social, political, military, and psychological problems. In the seventeenth century, infinitesimal calculus was erected as a rigorous framework of science as a result of, and in the context of, the revolutionary achievements that took place in the scientific discipline of celestial mechanics, whose protagonists were Nicolaus Copernicus, Galileo Galilei, Tycho Brahe, Johannes Kepler, and Isaac Newton. In its contemporary rigorous form, calculus was formulated independently in England by Sir Isaac Newton and in Germany by Gottfried Wilhelm Leibniz in the last quarter of the seventeenth century, using the algebraic set-up and, especially, the Cartesian set-up, which had been introduced and developed by their predecessors. Calculus consists of "differential calculus" (which is concerned with problems of tangents to curves) and "integral calculus" (which is concerned with problems of areas or volumes of regions).

Limit and Continuity of a Function

Whenever, by a known value of one quantity, we can find the value of another quantity, we say that there is a "functional dependence" between these quantities. For instance, if the length x of the side of a square is known, then its area can be found by the formula $A = x^2$. In this way, we specify the functional dependence between the length of the side of a square and its area.

As already explained, the specification of a "numerical function" requires a set of numbers X and a rule f, according to which every number x that belongs to the set X is associated with a certain number (the value of the function). An independent variable taking on values from the set X is said to be the "argument" of the function. Given a member a of the set X, the value of the function f for the argument a is denoted by $f(a)$.

If a function f is specified on a set X, then the set X is said to be the "domain" of this function, and the set of all the values of the function is said to be its "range." As already mentioned, a function $f: X \to Y$ assigns to each element $x \in X$ exactly one element $y \in Y$.

We can read the expression $y = f(x)$ as follows: "y is a function of x," meaning that, as the variable x varies, the variable y also varies according to some rule f; in this case, y is the dependent variable, and x is the independent variable.

Analytic representation of a function: Assume that we are given a collection of operations that must be performed with the argument x in order to obtain a function value. Then the function is said to be represented by an "analytic expression." For instance, consider the following functions: $y = x^2 + x + 1$, $x \in [0,1]$; $y = x^2 + x + 1$, $x \in [-2,3]$; and $y = x^2 + x + 1, x \in (-\infty, +\infty)$. Even though the analytic expressions of these functions are the same in form, we have three different functions, because they are defined on three different sets (their domains are different).

Graphical representation of a function: Assume that a function f is given by an analytic expression $f(x)$, that is, $y = f(x)$ with $x \in X$, where X is the corresponding real interval, on which f is defined. The "graph" of the function f is a set of points of the coordinate plane that have coordinates $\big(x, f(x)\big)$, where $x \in X$. If a function is even, then its graph is symmetric with respect to the axis of ordinates. If a function is odd, then its graph is symmetric about the origin.

A function $y = f(x)$ is defined to be "increasing" on its domain if, for any two of its points x_1 and x_2 such that $x_1 < x_2$, the inequality $f(x_1) < f(x_2)$ is satisfied; in other words, if to a greater value of the argument there corresponds a greater value of the function. A function $y = f(x)$ is defined to be "decreasing" on its domain if, for any two of its points x_1 and x_2 such that $x_1 < x_2$, the inequality $f(x_1) > f(x_2)$ is satisfied; in other words, if a smaller value of the function corresponds to a greater value of the argument.

A function f is said to have a "period" T if, for any value of x for which f is defined, the following equalities hold:

$f(x - T) = f(x) = f(x + T)$.

The aforementioned definition implies that, if a function f with period T is defined at the point x, it is also defined at the points $x + T$ and $x - T$. If a function f has a non-zero period T, then it is said to be "periodic." For instance, if time is measured in years, then the distance from the Earth to the Sun is given by a periodic function whose period is equal to 1.

One of the simplest functions is the "linear function" (or "linear equation"), where $y = mx + c$. In this, y and x are "variables" (that is, they can take on many values), while m and c are "constants" (that is, they have fixed values). As already explained, if we plot y against x on a diagram, the result will be a straight line, hence the name. A "non-linear function" ("non-linear equation") is any other sort of function (equation). For instance, $y = x^2$ is a quadratic equation that is downward-sloping for negative values of x and upward-sloping for positive values of x. Functions come in many forms, and they are very useful as models of the

real world when they are simple or can be satisfactorily approximated by, or manipulated into simple forms.

The concept of a limit, or a limiting process, is central to all mathematical analysis. In fact, one can argue that, from the perspective of mathematical analysis, "analysis" means taking limits. In his book entitled *Cours d'analyse*, the French mathematician Augustin-Louis Cauchy (1789–1857), one of the founders of modern mathematical analysis, explained the concept of a limit of a function in a clear, formal, and arithmetic, rather than geometric, way by arguing as follows: "when the successive values attributed to a variable approach indefinitely a fixed value so as to end by differing from it by as little as one wishes, this last is called the limit of all the others" (quoted in: Carl B. Boyer, *The History of Calculus and Its Conceptual Development*, New York: Dover, 1959, p. 272).

Consider an arbitrary function $f(x)$ defined at all values in an open interval of the number line $\mathbb{R}$ containing a point x_0, with the possible exception of x_0 itself, and let L be a real number. The "limit of a function" $f(x)$ at a point x_0 is L if and only if the values of x (where $x \neq x_0$) approach the number x_0 (notice that $f(x_0)$ may not be defined, since, according to the definition of a limit, x tends to x_0, but x never becomes equal to x_0). In other words, as x gets closer to x_0, $f(x)$ gets closer and stays close to L; symbolically:

$lim_{x \to x_0} f(x) = L.$

Remark: Let a be a real number and c a constant. Then

$lim_{x \to a} x = a$, and

$lim_{x \to a} c = c.$

Let us recall that the distance between any two points a and b on the number line $\mathbb{R}$ is $|a - b|$. Therefore, the statement

$$|f(x) - L| < \varepsilon$$

means that the distance between $f(x)$ and L is less than ε, and, by the definition of an absolute value, the statement

$$0 < |x - a| < \delta$$

is equivalent to the statement

$$a - \delta < x < a + \delta, \text{ so that } x \neq a.$$

Thus, the *Cauchy epsilon-delta definition of a limit* is the following: assume that, for all $x \neq a$, an arbitrary function $f(x)$ is defined over an open interval containing a. Then

$$lim_{x \to a} f(x) = L$$

if and only if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that, if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$. The statement (with the universal quantifier) "for every $\varepsilon > 0$" means "for every positive distance ε from L"; the statement (with the existential quantifier) "there exists a $\delta > 0$"

means that there is a positive distance δ from a; and the conditional statement "if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$" means that, if x is closer than δ to a, and $x \neq a$, then the value of $f(x)$ is closer than ε to L.

Consider a function f whose domain is D_f. Let a be an interior point of D_f. Then f is said to be "continuous at the point" a if

$lim_{x \to a} f(x)$ exists finitely and

$lim_{x \to a} f(x) = f(a)$,

meaning: if the limit of $f(x)$ as x tends to a is equal to the value of $f(x)$ at a. If a is a boundary point of D_f (i.e., in this case, an endpoint of a closed interval), then we distinguish the following two cases:

i. if $D_f = (x_1, a]$, then $f(x)$ is said to be "continuous from the left" at a if $lim_{x \to a^-} f(x) = f(a)$;
ii. if $D_f = [a, x_2)$, then $f(x)$ is said to be "continuous from the right" at a if $lim_{x \to a^+} f(x) = f(a)$.

The aforementioned definition of continuity can also be given in the following equivalent forms:

(i) A function f is continuous at $a \in D_f$ if and only if, for every sequence (x_n) with $lim_{n \to \infty} x_n = a$, where $x_n \in D_f$, it holds that $lim_{n \to \infty} f(x_n) = f(a)$. An infinite sequence (x_n) of real numbers $x_1, x_2, \ldots, x_n$ has a limit a if and only if the distance $|x_n - a|$ tends to zero as the indices of the terms of this sequence become greater than some value n_0. This means that, after a finite set of n_0 terms of this sequence, the remaining infinitely many terms of the given sequence, namely, $x_{n_0+1}, x_{n_0+2}, x_{n_0+3}, \ldots$, converge indefinitely to the value a. The sequential definition of continuity was originally developed by the German mathematician Heinrich Eduard Heine (1821–81).

(ii) A function f is continuous at $x = a \in D_f$ if and only if:
$\forall \varepsilon > 0, \exists \delta > 0 | |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$.

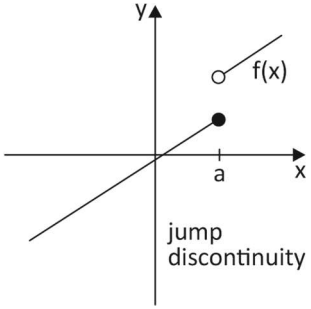
A function f is said to be "continuous over (or on, or in) an open interval" (x_1, x_2) if f is continuous at every point in that interval (x_1 may be $-\infty$, and/or x_2 may be $+\infty$). A function f is said to be "continuous over (or on, or in) the closed interval" $[x_1, x_2]$ if the following conditions hold: firstly, f is continuous at every x in the open interval (x_1, x_2); secondly, $f(x_1)$ and $f(x_2)$ both exist; and, thirdly, $lim_{x \to x_1^+} f(x) = f(x_1)$, and $lim_{x \to x_2^-} f(x) = f(x_2)$.

If we compare the definition of the limit of a function with the definition of the continuity of a function, we realize that they have the same structure, but they also have the following differences:

i. In the case of the limit of a function (Cauchy epsilon-delta definition), we have $0 < |x - a| < \delta$, or $x \neq a$, whereas, in the case of continuity, we have only $|x - a| < \delta$, meaning that the definition of continuity holds also when $x = a$.
ii. Instead of the value L that is used in the definition of the limit of a function, the definition of the continuity of a function uses the value $f(a)$, meaning that, in the case of the continuity of a function, the function must be defined at the point a. Indeed, it is meaningless to talk about the continuity (or the discontinuity) of a function at a point that does not belong to its domain of definition.
iii. In the definition of the limit of a function (Cauchy epsilon-delta definition), the point a must be an accumulation point of the domain of definition D_f of the corresponding function. Therefore, it may not belong to D_f. In the definition of the continuity of a function, meanwhile, the point a must belong to the domain of definition D_f of the corresponding function.

In intuitive terms, a function is said to be continuous if it varies with no abrupt breaks or jumps. Hence, points of continuity are characterized by the fact that, for small changes in the argument, the value of the function changes but little, whereas points of discontinuity are characterized by the fact that, for small changes in the argument, the function can change considerably. For instance, consider a load that is suspended on a thread above a table. Due to this load (supposed to be a material particle), the thread extends, and the distance l from the load to the point of thread suspension is a function of the mass m of the load, symbolically, $l = f(m)$, where $m \geq 0$. For small changes in the mass of the load, the distance l will change but little. But, if the mass of the load approaches the tensile strength m_0 of the thread, then a small increase in the mass of the load may cause a break in the thread. Thus, the distance l will increase jump-wise and become equal to the distance L from the suspension point to the surface of the table. On the half-closed interval $[0, m_0)$, the graph of the function $l = f(m)$ is a continuous line, and, at the point m_0, it suffers a discontinuity. Consequently, we get a graph consisting of two branches: at all points except m_0, the function $l = f(m)$ is continuous, in the sense that it exhibits a smooth change. At the point m_0, however, it has a discontinuity, in the sense that it exhibits a jump-wise change. In Figure 21, we see an example of a "jump discontinuity."

Figure 21: Jump Discontinuity.



Differential Calculus

Assume that a function $y = f(x)$ is defined at the points x and x_1. The difference $x_1 - x$ is called the "increment of the argument," and it is denoted by Δx. The difference $f(x_1) - f(x)$ is called the "increment of the function," and it is denoted by Δf or Δy. Therefore, $\Delta x = x_1 - x \Leftrightarrow x_1 = x + \Delta x$, and $\Delta f = f(x_1) - f(x) = f(x + \Delta x) - f(x)$. Using this formula, we can compute the value of Δf for any given x and Δx. Moreover, notice that a function $y = f(x)$ is continuous at a point $x = a$ if and only if $lim_{\Delta x \to 0} \Delta f = 0$, where $\Delta x = x - a$ and $\Delta f = f(x) - f(a)$.

Assume that, for function $y = f(x)$, at a given point x, there exists the limit of the ratio of the increment of the function, Δf, to the increment of the argument, Δx, provided that $\Delta x \to 0$. Then the function $y = f(x)$ is said to be "differentiable at the point x," and this limit is called the "derivative of the function $y = f(x)$" at the point x, and it is denoted by $\frac{df(x)}{dx}$, or $f'(x)$, or y'. Symbolically:

$\frac{df(x)}{dx} \equiv f'(x) \equiv y' = lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$.

Notice that $f'(x)$ is a new function defined at every such point x at which the indicated limit exists; this function is called the "derivative of the function $y = f(x)$," and it measures the rate of change of y with regard to x.

Let $s = s(t)$ denote the distance travelled by a point moving in a straight line on which a reference point, the unit of measurement, and the direction are chosen (notice that $s(t)$ is the position of the point on the straight line at instant t). In physics, the "average velocity" of motion during a time interval is defined as the ratio of the net displacement to the elapsed time—that is, the average velocity during the time interval from t_1 to t_2 is expressed by the quantity

$v_{av} = \frac{s(t_2)-s(t_1)}{t_2-t_1}$.

If we set $t_1 = t$, then we obtain

$v_{av} = \frac{s(t+\Delta t)-s(t)}{\Delta t} = \frac{\Delta s}{\Delta t}$.

Suppose that the average velocity of a particle is measured for a number of different time intervals, and that it is not constant. In other words, the particle under consideration is moving with varying velocity. We then have to compute the velocity of the particle at any given instant of time. This is called the instantaneous velocity. The (numerical value of the) "instantaneous velocity," or the (numerical value of the) velocity at instant t, is defined as the limit of the average velocity of motion during the time interval $[t, t + \Delta t]$ provided that $\Delta t \to 0$, symbolically:

$v_{inst} = lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$,

which is the derivative of displacement $s = s(t)$ with respect to time,

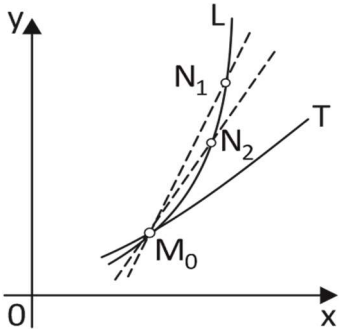
$v_{inst} = \frac{ds(t)}{dt}$.

Similarly, we can compute the instantaneous rate of change of any other physical or (quantifiable) socio-economic phenomenon with respect to its independent variable. For instance, in economics, inflation is defined as the derivative of price (as a function of time) with respect to time; the rate of change of demand with respect to price is defined as the derivative of the quantity demanded (as a function of price) with respect to price (dQ/dP, where $Q = f(P)$); and the point price-elasticity of demand, which measures the degree to which the desire for something changes as its price changes within the same demand curve, is equal to the absolute value of the derivative of the quantity demanded with respect to price multiplied by the point's price divided by its quantity ($\left|\frac{dQ}{dP}\right| \frac{P}{Q}$).

Given a function $y = f(x)$, we realize that, in order to find the rate of change of y with regard to x at a particular point, we need to find the slope of the tangent line to the curve at that point. In differential calculus, a main objective is to try to understand tangents to curves, as illustrated in Figure5-22. Hence, it is important to define a tangent line to an arbitrary plane curve in a rigorous way. A tangent line cannot be rigorously defined as a straight line having only one common point with the corresponding curve. In order to define a tangent line to an arbitrary plane curve in a rigorous way, we must use the concept of a limit. Let L be an arc of some curve, and M_0 be a point of this curve. We draw a secant M_0N through the point M_0. If the point N, moving in the curve, approaches the point M_0, then the secant M_0N turns about the point M_0. Thus, it may so happen that, as the point N approaches M_0, the secant tends to a certain limit position

M_0T, so that M_0T is referred to as the "secant" to the curve L at the point M_0, as illustrated in Figure 22. Then the "tangent line" to the curve L at the point M_0 is defined as the limit position of the secant M_0N as $N \to M_0$.

Figure 22: A tangent line to a curve.



Let us try to compute the slope of the tangent line for the case when the curve L is the graph of a certain function $y = f(x)$. Let M_0 be a point of the graph with abscissa x_0 and ordinate $y_0 = f(x_0)$. Assuming that the tangent line to the curve L at the point M_0 does exist, we take one more point $N(x_0 + \Delta x, y_0 + \Delta y)$ on the curve, as illustrated in Figure 23, and we draw a straight line through the points M_0 and N. If φ is the slope of this secant to the positive direction of the x-axis, then

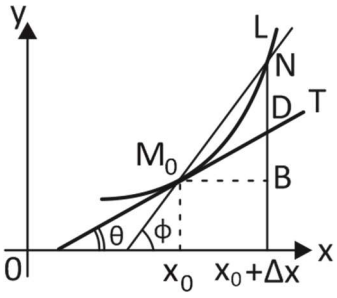
$|BN| = \Delta y$, $|M_0B| = \Delta x$, and $tan\varphi = \frac{|BN|}{|M_0B|} = \frac{\Delta y}{\Delta x}$,

so that the slope of this secant is $k_{tan} = lim_{N \to M_0} tan\varphi = lim_{\Delta x \to 0} tan\varphi$.

If we denote the slope of the tangent line to the axis of abscissas with θ, as shown in Figure 23, then the slope of the tangent line is

$k_{tan} = tan\theta = lim_{\Delta x \to 0} tan\varphi = lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$.

Figure 23: The slope of a tangent line.



Consequently, in order to draw a non-vertical tangent line to the graph of the function $y = f(x)$ at a point with abscissa x_0, it is necessary and sufficient that, at this point, the limit $lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ exists finitely; in fact, this limit is equal to the slope of the tangent line. In other words, we create an infinite sequence of slopes, and then we say that the slope of the given tangent line is the infinite limit of this sequence. Hence, infinitesimal calculus provides us with abstract objects (such as a tangent to a curve) at which only infinite tasks can arrive through the concept of a limit. The concept of a limit has a deep philosophical significance, because it secures the theoretical convenience of being able to do an infinite number of tasks through a theoretical concept—namely, that of a limit—without actually doing each one of them, which would be practically impossible. This abstraction underpins the foundations of calculus as it was articulated by Newton and Leibniz in the seventeenth century. In view of the foregoing, the slope of the tangent line to the graph of a function $y = f(x)$ at the point x_0 is equal to the value of the derivative at the point of tangency; symbolically: $k_{tan} = f'(x)$. *This is the geometric significance of the derivative.*

Basic rules of differentiation: Let $X \subseteq \mathbb{R}$ be an interval, $a \in X$, and $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ be functions that are differentiable at a. Then the following relations hold:

If $k \in \mathbb{R}$, then the function kf is differentiable at a, and

$$(kf)'(a) = kf'(a).$$

The function $f + g$ is differentiable at a, and

$$(f + g)'(a) = f'(a) + g'(a).$$

The function $f \cdot g$ is differentiable at a, and

$$(f \cdot g)'(a) = f'(a)g(a) + f(a)g'(a).$$

If $g(a) \neq 0$, then the function $\frac{f}{g}$ is differentiable at a, and

$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a)-f(a)g'(a)}{g(a)^2}$.

Power rule: $\frac{d}{dx}x^n = nx^{n-1}$, where n is an arbitrary real number.

Obviously, $\frac{d}{dx}(c) = 0$ for any constant c; and $\frac{d}{dx}(x) = 1$ for any $x \in \mathbb{R}$.

Differentiation of a composite function.: $\big(f\big(g(x)\big)\big)' = f'\big(g(x)\big) \cdot g'(x)$.

In other words, if $y = y\big(u(x)\big)$, then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

Higher order derivatives: It is evident that the first derivative $\frac{dy}{dx}$ expresses the rate of change of y with respect to x (e.g., velocity). Then $\frac{d}{dx}\left(\frac{dy}{dx}\right) \equiv \frac{d^2y}{dx^2} \equiv y''$ expresses the rate of change of the first derivative of y with respect to x (e.g., acceleration), and $\frac{d^3y}{dx^3} \equiv y''' \equiv y^{(3)}$ expresses the rate of change of the second derivative of y with respect to x (e.g., jerk). Of course, we can compute the nth derivative of $y = f(x)$, denoted by $\frac{d^ny}{dx^n} \equiv y^{(n)}$, where n is called the order of the derivative.

Basic differentiation formulas (following directly from the definition of the derivative of a function):

i. $\frac{d}{dx}(a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0) = a_n \cdot nx^{n-1} + a_{n-1} \cdot (n-1)x^{n-2} + \cdots + a_1$.

ii. $\frac{d}{dx}(e^x) = e^x$.

iii. $\frac{d}{dx}(lnx) = \frac{1}{x}$.

iv. $\frac{d}{dx}(a^x) = a^x lna$.

v. $\frac{d}{dx}(x^x) = x^x(1 + lnx)$; notice that $y = x^x \Leftrightarrow lny = lnx^x = xlnx$ (and we apply the product rule for derivatives).

vi. $\frac{d}{dx}(log_a x) = \frac{1}{xlna}$.

vii. $\frac{d}{dx}(sinx) = cosx$; and $\frac{darcsinx}{dx} = \frac{1}{\sqrt{1-x^2}}$ for $-1 < x < 1$.

viii. $\frac{d}{dx}(cosx) = -sinx$; and $\frac{darccosx}{dx} = \frac{-1}{\sqrt{1-x^2}}$ for $-1 < x < 1$.

ix. $\frac{d}{dx}(tanx) = \frac{1}{cos^2x} = sec^2x$; and $\frac{darctanx}{dx} = \frac{1}{1+x^2}$.

x. $\frac{d}{dx}(cotx) = -\frac{1}{sin^2x} = -csc^2x$; and $\frac{darccotx}{dx} = \frac{-1}{1+x^2}$.

Investigation of the behavior of a function using differential calculus: If a function $y = f(x)$ is differentiable on an interval (a, b), then:

i. f is increasing on the interval (a, b) if and only if its derivative is non-negative in this interval; symbolically: $f'(x) \geq 0 \; \forall \, x \in (a, b)$;
ii. f is decreasing on the interval (a, b) if and only if its derivative is non-positive in this interval; symbolically: $f'(x) \leq 0 \; \forall \, x \in (a, b)$.

Geometric significance: A differentiable function increases where its graph has positive slopes, and decreases where its graph has negative slopes. If $f'(x) = 0$, then $f(x)$ is constant (in a sense, it increases and decreases *simultaneously*).

We often have to solve optimization problems—that is, to choose from various variants the best one for some reasons. For instance, builders must know how to select the dimensions of a square beam in order to ensure its best tensile strength, aircraft builders must know what orbit ensures minimum fuel consumption, agronomists must know what seeding rate will guarantee the richest harvest, logistics managers must know how to minimize the transportation cost, production managers must know how to minimize costs and maximize utility, artillery officers must know what inclination of a gun tube will result in the greatest range of fire, and so on. Most optimization problems reduce to finding the extreme values, meaning the greatest and the lowest values, of a function.

Assume that a function $y = f(x)$ is continuous at a point $x = c$, and that there exists a neighborhood $(c - \delta, c + \delta)$ of this point such that the inequality $f'(x) > 0$ holds in the interval $(c - \delta, c)$, and the inequality $f'(x) < 0$ holds in the interval $(c, c + \delta)$. Then $x = c$ is a "point of maximum" for $f(x)$. In other words, if $f(x)$ increases in the interval $(c - \delta, c)$ to the left of c, and decreases in the interval $(c, c + \delta)$ to the right of c, then $x = c$ is a "point of maximum" for $f(x)$.

On the other hand, assume that a function $y = f(x)$ is continuous at a point $x = c$, and that, for some $\delta > 0$, it holds that $f'(x) < 0$ in the interval $(c - \delta, c)$, and $f'(x) > 0$ in the interval $(c, c + \delta)$. Then $x = c$ is a "point of minimum" for $f(x)$. In other words, if $f(x)$ decreases in the interval $(c - \delta, c)$ to the left of c, and increases in the interval $(c, c + \delta)$ to the right of c, then $x = c$ is a "point of minimum" for $f(x)$.

Consequently, we obtain the following algorithm for investigating a function $y = f(x)$ for an extremum (maximum or minimum):

i. Find the derivative $f'(x)$.
ii. Find the critical points, that is, the points at which the function is continuous and the derivative $f'(x)$ is either equal to zero or does not exist.

iii. Consider the neighborhood of each critical point found that does not contain another critical point and investigate the sign of the derivative to the left and to the right of the critical point under consideration.
iv. Using the aforementioned sufficient conditions for a maximum and a minimum, draw relevant conclusions (when passing through a maximum, the derivative changes sign from plus to minus, whereas, when passing through a minimum, the derivative changes sign from minus to plus).

For instance, let us investigate the function $f(x) = x^3 - 9x^2 + 24x$ for an extremum. We work as follows:

i. We have $f'(x) = 3x^2 - 18x + 24$.
ii. Equating the derivative to zero, we find the two roots (solutions) of the equation $3x^2 - 18x + 24 = 0$, namely: $x_1 = 2$ and $x_2 = 4$ (the curve has horizontal tangents at these values). In this case, the derivative is defined everywhere, and, therefore, there are no other critical points.
iii. We study the behavior of the function in a neighborhood of the point $x_1 = 2$ and in a neighborhood of the point $x_2 = 4$. We see the following: when passing through the point $x_1 = 2$, the derivative changes sign from plus to minus, whereas, when passing through the point $x_2 = 4$, the derivative changes sign from minus to plus.
iv. At $x_1 = 2$, the function has a maximum $y_{max} = 20$. At $x_2 = 4$, the function has a minimum $y_{min} = 16$.

Notice that, if x is a critical point of $f(x)$ and the second derivative of $f(x)$ is positive (resp. negative), then x is "local minimum" (resp. "local maximum") of $f(x)$.

A function $f(x)$ is said to be "concave up" on an interval X if all the tangents to $f(x)$ on X are below the graph of $f(x)$, as shown, for instance, in Figure 24 (i.e., it "opens" up). A function $f(x)$ is said to be "concave down" on an interval X if all the tangents to $f(x)$ on X are above the graph of $f(x)$, as shown, for instance, in Figure 25 (i.e., it "opens" down).

Figure 24: A Concave-Up Function.

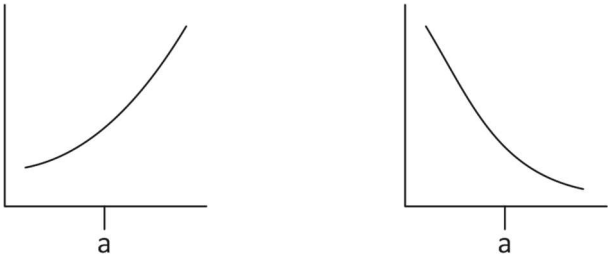
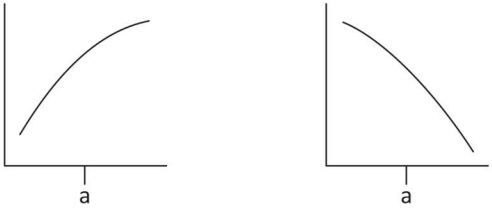


Figure 25: A Concave-Down Function.



Let f be a function differentiable on (a,b). (i) If f' is increasing (namely, if $f''(x)>0$ on (a,b)), then f is concave up on (a,b). (ii) If f' is decreasing (namely, if $f''(x)<0$ on (a,b)), then f is concave down on (a,b). (iii) If f' is constant, then the graph of f has no concavity.

If $f:(a,b)\to\mathbb{R}$ changes its direction of concavity at x_0, then the point $(x_0,f(x_0))$ is said to be a "point of inflection." In other words, x_0 is a point of inflection if $x_0\in(a,b)$ so that either f is concave down in (a,x_0) and concave up in (x_0,b), or f is concave up in (a,x_0) and concave down in (x_0,b).

Rolle's Theorem: Let $f:[a,b]\to\mathbb{R}$ be a function satisfying the following conditions:

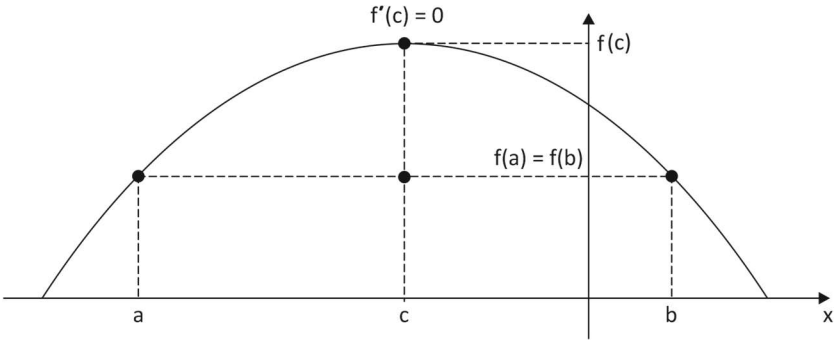
i. f is continuous on the closed interval $[a,b]$,
ii. f is differentiable on the open interval (a,b), and
iii. $f(a)=f(b)$.

Then there exists at least one point $c\in(a,b)$ such that $f'(c)=0$.

Geometric interpretation of Rolle's Theorem: Under the above conditions, there exists a point c at which the tangent line to the graph of $y=f(x)$ is parallel to the x-axis, as shown in Figure 26. In particular, conditions (i) and (ii) imply that the curve $y=f(x)$ is continuous from $x=a$ to $x=b$,

and it has a definite tangent at each point between $x = a$ and $x = b$; and condition (iii) implies that the ordinates at the endpoints a and b are equal.
Algebraic interpretation of Rolle's Theorem: Since, according to condition (iii), $f(a) = f(b)$, let $f(a) = f(b) = 0$. Then Rolle's Theorem means that, if $f(x)$ is a polynomial in x, and if a and b are two roots of the equation $f(x) = 0$, then the equation $f'(x) = 0$ has at least one root between a and b. In fact, the French mathematician Michel Rolle, after whom the above theorem is named, proved the given theorem in 1691 only in the case of polynomial functions, and a general proof of this theorem was achieved and published by Augustin-Louis Cauchy in 1823. The name "Rolle's Theorem" was first used by the German mathematician, logician, psychologist, and philosopher Moritz Wilhelm Drobisch in the 1830s.

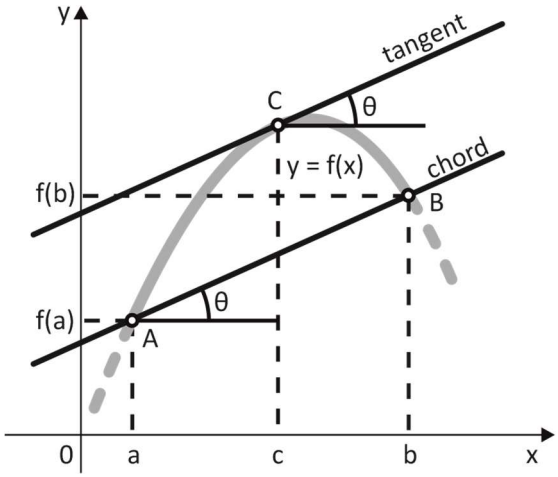
Figure 26: Rolle's Theorem.



In mathematical analysis, the mean value theorems play a very important role, because they examine the relationship between the values of a function and the values of the derivative of the given function. The Italian-French mathematician and astronomer Joseph-Louis Lagrange (1736–1813) has proved the following mean value theorem, which allows us to express the increment of a function on an interval through the value of the derivative at an intermediate point of the corresponding segment:
Lagrange's Mean Value Theorem: If $f: [a, b] \to \mathbb{R}$ is a function continuous on $[a, b]$ and differentiable on (a, b), then there exists a $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.
Geometric interpretation of Lagrange's Mean Value Theorem: As shown in Figure 27, Lagrange's Mean Value Theorem implies that the slope of the chord passing through the points of the graph corresponding to the

ends of the segment a and b is equal to $k = tan\theta = \frac{f(b)-f(a)}{b-a}$, and then there exists a point $x = c$ inside the closed interval $[a, b]$ such that the tangent to the graph at $x = c$ is parallel to the chord. In other words, if a function f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a point c in the interval (a, b) such that $f'(c)$ is equal to the function's average rate of change over $[a, b]$.

Figure 27: Lagrange's Mean Value Theorem.



For instance, given $f(x) = x^2 + x + 1$, if we are asked to find the point c at which $f'(x)$ gets its mean value over $[0,2]$, then we work as follows: we confirm that the hypotheses of Lagrange's Mean Value Theorem are satisfied, and, therefore, $\exists c \in (a, b) | \frac{f(b)-f(a)}{b-a} = f'(c) \Rightarrow \frac{f(2)-f(0)}{2-0} = 3 = f'(c) = 2c + 1 \Rightarrow c = 1$.

Optimization: (i) If we are enclosing a rectangular field (whose length is x and whose width is y) with $100\ ft$ of fence material, and one side (x) of the field is a building, then we can determine the dimensions that will maximize the enclosed area as follows: we have to maximize the function $A = xy$ subject to the constraint of $x + 2y = 100$. Hence, $x = 100 - 2y$, and then $A = (100 - 2y)y = 100y - 2y^2$. We differentiate and find the critical point(s). In fact, $A' = 100 - 4y$; thus, the critical point is $y = 25$, and this is a maximum, since $A''(25) < 0$. Finally, we find x as follows: $x = 100 - 2(25) = 50$. Consequently, the required dimensions are $50 \times$

25. (ii) We can determine the points on $y = x^2 + 1$ that are closest to the point $(0,2)$ as follows: we have to minimize the distance function $f = d^2 = (x-0)^2 + (y-2)^2$ subject to the constraint of $y = x^2 + 1$. Hence, $x^2 = y - 1$, and then the distance function becomes $f = y^2 - 3y + 3$. We differentiate and find the critical point(s). In fact, $f' = 2y - 3$; thus, the critical point is $y = \frac{3}{2}$, and this is a minimum, since $f''\left(\frac{3}{2}\right) > 0$. Finally, we find x as follows: $x^2 = \frac{3}{2} - 1 = \frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}}$. Consequently, the points on $y = x^2 + 1$ that are closest to the point $(0,2)$ are $\left(\frac{1}{\sqrt{2}}, \frac{3}{2}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \frac{3}{2}\right)$.

The curvature of a curve: By the term "curvature," we refer to the measure of how sharply a curve bends. If $y = f(x)$ is a plane curve, then the curvature at any point $P(x, y)$ is expressed in terms of the first and the second derivatives of the function $f(x)$ by the formula

$$K = \frac{|f''(x)|}{[1 + (f'(x))^2]^{\frac{3}{2}}}$$

where K characterizes the speed of rotation of the tangent to the curve at the given point. Curvature is one of the key concepts of differential geometry. Differential geometry is a combination of calculus and analytic geometry applied to curves and surfaces. The pioneers of differential geometry are C. Huygens, A. C. Clairaut, L. Euler, A.-L. Cauchy, and G. Monge. In the twentieth century, curvature played a very important role in the development of modern physics due to the theory of relativity.

Integral Calculus

As already mentioned, in calculus, we start with two general questions about functions. Firstly, how steep is a function at a point? Secondly, what is the area underneath a graph over some region? The first question is answered using a tool called the "derivative." In other words, the derivative measures the rate of change of a function at a point. The second question is answered using a tool called the "integral."

Integration can be construed as the inverse of differentiation. Let $f: I \to \mathbb{R}$ be a function, where I is an interval; in fact, I may have one of the following forms:

$[a,b], [a,b), (a,b], (a,b), [a,+\infty), (a,+\infty), (-\infty,b], (-\infty,b), (-\infty,+\infty)$

where $a, b \in \mathbb{R}$.

If $F: I \to \mathbb{R}$ is a function such that $F'(x) = f(x)\ \forall x \in I$, then F is called the "antiderivative" of f in I, and it is denoted by

$F(x) = \int f(x)dx$, where $x \in I$,

according to Leibniz's notation. In other words, $\int f(x)dx = F(x) + c$ if and only if $[F(x) + c]' = f(x)$. The aforementioned definition implies that the "indefinite integral" of a given function with respect to x is a new function plus a constant if and only if the derivative of the new function and of the constant equals the given function. Thus, differentiation can be used in order to verify the result of an integral.

Examples:

i. $\int adx = ax + c$, because $(ax + c)' = a$;

ii. $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ over the following intervals: (i) $n \neq -1, x > 0$; (ii) $n \neq -1, x < 0$; and (iii) $n \geq 0, x \in \mathbb{R}$. For instance, $\int \sqrt{x}dx = \int x^{1/2}dx = \frac{x^{3/2}}{\frac{3}{2}} + c = \frac{2}{3}x^{3/2} + c$, and $\int xdx = \frac{x^2}{2} + c$. If $n = -1$, then $\int x^n dx = ln|x| + c$; i.e., $\int \frac{dx}{x} = ln|x| + c$;

iii. $\int a^x dx = \frac{a^x}{lna} + c$;

iv. $\int lnaxdx = x(lnax - 1) + c$;

v. $\int log_a xdx = xlog_a x - \frac{x}{lna} = \frac{xlnx-}{lna}$;

vi. $\int sinxdx = -cosx + c$;

vii. $\int cosxdx = sinx + c$;

viii. $\int tanxdx = -ln|cosx| + c$ (notice that $\int tanxdx = \int \frac{sinx}{cosx}dx$ and set $u = cosx$ and $du = -sinxdx$, so that $\int \frac{sinx}{cosx}dx = -\int \frac{du}{u} = -ln|u| + c$, where $u = cosx$).

ix. $\int cotxdx = ln|sinx| + c$.

Let $f: I \to \mathbb{R}$ and $g: I \to \mathbb{R}$ be two functions. If their indefinite integrals exist over I, then there exists the indefinite integral of $af + bg$, where a and b are constants, and

$\int [af(x) + bg(x)]dx = a\int f(x)dx + b\int g(x)dx$.

If the integral includes the expression $\sqrt{a^2 - x^2}$, then we set $x = |a|sin\theta$ or $x = |a|cos\theta$, so that: (i) if $x = |a|sin\theta$, then $dx = |a|cos\theta d\theta$ and $\sqrt{a^2 - x^2} = |a|cos\theta$; (ii) if $x = |a|cos\theta$, then $dx = -|a|sin\theta d\theta$ and $\sqrt{a^2 - x^2} = |a|sin\theta$.

If the integral includes the expression $\sqrt{a^2 + x^2}$, then we set $x = |a|tan\theta$ or $x = |a|cot\theta$. If $x = |a|tan\theta$, then $dx = \frac{|a|}{cos^2\theta}d\theta$ and $\sqrt{a^2 + x^2} = \frac{|a|}{cos\theta}$.

If the integral includes the expression $\sqrt{x^2 - a^2}$, then we set $x = |a|\frac{1}{cos\theta}$, so that $dx = |a|\frac{si}{cos^2\theta}d\theta$ and $\sqrt{x^2 - a^2} = |a|\frac{sin\theta}{co} = |a|tan\theta$.

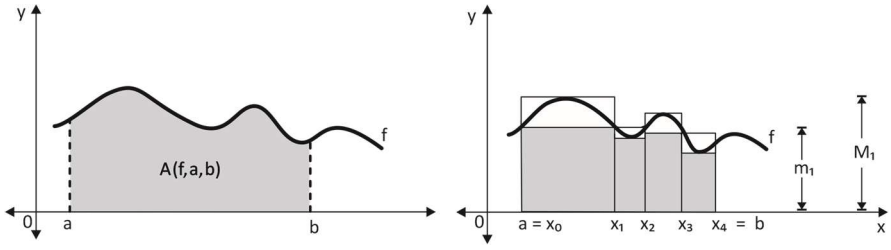
If the integral includes the expression $\sqrt{ax + b}$, then we set $\sqrt{ax + b} = t$.

Integration by parts: $\int udv = uv - \int vdu$,
where: $u = u(x)$ and $du = u'(x)dx$ while $v = v(x)$ and $dv = v'(x)dx$.
For instance, given the integral $\int xe^x dx$, we set $u = x$ and $dv = e^x dx$. Then $du = dx$ and $v = \int e^x dx = e^x$. Hence, $\int xe^x dx = \int xde^x = xe^x - \int e^x dx = xe^x - e^x + c$.
The "definite integral" is written as

$$\int_a^b f(x)dx$$

and represents the area bounded by the curve $y = f(x)$, the x-axis, and the ordinates $x = a$ and $x = b$ if $f(x) \geq 0$. If $f(x)$ is sometimes positive and sometimes negative, then the definite integral represents the algebraic sum of the areas above and below the x-axis. In particular, the areas that are above the x-axis are considered to be positive, whereas the areas that are below the x-axis are considered to be negative.
As shown in Figure 28, the definite integral $\int_a^b f(x)dx$ can be defined as follows:
We subdivide the closed interval $[a, b]$ into n subintervals
$[a, x_1], [x_1, x_2], \dots, [x_{k-1}, x_k], \dots, [x_{n-1}, b]$
by means of the points $x_1, x_2, \dots, x_{n-1}$, which have been chosen arbitrarily. Hence, the set of points $P = \{a = x_0, x_1, x_2, \dots, x_{k-1}, x_k, \dots, x_{n-1}, x_n = b\}$ is a "partition" of
$[a, b]$. Let Δx_k be the length of the kth subinterval, that is, $\Delta x_k = x_k - x_{k-1}$. Then the "norm" of partition P is denoted by $\|P\|$, and it is equal to $max\{\Delta x_k | k = 1,2, \dots, n\}$.

Figure 28: The Integral as the Limit of a Sum.



In each of the n subintervals mentioned in the aforementioned partition, we choose points $c_1, c_2, \dots, c_n$ in an arbitrary way, and we form the sum

$$S(P, f, c_k) = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \cdots + f(c_k)\Delta x_k + \cdots + f(c_n)\Delta x_n = \sum_{k=1}^{n} f(c_k)\Delta x_k.$$

Notice that, as the number of subdivisions n increases, $\|P\|$ vanishes—that is, $\|P\| \to 0$ as $n \to \infty$. Hence, if $lim_{\|P\|\to 0} S(P, f, c_k)$ exists and is independent of the mode of subdivision of $[a, b]$, then this limit is said to be the integral of f on $[a, b]$; symbolically:

$$lim_{\|P\|\to 0} S(P, f, c_k) = \int_a^b f(x)dx$$

where $f(x)dx$ is called the "integrand," $[a, b]$ is called the "range of integration," and a and b are called the lower and the upper "limit of integration" respectively. Notice that the aforementioned limit exists if $f(x)$ is continuous (or sectionally continuous) on $[a, b]$. Leibniz symbolized the definite integral of a function $f(x)$ on $[a, b]$ as $\int_a^b f(x)dx$, because the sign $\int$ is an elongated S standing for the word "sum," since Leibniz defined $\int_a^b f(x)dx$ as the sum of infinitely many rectangles of height $f(x)$ and infinitesimally small width dx.

The Fundamental Theorem of Infinitesimal Calculus states that: (i) For a function f, an antiderivative (or indefinite integral) F may be obtained as the integral of f over an interval with a variable upper bound. Symbolically: if f is a continuous real-valued function defined on a closed interval $[a, b]$ and $F(x) = \int_a^x f(t)dt$, then F is continuous for all x in $[a, b]$ and differentiable on (a, b), and $F'(x) \equiv \frac{dF(x)}{dx} = f(x)$. (ii) The integral of function f over a fixed interval is equal to the change of any antiderivative F between the ends of the interval. Symbolically: if f is a continuous real-valued function defined on a closed interval $[a, b]$, if F is a continuous function on $[a, b]$, and if F is an antiderivative of f on (a, b), then $F'(x) \equiv \frac{dF(x)}{dx} = f(x)$.

Example: We calculate the value of $\int_2^3 x^2 dx$ as follows:

$$\int_2^3 x^2 dx = \frac{x^3}{3}|_2^3 =$$

$$\left(value\ of \frac{x^3}{3} when\ x = 3\right) - \left(value\ of \frac{x^3}{3} when\ x = 2\right) = \frac{3^3}{3} - \frac{2^3}{3} = \frac{19}{3}.$$

The average value of a function $f(x)$ on the closed interval $[a, b]$ is

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

(where $b - a$ is the length of the interval over which we are averaging, and $\int_a^b f(x)\,dx$ is the area underneath the curve that we are averaging).

When the integrand $f(x)$ is known only at certain points (e.g., those obtained by sampling), or when a formula for the integrand is known but it is difficult or impossible to find an antiderivative that is an elementary function, we may use numerical methods of integration—that is, approximate formulas for definite integrals. The simplest approximate formula for definite integrals is

$\int_a^b f(x)dx \approx \frac{1}{2}(b-a)[f(a)+f(b)]$,

which is exact when $f(x)$ is linear. However, a much better approximate formula for definite integrals is

$\int_a^b f(x)dx \approx \frac{1}{6}(b-a)\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$,

which is known as "Simpson's Rule" (named after the eighteenth-century British mathematician Thomas Simpson, who formulated it. Before him, however, Johannes Kepler had already used similar formulas. For this reason, "Simpson's Rule" is sometimes called "Kepler's Rule"). Simpson's Rule derives from the observation that, if $p(x) = Ax^2 + Bx + C$, then $\int_a^b p(x)\,dx = \frac{b-a}{6}\left[p(a)+4p\left(\frac{a+b}{2}\right)+p(b)\right]$, and it is used in order to approximate any integral $\int_a^b f(x)dx$, where f is an arbitrary function, and not necessarily a quadratic polynomial (i.e., parabola).

The calculation of the area between two arbitrary curves: In the first case, we want to determine the area A between the equations $y = f(x)$ and $y = g(x)$ over the interval $[a,b]$ under the assumption that $f(x) \geq g(x)$, meaning that the graph of $f(x)$ is above the graph of $g(x)$. Then

$A = \int_a^b [(upperfunction) - (lowerfunction)]dx = \int_a^b [f(x) - g(x)]\,dx$,

where $a \leq x \leq b$.

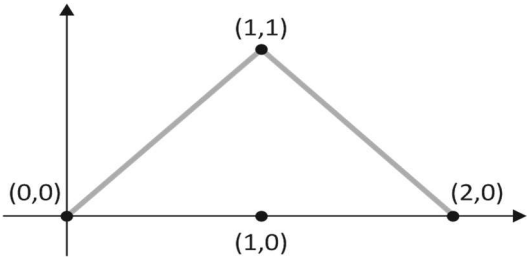
In the second case, we want to determine the area A between the equations $x = f(y)$ and $x = g(y)$ over the interval $[c,d]$ under the assumption that $f(y) \geq g(y)$, namely, $x = f(y)$ is on the right-hand side of $x = g(y)$. Then

$A = \int_c^d [(rightfunction) - (leftfunction)]dy = \int_c^d [f(y) - g(y)]\,dy$,

where $c \leq y \leq d$.

The area of a triangle can be calculated as follows: A triangle consists of tree lines connecting the three vertices. In order to find the area bounded by these three lines, we must find the equations of these three lines and integrate their differences. For instance, in order to find the area of the triangle with vertices (0,0), (1,1), and (2,0), we notice that it consists of the following three lines: $y = 0$, $y = x$, and $y = 2 - x$, as shown in Figure 29.

Figure 29: Calculating the area of a triangle.



For the left half of the triangle (i.e., between the points $x = 0$ and $x = 1$), we need to find the area between $y = x$ and $y = 0$. For the right half of the triangle (i.e., between the points $x = 1$ and $x = 2$), we need to find the area between $y = 2 - x$ and $y = 0$. Hence, finally, we calculate $A(triangle) = A(left\ half) + A(right\ half)$, symbolically:

$\int_0^1 (x - 0)\, dx + \int_1^2 [(2 - x) - 0]\, dx = 1\ square\ unit.$

The area of a square can be calculated as follows: If a is the length of the side of the square, then the area of the square is given by $A(square) = \int_0^a adx = ax|_0^a = a^2$.

The area of a circle whose radius is r can be calculated as follows: If we chop up the circle into triangular pie wedges with base dx and height r, then the area of each triangle is one-half the base dx times the height r. Adding them up, we obtain:

$$\int_0^{2\pi r} \frac{1}{2} r dx = \frac{1}{2} rx|_0^{2\pi r} = \pi r^2$$

given that the circumference of a circle of radius r is $2\pi r$. Similarly, we can work as follows: if we chop up the circle into circular rings with radius x and thickness dx, then the area of each ring is its circumference $2\pi x$ times the thickness dx. Adding them up, we obtain:

$\int_0^r 2\pi x dx = \pi x^2\,|_0^r = \pi r^2$.

The calculation of the volume of a solid of revolution: In order to obtain a solid of revolution, we start out with a curve $y = f(x)$ on an interval $[a, b]$, as shown, for instance, in Figure 30, and then we rotate this curve (360°) about a given axis, so that a volume is generated, as shown, for instance, in Figure 31.

In order to determine the volume of a solid of revolution on the interval $[a, b]$, we work as follows: we divide the interval $[a, b]$ into n

subintervals, each of which has width $\Delta x = \frac{b-a}{n}$, and then we choose a point x_k^* (where $k = 1,2, \dots, n$) from each subinterval. When we want to determine the area between two curves, we approximate the area by using rectangles on each subinterval. Understandably, when we want to compute the volume of a solid of revolution, we use disks on each subinterval to approximate the area. The area of the face of each disk is given by $A(x_k^*)$, and the volume of each disk is given by $V_k = A(x_k^*)\Delta x$. Hence, the volume of the corresponding solid of revolution on the interval $[a, b]$ can be approximated by $V \approx \sum_{k=1}^{n} A(x_k^*)\Delta x$. Then, its exact volume is

$V = lim_{n\to\infty} \sum_{k=1}^{n} A(x_k^*)\Delta x = \int_a^b A(x)dx,$

where $a \leq x \leq b$.

In other words, in this case, the volume is the integral of the cross-sectional area $A(x)$ at any x, and $x \in [a, b]$. Given that $A = \pi r^2$, $r = f(x)$, and $f(x)$ is a non-negative continuous function from $[a, b]$ to $\mathbb{R}$, the volume of the solid generated by a region under $y = f(x)$ bounded by the x-axis and the vertical lines $x = a$ and $x = b$ via revolution about the x-axis is

$V = \pi \int_a^b [f(x)]^2 \, dx;$

we take disks with respect to x, and $r = y = f(x)$; dx indicates that the area is rotated about the x-axis.

Figure 30: A Curve.

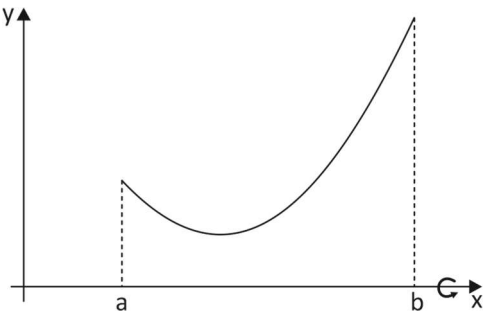
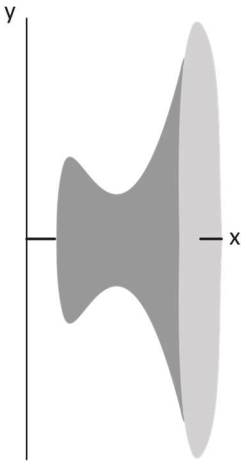


Figure 31:A Solid of Revolution.



If we rotate a curve about the y-axis, thus obtaining a cross-sectional area that is a function of y instead of x, then the aforementioned formula becomes

$V = \int_c^d A(y)dy,$

where $c \leq y \leq d$. Given that, in this case, $A = \pi r^2$, and $r = f(y)$, the volume of the solid generated by a region under $x = f(y)$ bounded by the y-axis and the horizontal lines $y = c$ and $y = d$ via revolution about the y-axis is

$V = \pi \int_c^d [f(y)]^2 \, dy;$

we take disks with respect to y, and $r = x = f(y)$; dy indicates that the area is rotated about the y-axis.

If we have two curves y_1 and y_2 that enclose some area, and we rotate that area about the x-axis, then the volume of the solid formed is given by

$V = \pi \int_a^b [(y_2)^2 - (y_1)^2] \, dx.$

The volume of a sphere can be calculated as follows: A sphere of radius r centered at the origin $(0,0,0)$ can be generated by revolving the upper semicircular disk enclosed between the x-axis and $x^2 + y^2 = r^2$ about the x-axis. If we revolve the semi-circle given by

$$y = f(x) = \sqrt{r^2 - x^2}$$

about the x-axis, we obtain a sphere of radius r. A cross-section of the sphere is a circle with radius $f(x)$ and area $\pi[f(x)]^2$. If we slice the sphere vertically into disks, then each disk has infinitesimal thickness dx,

and the volume of each disk is approximately $\pi[f(x)]^2 dx$. If we add up the volumes of the disks, then we obtain the volume of the sphere—namely:

$V = \pi \int_a^b [f(x)]^2\, dx = \pi \int_{-r}^{r} (r^2 - x^2) dx = \pi \left(r^2 x - \frac{x^3}{3} \right) |_{-r}^{r} =$

$\pi \left(\frac{2}{3} r^3 \right) - \pi \left(-\frac{2}{3} r^3 \right) = \frac{4}{3} \pi r^3.$

The volume of a cone can be calculated as follows: A cone with base radius r and height h can be formed by rotating a straight line through the origin $(0,0,0)$ about the x-axis. The slope of the straight line is $tan\theta = \frac{r}{h}$, so that the equation of the line is $y = \frac{r}{h} x$, and the limits of integration are $x = 0$ and $x = h$. Therefore, the volume of the corresponding cone is

$V = \pi \int_0^h \left(\frac{r}{h} x \right)^2 dx = \frac{\pi r^2}{h^2} \left(\frac{x^3}{3} \right) |_0^h = \frac{1}{3} \pi r^2 h.$

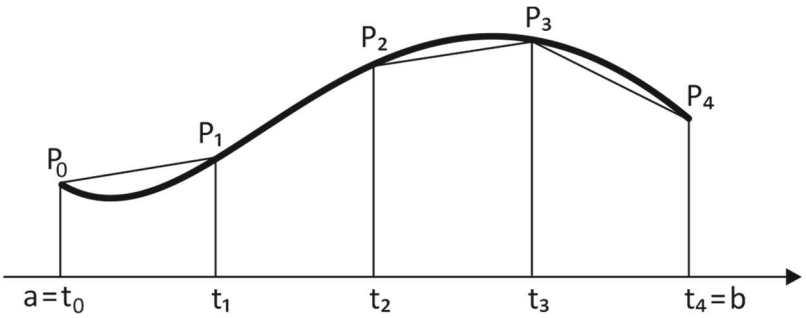
The volume of a cylinder with base radius r and height h (assuming that the plane xOy is the cylinder's base plane) is $V = \pi \int_0^h r^2 dx = \pi r^2 h$.

The calculation of the arc length of a curve: Let us consider a curve γ defined by the parametric equations

$x = g(t)$ and $y = f(t)$, where $t \in [a, b]$,

as shown, for instance, in Figure 32, and let $P = \{t_0, t_1, \dots, t_n\}$ be a partition of $[a, b]$. Intuitively, if we regard parameter t as the time variable, then the curve may be thought of as the path of a moving point whose position vector at time t is $\gamma(t) = \big(g(t), f(t)\big)$.

Figure 32: The Arc Length of a Curve.



Let $A_k = [g(t_k), f(t_k)]$, where $k = 1,2, \dots, n$, be the corresponding points of γ, as shown in Figure 32. Then these points define a polygonal line. The sum

$$L_P = \sum_{i=1}^{n} \sqrt{[g(t_i) - g(t_{i-1})]^2 + [f(t_i) - f(t_{i-1})]^2}$$

is the length of the polygonal line that is defined by the points A_k (corresponding to a partition P); and the finer the partition P, the more the corresponding polygonal line tends to be identified with the curve γ. Now, let us consider the set L of all numbers L_P, which correspond to all possible partitions P of $[a, b]$, symbolically:

$L = \{L_P | P \text{ is a partition of } [a, b]\}$.

If this set L is bounded, then the curve is said to be "alignable," and the supremum $S = L(\gamma)$ of this set is said to be the length of the curve γ. Moreover, we write $S = L_a^b(\gamma)$ in order to denote the length of the arc of the curve that is defined on the interval $[a, b]$.

Notice that, if γ is an alignable curve on $[a, b]$, and if $a < c < b$, then

$L_a^b(\gamma) = L_a^c(\gamma) + L_c^b(\gamma)$.

If the derivatives g' and f' are continuous on $[a, b]$, then the curve γ is alignable on $[a, b]$, and its length is given by

$S = L(\gamma) = \int_a^b \sqrt{[g'(t)]^2 + [f'(t)]^2}\, dt$,

where $t \in [a, b]$. If γ is defined by $y = f(x)$, where $x \in [a, b]$, and if the derivative $f'(x)$ exists and is continuous on $[a, b]$, then, setting $x = t$ and $y = f(t)$ in the aforementioned equation, we obtain the following formula:

$S = \int_a^b \sqrt{1 + [f'(x)]^2 dx}$,

where $x \in [a, b]$.

The physical significance of the integral

The development of infinitesimal calculus by Newton and Leibniz is intimately related to the study of celestial mechanics, and physics in general, by them. Infinitesimal calculus, known also as the differentiation–integration method, is concerned with the limits of applicability of physical laws. Physical laws are not absolute, and the validity of a law is restricted to the framework of the applicability limits (i.e., certain conditions). However, a physical law can be expanded by changing its form beyond the limits of applicability by means of infinitesimal calculus. This method is based on the following two principles: (i) the principle that a law can be represented in differential form, and (ii) the superposition principle, according to which the quantities that enter into the law are additive.

Suppose that a physical law has the form

$X = YZ$, (∗)

where X, Y, and Z are physical quantities, and, in particular, Y is a constant representing the given law's limits of applicability. We can generalize the given law to the case where Y is not a constant but a function of Z, that is, $Y = Y(Z)$, as follows: As shown in Figure 33, we isolate an interval dZ so small that the variation of Z over this interval can be ignored. Hence, in the interval ("infinitesimal") dZ, we can approximately assume that Y is constant, and that the law $(*)$ is valid in this interval. Therefore,

$$dX = Y(Z)dZ, \qquad (**)$$

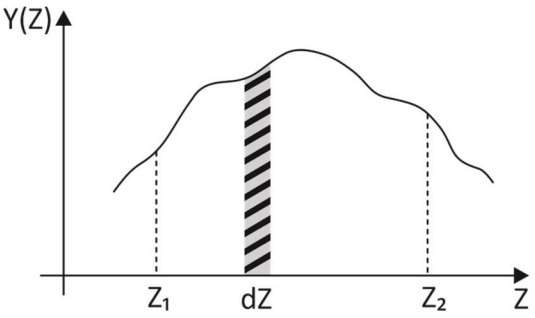
where dX is the variation of X over dZ. Due to the superposition principle, that is, by summing the quantities $(**)$ over all the intervals of variation of Z, we obtain an expression for X in the form

$$X = \int_m^M Y(Z)dZ,$$

where m and M are the initial and the final values of Z, respectively.

As a conclusion, the method of infinitesimal calculus consists of two parts: in the first part of the method, we find the differential $(**)$ of the quantity under investigation; in the second part of the method, we sum, or "integrate," having adequately determined the integration variable and the limits of integration (in order to determine the integration variable, we must analyze the quantities on which the differential of the investigated quantity depends and choose the most important variable; and the limits of integration are the lower and the upper values of the integration variable).

Figure 33: The method of infinitesimal calculus.



Differentiation and Integration of Multivariable Functions

So far, we have studied exclusively functions of a single (independent) variable x, but we can also apply the concept of differentiation to

functions of several variables $x, y, \ldots$ Suppose that $f(x,y)$ is a function of two variables x and y, and that the limits

$$lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

and

$$lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

exist for all values of x and y in question—that is, $f(x,y)$ possesses a derivative $\frac{df}{dx}$ with respect to x and a derivative $\frac{df}{dy}$ with respect to y. These derivatives are called the "partial derivatives" of f, and they are respectively denoted by

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ or

$f'_x(x,y), f'_y(x,y)$.

Similarly, we can differentiate functions of three or more variables.

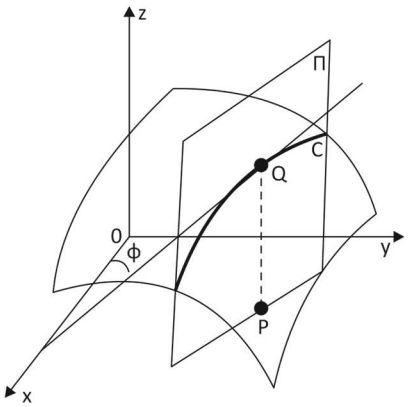
In general, when calculating partial derivatives, we treat all independent variables other than the variable with respect to which we differentiate as constants. For instance, if $f(x,y) = x^2 - 3xy - 5$, then

$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 - 3xy - 5) = \frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial x}(3xy) - \frac{\partial}{\partial x}(5) = 2x - 3y$, and

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 - 3xy - 5) = \frac{\partial}{\partial y}(x^2) - \frac{\partial}{\partial y}(3xy) - \frac{\partial}{\partial y}(5) = -3x$.

The geometric significance of $\frac{\partial f}{\partial x}|_{(x_0,y_0)}$ and $\frac{\partial f}{\partial y}|_{(x_0,y_0)}$ is illustrated in Figure 34. Let us consider a function $z = f(x,y)$, whose graph in $\mathbb{R}^3$ is a surface. We suppose that $P(x_0, y_0)$ is an arbitrary point of the domain of f. Notice that, in $\mathbb{R}^3$, the equation $y = y_0$ represents a plane Π that is perpendicular to the y-axis. This plane intersects the surface $z = f(x,y)$ by a curve C whose equation is $z = f(x, y_0)$. If $Q(x_0, y_0, z_0)$ is a point belonging to C, so that its orthogonal projection to the plane xOy is the point P, then the slope of the tangent to the curve C at Q is equal to $\frac{\partial f}{\partial x}|_{(x_0,y_0)} = tan\varphi$, where φ is the angle formed by the x-axis and the tangent to the curve C at Q, as shown in Figure 34. In the same way, we can show that the slope of the tangent to the curve C at Q is equal to $\frac{\partial f}{\partial y}|_{(x_0,y_0)} = tan\theta$, where θ is the angle formed by the y-axis and the tangent to the curve C at Q.

Figure 34: The geometric significance of a partial derivative



Generalization: If $f: \mathbb{R}^n \to \mathbb{R}$ is a function,
$\mathbb{R}^n \ni (x_1, x_2, \dots, x_n) \to f(x_1, x_2, \dots, x_n) \in \mathbb{R}$,
then

$$\frac{\partial f(x_1, x_2, \dots, x_i, \dots, x_n)}{\partial x_i} = lim_{\Delta x_i \to 0} \frac{f(x_1, x_2, \dots, x_i + \Delta x_i, \dots, x_n) - f(x_1, x_2, \dots, x_i, \dots, x_n)}{\Delta x_i}$$

is the partial derivative of $f(x_1, x_2, \dots, x_n)$ with respect to x_i, where $i = 1,2, \dots, n$.

We take for granted the obvious generalizations of the theorems of differentiation to two or more variables.

We can integrate functions of several variables as follows: suppose that the domain of a bivariate function is the Cartesian product of two closed intervals—that is, a rectangle—say
$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 | a \le x \le b, c \le y \le d\}$.

If $R = [a, b] \times [c, d]$, whenever the integrand is $f(x, y)$, we have to integrate over two variables, x and y, so that, for each variable, we have an integration sign. In order to indicate the variables involved, we have dx and dy, symbolically:

$\iint_R f(x, y)\, dxdy \equiv \int_c^d \int_a^b f(x, y) dxdy$,

where $f(x, y)$ is an integrable function of two real variables. In this case, we compute the innermost integral first, and then we work our way outward. In particular, we compute the dx integral inside first, while treating y as a constant, and then we integrate the result over y as we

would do with any variable. One interpretation of the double integral of $f(x,y)$ over the rectangle R is the volume under the function (surface) $f(x,y)$ and above the xy-plane. For instance, $\int_0^2 \int_0^1 x^2y^2 dxdy$ can be calculated as follows: We focus on the inner integral first: $\int_0^2 \left[\int_0^1 x^2y^2 dx\right] dy$; and, treating y as a constant, we integrate normally for x^2dx, thus obtaining $\int_0^2 \left[\frac{x^3y^2}{3}|_0^1\right] dy = \int_0^2 \left[\frac{1^3y^2}{3} - \frac{0^3y^2}{3}\right] dy = \int_0^2 \left[\frac{y^2}{3}\right] dy$. Now, we are left with a normal definite integral: $\int_0^2 \frac{y^2}{3} dy = \frac{y^3}{3\cdot 3}|_0^2 = \frac{y^3}{9}|_0^2 = \frac{2^3}{9} - \frac{0^3}{9} = \frac{8}{9}$. Therefore, $\int_0^2 \int_0^1 x^2y^2 dxdy = \frac{8}{9}$.

Furthermore, double integrals can be used in order to compute areas as follows: the area A of a plane region R is given by the formula

$A = \iint_R dxdy,$

in Cartesian coordinates. Notice that, if

$R = \{(x,y) \in \mathbb{R}^2 | a \leq x \leq b, h_1(x) \leq y \leq h_2(x)\},$

then the area of R is

$A = \iint_R dxdy = \int_a^b \int_{h_1(x)}^{h_2(x)} dy\, dx = \int_a^b [h_2(x) - h_1(x)]\, dx,$

where $h_1(x)$ and $h_2(x)$ are two curves.

The order in which we do the integrations does not matter, provided that we keep track of the limits of integration of each variable. For instance, in the double integral $\int_c^d \int_a^b f(x,y)dxdy$, dx is associated with the x integrand, which runs from a to b, while dy is associated with the y integrand, which runs from c to d, and, therefore,

$$\int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx$$

meaning that the limits of integration of each integrand remain the same. This result is known as Fubini's Theorem: given that a definite double integral can be thought of as a process of adding up all the infinitesimal elements of a (Cartesian) area $dxdy$ (imagine little rectangles) over the required region, thus obtaining the area of that region, the equality between the aforementioned two iterated integrals (i.e., Fubini's Theorem) can be thought of as an infinite version of the idea that addition is commutative and associative.

Increasing the number of integrals in the context of multiple integration is the same as increasing the number of dimensions, so that a single integral gives a two-dimensional area, a double integral gives a three-dimensional volume, a triple-integral gives a four-dimensional hypervolume, etc. In general, the multiple integral of a function $f(x_1, \ldots, x_n)$ in n variables over a domain U is represented by n nested integral signs in the reverse order of

computation (in the sense that the leftmost integral is computed last), followed by the function and the integrand arguments in such an order that indicates that the integral with respect to the rightmost argument is computed last; and the domain of integration is either represented symbolically for every argument over each integral sign or it is indicated by a characteristic letter (variable) at the rightmost integral sign:

$$\int \dots \int_U f(x_1, \dots, x_n) d\, x_1 \dots dx_n$$

($x_1, \dots, x_n \in U$). We take for granted the obvious generalizations of the theorems of integration to two or more variables.

Chapter 10
Complex Numbers and Complex Analysis

As we have already mentioned, the concept of a number has been extended from natural to real numbers, both because of human practice and because of the needs of mathematics itself. In particular, the concept of a number grew out of the counting of objects. Counting gave rise to the numbers 1, 2, 3, and so on, which are called natural numbers. Then the necessity of performing the operation of division led to the concept of positive fractional numbers; furthermore, the necessity of performing the operation of subtraction led to the concepts of zero and negative numbers; finally, the necessity of taking roots of positive numbers led to the concept of irrational numbers. The aforementioned operations are feasible in the set of real numbers. However, there are still impracticable operations—for instance, taking a square root of a negative number. Hence, there is a need to extend the concept of a number even further, specifically, to invent new numbers different from the real numbers.

Indeed, if we adjoin to the real system $\mathbb{R}$ a root i of the polynomial $x^2 + 1 = 0$, which is irreducible to $\mathbb{R}$, we obtain the system of complex number $\mathbb{C} \equiv \mathbb{R}(i)$. In other words, a (two-dimensional) number of the form $z = a + bi$, where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$, is called a "complex number"; the number a is called the "real part" of $z = a + bi$, and it is denoted by $Re(z)$; the number b is called the "imaginary part" of $z = a + bi$, and it is denoted by $Im(z)$; and $i = \sqrt{-1}$ is called the "imaginary unit." Hence, any polynomial equation with coefficients can be solved in the system of complex numbers; indeed, the system of complex numbers is the fundamental connection between geometry and algebra.

We picture the complex number $z = a + bi$ by putting a on the x-axis and b (or rather bi) on the y-axis.

The "modulus" or "absolute value" of $a + bi$ is $\sqrt{a^2 + b^2}$, and it is denoted by $mod(a + bi)$ or $|a + bi|$. The square of the modulus of a complex number $z = a + bi$ is called its "norm," and it is denoted by $Nm(z)$; so that, if $z = a + bi$, then $Nm(z) = a^2 + b^2$.

The "argument" of $z = a + bi$, denoted by $arg(z)$, is a quantity θ such that $cos\theta = \frac{a}{|z|}$ and $sin\theta = \frac{b}{|z|}$. It is many-valued and determined only up to multiples of 2π.

If we define an origin O and rectangular axes $X'OX$ and $Y'OY$, and mark the point P whose coordinates referred to these axes are x and y respectively, then the complex number $z = x + yi$ may be considered to be represented either by the point P or by the vector $\overrightarrow{OP}$, drawn to P from

the origin. The numbers 1 and $i = \sqrt{-1}$ will be, respectively, represented by points A and B on OX and OY (respectively) at unit distance from O. Therefore, $z = x + yi$ denotes the operation by which the vector $\overrightarrow{OA} = 1$ is converted into the vector $\overrightarrow{OP} = x + yi$. The modulus and the argument of $x + yi$ are, respectively, equal to the radius vector and the vectorial angle of P, assuming that OX is the initial line, and that the radian is the unit angle; so that, calling these r and θ, we obtain $x + yi = r(cos\theta + isin\theta) = re^{\theta i}$.

In particular, the number $i = \sqrt{-1}$ signifies a 90^o rotation about the real axis, turning 1 into -1. Hence, $i = \sqrt{-1}$ done twice, or squared, is equal to -1. From the perspective of real numbers, we are used to thinking of numbers on a line, but, with the introduction of imaginary numbers, we come up with two-dimensional numbers. Indeed, complex numbers are two-dimensional, since they are made up of the real axis and the imaginary axis, and we can transit from one to the other by rotating.

In 1833 at the Royal Irish Academy, the Irish mathematician and astronomer Sir William Rowan Hamilton presented the complex numbers as ordered pairs of real numbers, thus denoting a complex number by an ordered pair (a, b), and denoting the imaginary unit by $i = \sqrt{-1}$, so that $i^2 = (0,1) \cdot (0,1) = (-1,0) = -1$.

The zero of $\mathbb{C}$ is $(0,0)$, and the unit of $\mathbb{C}$ is $(1,0)$.

In fact, as Hamilton has originally shown, the complex number system $\mathbb{C}$ is the set $\mathbb{R} \times \mathbb{R}$ with operations defined as follows:

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$
$$(a + bi) - (c + di) = (a - c) + (b - d)i,$$
$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i, \text{ and}$$
$$\frac{(a + bi)}{(c + di)} = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}$$

where $a, b \in \mathbb{R}$, and $i = \sqrt{-1}$.

The (complex) "conjugate" of $a + bi$ is $a - bi$, and the conjugate of a complex number z is denoted by $\bar{z}$ or by z^*; so that, if $z = a + bi$, then: $z + \bar{z} = 2a$, $z - \bar{z} = 2ib$, $z\bar{z} = a^2 + b^2$.

If a function g takes real inputs and gives complex outputs, then the derivative with respect to its real input is computed by taking the derivatives of the real and the imaginary parts separately, namely:

$$\frac{dg}{dx} = \frac{dRe(g)}{dx} + i\frac{dIm(g)}{dx}$$

where $i = \sqrt{-1}$, $Re(g)$ is the real part of g, and $Im(g)$ is the imaginary part of g.

The integral of a complex function $f(x) = g(x) + ih(x)$ between the limits a and b is defined by

$$\int_a^b f(x)dx = \int_a^b [g(x) + ih(x)]dx = \int_a^b g(x)dx + i\int_a^b h(x)dx$$

where $i = \sqrt{-1}$, and x is a real variable. Obviously, the properties of such integrals may be deduced from the properties of the real integrals.

Chapter 11
Vector Calculus

When a function takes a real number and sends it to a vector, then it is said to be a vector-valued function. In the real plane, or in the xy-plane, the general form of a vector-valued function is the following:

$\vec{r}(t) = f(t)\hat{\imath} + g(t)\hat{\jmath}$; (1)

and, in the real 3-dimensional space, or in the xyz-space, the general form of a vector-valued function is the following:

$\vec{r}(t) = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$; (2)

where the component functions f, g, and h are real-valued functions of the parameter t, and $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are the corresponding unit vectors on the x-axis, the y-axis, and the z-axis respectively. The standard unit vectors in the direction of the x, y, and z axes of a 3-dimensional Cartesian coordinate system are

$\hat{\imath} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\hat{\jmath} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\hat{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

The "limit" of a vector-valued function $\vec{r}(t)$ is $\vec{L}$ as t tends to a, symbolically:

$$lim_{t\to a}\vec{r}(t) = \vec{L}$$

if and only if

$$lim_{t\to a}\left\|\vec{r}(t) - \vec{L}\right\| = 0.$$

Therefore, (1) implies that

$lim_{t\to a}\vec{r}(t) = [lim_{t\to a}f(t)]\hat{\imath} + [lim_{t\to a}g(t)]\hat{\jmath}$,

and (2) implies that

$lim_{t\to a}\vec{r}(t) = [lim_{t\to a}f(t)]\hat{\imath} + [lim_{t\to a}g(t)]\hat{\jmath} + [lim_{t\to a}h(t)]\hat{k}$,

provided that the limits of the component functions f, g, and h as $t \to a$ exist. Similarly, we can define the limit of a vector-valued function of n component functions for $n > 3$.

A vector-valued function $\vec{r}(t)$, where $t \in [a, b]$, is said to be "continuous" at a point $t_0 \in [a, b]$ if $lim_{t\to t_0}\vec{r}(t) = \vec{r}(t_0)$; and $\vec{r}(t)$ is said to be continuous on $[a, b]$ if it is continuous at every point of $[a, b]$.

The derivative of a vector-valued function $\vec{r}(t)$, where $t \in [a, b]$, is defined as follows:

$$\overrightarrow{r'}(t) = lim_{\Delta t\to 0}\frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t}$$

provided that the limit exists. If $\overrightarrow{r'}(t)$ exists, then $\vec{r}(t)$ is said to be differentiable at t. If $\overrightarrow{r'}(t)$ exists $\forall t \in (a, b)$, then $\vec{r}(t)$ is said to be differentiable on the interval (a, b). In order for $\vec{r}(t)$ to be differentiable

on $[a,b]$, $\vec{r}(t)$ must be differentiable on the interval (a,b), and the following two limits must exist as well:
$\overrightarrow{r'}(a) = lim_{\Delta t \to 0^+} \frac{\vec{r}(a+\Delta t)-\vec{r}(a)}{\Delta t}$ and
$\overrightarrow{r'}(b) = lim_{\Delta t \to 0^-} \frac{\vec{r}(b+\Delta t)-\vec{r}(b)}{\Delta t}$.
Consequently, (1) implies that
$\overrightarrow{r'}(t) = f'(t)\hat{\imath} + g'(t)\hat{\jmath}$,
and (2) implies that
$\overrightarrow{r'}(t) = f'(t)\hat{\imath} + g'(t)\hat{\jmath} + h'(t)\hat{k}$.
The properties of the derivative of a vector-valued function are analogous to those of the derivative of a scalar-valued function.
Let f, g, and h be integrable real-valued functions on $[a,b]$. Then (1) implies that the indefinite integral of a vector-valued function $\vec{r}(t) = f(t)\hat{\imath} + g(t)\hat{\jmath}$ is
$\int [f(t)\hat{\imath} + g(t)\hat{\jmath}]\, dt = [\int f(t)dt]\hat{\imath} + [\int g(t)dt]\hat{\jmath}$,
and the definite integral of a vector-valued function $\vec{r}(t) = f(t)\hat{\imath} + g(t)\hat{\jmath}$ is
$\int_a^b [f(t)\hat{\imath} + g(t)\hat{\jmath}]dt = \left[\int_a^b f(t)dt\right]\hat{\imath} + \left[\int_a^b g(t)dt\right]\hat{\jmath}$.
By analogy, (2) implies that
$\int \left[f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}\right] dt = [\int f(t)dt]\hat{\imath} + [\int g(t)dt]\hat{\jmath} + [\int h(t)dt]\hat{k}$,
and
$\int_a^b \left[f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}\right]dt = \left[\int_a^b f(t)dt\right]\hat{\imath} + \left[\int_a^b g(t)dt\right]\hat{\jmath} + \left[\int_a^b h(t)dt\right]\hat{k}$.
The properties of the integral of a vector-valued function are analogous to those of the integral of a scalar-valued function.
Let us consider a function $f(x,y)$; f depends on both x and y, and its graph is a surface in space. Then, in order to interpret and compute the rate of change of $f(x,y)$, we find the rate of change of $f(x,y)$ in a specific direction independently. If we want the rate of change in the x-direction, we differentiate $f(x,y)$ with respect to x while treating y as a constant. In other words, we compute the partial derivative $\frac{\partial f(x,y)}{\partial x}$. Similarly, if we want the rate of change in the y-direction, we differentiate $f(x,y)$ with respect to y while treating x as a constant. In other words, we compute the partial derivative $\frac{\partial f(x,y)}{\partial y}$. The "gradient" of $f(x,y)$ is denoted by ∇f, and it is a concept that combines the two aforementioned partial derivatives; specifically, the gradient of $f(x,y)$ is a vector consisting of both partial derivatives of f in their associated positions, symbolically:

$grad f \equiv \nabla f = \frac{\partial f(x,y)}{\partial x}\hat{\imath} + \frac{\partial f(x,y)}{\partial y}\hat{\jmath}$,

where $\hat{\imath}$ is the unit vector in the x-direction, and $\hat{\jmath}$ is the unit vector in the y-direction. By analogy, we can define the gradient of a function $f(x,y,z)$, etc. The gradient of a function is a vector field. A vector field V in $\mathbb{R}^3$ is a rule that assigns a tangent vector V_p to every point p of $\mathbb{R}^3$; and tangent vector V_p is said to be the value of the vector field V at the point p.

Let us consider a vector-valued function $\vec{r}(x,y,z) = f(x,y,z)\hat{\imath} + g(x,y,z)\hat{\jmath} + h(x,y,z)\hat{k}$ such that the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial g}{\partial y}$, and $\frac{\partial h}{\partial z}$ exist and are continuous on $U \subseteq \mathbb{R}^3$. Then the "divergence" of $\vec{r}(x,y,z)$ is a vector operator that operates on a vector field, producing a scalar field that gives the quantity of the vector field's source at each point; and it is defined as follows:

$div\vec{r} \equiv \vec{\nabla}\vec{r} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$.

In other words, the divergence of a function tells us how the corresponding vector field behaves towards or away from a point.

Chapter 12
Differential Equations and Mathematical Modelling

The Fundamental Theorem of Infinitesimal Calculus is a rigorous explanation of the dialectical relationship between integration and differentiation, and, thus, it is a major underpinning of the theory of differential equations.

By the term "ordinary differential equation," we refer to any equation that contains an unknown function, some of its derivatives, and an independent variable. The "order" of a differential equation is the order of the highest ordered derivative occurring in the given differential equation. The fundamental problem of the theory of differential equations is to find all of the functions $y = f(x)$ that satisfy some differential equation. Every function $y = f(x)$ that satisfies some differential equation is said to be a "solution" of the given differential equation.

A family of functions

$$y = f(x, c) \qquad (*)$$

where c is a constant belonging to $A \subseteq \mathbb{R}$, is said to be a "general solution" of a differential equation

$$y' \equiv \frac{dy}{dx} = F(x, y) \qquad (**)$$

if, for every $c \in A$, $(*)$ is a solution of $(**)$. The solution that we obtain for each particular value of c is said to be a "partial solution" of the differential equation $(**)$.

The theory of differential equations is a branch of mathematics in which the study of theoretical problems can hardly be distinguished from the study of practical problems, and dynamicity, which is a major characteristic of modern mathematics, is clearly manifested. Moreover, the theory of differential equations has played an important role in the transition from the eighteenth-century infinitesimal calculus to advanced mathematical analysis and modern geometry. One of the major advantages of differential equations is that they constitute one of the major underpinnings and instruments of the "mathematization" (i.e., of the "mathematical modelling") of many problems both in the context of the natural sciences and in the context of the social sciences.

Example: If $s(t)$ gives the position of a moving particle as a function of time, then velocity, $v(t)$, is given by the formula

$$s'(t) \equiv \frac{ds(t)}{dt} = v(t),$$

and acceleration, $a(t)$, is given by the formula

$v'(t) \equiv \frac{dv(t)}{dt} = a(t)$.

Suppose that we wish to study the motion of an object of mass m that is in free fall in vacuum (this hypothesis is a simplification of physical reality in order to facilitate the mathematization of this problem, which can easily lead to generalizations that, under certain conditions, provide satisfactory approximations of the actual state of affairs). Moreover, for reasons of simplicity, we shall assume that the orbit of the object's fall is so small in comparison with the radius of the Earth that (without a significant error in the conclusion) we can suppose that the object's weight (i.e., the force acting on it due to gravity) is constant. Then the object's acceleration is constant, too, and it is denoted by g (g is said to be the "gravitational acceleration," namely, the free fall acceleration of an object in vacuum; $g \approx 9.80\ m/sec^2$). In view of the foregoing, we have:

$v'(t) = -g$ (the negative sign indicates that the object's motion is accelerating downward), and

$\int v'(t)dt = \int(-g)\,dt \Rightarrow v(t) = -gt + c_1$, where c_1 is a constant.

The last equation gives the value of velocity if we know the constant c_1. Furthermore, we obtain:

$s'(t) = v(t) \Rightarrow \int s'(t)dt = \int(-gt + c_1)\,dt \Rightarrow s(t) = -\frac{1}{2}gt^2 + c_1 t + c_2$, where c_2 is a constant.

Hence, we can determine displacement, too, provided that we know c_2. In general, constants are determinable quantities.

In physics, constants are functions of the initial conditions of the phenomenon under investigation. For instance, in the aforementioned phenomenon of free fall in vacuum, we must take into consideration whether the object was left to fall, in which case its initial velocity is $v_0 = 0$, or whether it was given a non-zero initial velocity $v_0 = v(t_0)$. In any case, applying the formula of velocity for $t_0 = 0$, we obtain $v(t_0) = -g0 + c_1 \Rightarrow c_1 = v_0$, and, therefore, $v(t) = -gt + v_0$. By analogy, regarding displacement, we have: $s(t_0) = -\frac{1}{2}g0^2 + c_1 0 + c_2$, and, therefore, setting $s(t_0) = s_0$, we obtain $s(t) = -\frac{1}{2}gt^2 + v_0 t + s_0$, which is the formula of "uniformly accelerated motion."

The aforementioned results are based on the hypothesis that we study motion in vacuum. If, however, we decide to take account of the resistance of the Earth's atmosphere during the object's fall, then we must modify the aforementioned model as follows: we assume that a force due to the resistance of the Earth's atmosphere is applied on the moving object in the direction opposite to the object's motion (for which reason this force has a negative sign), and that this force is proportional to the moving object's

speed. In other words, for a suitable $k > 0$, this force is equal to $-kv(t)$. Then we assume that the total force that is applied on the moving body is $-mg - kv(t)$, that is, weight and air resistance. Consequently, according to Newton's Second Law of Motion (i.e., $Force = Mass \times Acceleration$), we obtain the linear differential equation

$$-mg - kv(t) = mv'(t) \Rightarrow v'(t) + \frac{k}{m}v(t) + g = 0,$$

so that, in this case, we must solve the given linear differential equation in order to find v.

Separation of Variables: If a differential equation may be written in the form

$$\frac{dy}{dx} = f(x)g(y),$$

then it is said to be solvable by "separation of variables" as follows:

$$\int \frac{dy}{g(y)} = \int f(x)\,dx.$$

Remark: In case we have a differential equation of the form

$$y^{(n)} = f(x) \Leftrightarrow \frac{d^n y}{dx^n} = f(x), \quad (1)$$

then, by integrating (1), we obtain

$$\frac{d^{n-1}y}{dx^{n-1}} = \int f(x)dx + c_1. \quad (2)$$

By setting $\int f(x)dx = f_1(x)$ and then integrating (2), we obtain

$$\frac{d^{n-2}y}{dx^{n-2}} = \int f_1(x)dx + c_1 x + c_2.$$

Repeating the same process, we obtain the general solution of (1), which is of the form

$$y = w(x) + \frac{c_1}{(n-1)!}x^{n-1} + \frac{c_2}{(n-2)!}x^{n-2} + \cdots + c_n,$$

meaning that the general solution of $y^{(n)} = f(x)$ can be obtained through n successive integrations.

For instance, let us find the general solution of the differential equation $x^2dy - ydx = 0$ as well as its partial solution that satisfies the condition $y(2) = 4$ (i.e., the integral curve that passes through the point $P(2,4)$). We shall apply the method of separation of variables:

$$x^2dy - ydx = 0 \Rightarrow \frac{dy}{y} = \frac{dx}{x^2} \Rightarrow \frac{dy}{y} = x^{-2}dx \Rightarrow \int \frac{dy}{y} = \int x^{-2}dx \Rightarrow lny = \frac{x^{-1}}{-1} + c \Rightarrow lny = -\frac{1}{x} + c \Rightarrow y = e^{-\frac{1}{x}+c} \Rightarrow y = e^c e^{-\frac{1}{x}} \Rightarrow y = ke^{-\frac{1}{x}},$$

which is the general solution of the given differential equation. In order to find the partial solution for which $x = 2 \Rightarrow y = 4$ (i.e., the integral curve that passes through the point $P(2,4)$), we must determine the constant k. If we substitute $x = 2$ and $y = 4$ into the general solution, then we obtain $4 = ke^{-\frac{1}{2}} \Rightarrow k = 4e^{\frac{1}{2}} = 4\sqrt{e}$. Hence, if we substitute this value of k into the

general solution, then we shall obtain the required partial solution, namely, $y = 4\sqrt{e}e^{-\frac{1}{x}}$.

Homogeneous Differential Equations: A differential equation is said to be "homogeneous" if it may be written in the form

$$f(x,y)dx + g(x,y)dy = 0, \quad (1)$$

where the functions $f(x,y)$ and $g(x,y)$ are homogeneous with respect to x and y of the same degree of homogeneity, meaning that

$f(x,y)$ may be written in the form $x^m A\left(\frac{y}{x}\right)$ and (2)

$g(x,y)$ may be written in the form $x^m B\left(\frac{y}{x}\right)$. (3)

Thus, due to (2) and (3), (1) becomes (for $x^m \neq 0$):

$$A\left(\frac{y}{x}\right)dx + B\left(\frac{y}{x}\right)dy = 0 \Rightarrow \frac{dy}{dx} = -\frac{A\left(\frac{y}{x}\right)}{B\left(\frac{y}{x}\right)},$$

which ultimately reduces to the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \Leftrightarrow y' = f\left(\frac{y}{x}\right), \quad (4)$$

where $f\left(\frac{y}{x}\right)$ is a homogeneous function whose degree of homogeneity is equal to zero. In order to find the general solution of (4), we set

$$\frac{y}{x} = w \Leftrightarrow y = xw \quad (5)$$

where w is a function of the independent variable x, that is, $w = w(x)$.

By differentiating (5), we obtain

$$dy = wdx + xdw,$$

and, after dividing by dx, we obtain

$$\frac{dy}{dx} = w + x\frac{dw}{dx}. \quad (6)$$

Therefore, due to (5) and (6), the original differential equation becomes

$$w + x\frac{dw}{dx} = f(w) \Rightarrow x\frac{dw}{dx} = f(w) - w \Rightarrow \frac{dw}{f(w)-w} = \frac{dx}{x}. \quad (7)$$

The differential equation (7), which is equivalent to (1), can be solved by the method of separation of variables. In particular, (7) gives

$$\int \frac{dw}{f(w)-w} = lnx + lnc, \text{ or } \int \frac{dw}{f(w)-w} = lncx, \text{ or } cx = e^{\int \frac{dw}{f(w)-w}}. \quad (8)$$

In (8), we have to compute the integral $\int \frac{dw}{f(w)-w}$ and then to make the substitution $w = \frac{y}{x}$ in order to ultimately find the general solution of (1).

For instance, let us solve the differential equation $(x^2 - y^2)dx + 2xydy = 0$. This differential equation is homogeneous, because the expressions $f(x,y) = x^2 - y^2$ and $g(x,y) = 2xy$ are homogeneous with respect to x and y, and their degree of homogeneity is 2. We set

$\frac{y}{x}=w\Leftrightarrow y=xw$ $(*)$

where $w=w(x)$. By differentiating $(*)$ with respect to x, we obtain

$y^{'}=w+xw^{'}$. $(**)$

Due to $(*)$ and $(**)$, the given differential equation becomes $(x^2-x^2w^2)+2x^2w(w+xw^{'})=0\Rightarrow x^2(1-w^2)+2x^2w(w+xw^{'})=0$, and, because, by $(*)$, $x\neq 0$, we divide the last expression by x^2 to obtain

$$(1-w^2)+2w(w+xw^{'})=0\Rightarrow 1-w^2+2w^2+2xw\frac{dw}{dx}=0\Rightarrow$$
$$1+w^2+2xw\frac{dw}{dx}=0\Rightarrow\frac{2wd}{w^2+1}=-\frac{dx}{x}\Rightarrow\int\frac{2wdw}{w^2+1}=$$
$$-\int\frac{dx}{x}\Rightarrow ln(w^2+1)=-lnx+lnc\Rightarrow ln(w^2+1)=ln\left(\frac{c}{x}\right)\Rightarrow$$
$$w^2+1=\frac{c}{x}.$$

By the substitution $w=\frac{y}{x}$, we realize that the general solution of the given differential equation is $y^2+x^2=cx$.

Differential Equations Reducible to Homogeneous Differential Equations: The differential equations of the form

$\frac{dy}{dx}=f\left(\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}\right)$, $(*)$

where a_1,b_1,c_1,a_2,b_2,c_2 are real constants, are reducible to homogeneous differential equations. In order to solve $(*)$ by reducing it to a homogeneous differential equation, we distinguish the following two cases:

Case I: If $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}\Leftrightarrow a_1b_2-a_2b_1\neq 0$, then we can find the general solution of $(*)$ as follows: We solve the system of equations

$$\left.\begin{matrix}a_1x+b_1y+c_1=0\\a_2x+b_2y+c_2=0\end{matrix}\right\}. \quad (1)$$

Let $(x,y)=(x_0,y_0)$ be the solution of (1). Then we set

$$\left.\begin{matrix}x=x_0+w\\y=y_0+v\end{matrix}\right\}, \quad (2)$$

and, by differentiating (2), we obtain

$$\left.\begin{matrix}dx=dw\\dy=dv\end{matrix}\right\}, \quad (3)$$

so that, by (2) and (3), the differential equation $(*)$ becomes

$\frac{dv}{dw}=f\left(\frac{a_1(x_0+w)+b_1(y_0+v)+c_1}{a_2(x_0+w)+b_2(y_0+v)+c_2}\right)\Rightarrow\frac{dv}{dw}=f\left(\frac{a_1x_0+b_1y_0+c_1+a_1w+b_1v}{a_2x_0+b_2y_0+c_2+a_2w+b_2v}\right)$.

But $a_1x_0+b_1y_0+c_1=0$ and $a_2x_0+b_2y_0+c_2=0$, because (x_0,y_0) is the solution of (1), and, therefore,

$\frac{dv}{dw} = f\left(\frac{a_1w+b_1v}{a_2w+b_2v}\right)$. (4)

The differential equation (4) is homogeneous with respect to v and w, and, in order to find its general solution, we set $\frac{v}{w} = z \Leftrightarrow v = wz$, where $z = z(w)$, and we work according to the method of solving homogeneous differential equations, which I have already explained. When we find the general solution of (4), we set $z = \frac{v}{w}$, and then, by (2), $w = x - x_0$ and $v = y - y_0$ in order to ultimately find the general solution of $(*)$.

Case II: If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \lambda \Rightarrow a_1b_2 - a_2b_1 = 0$, then we can find the general solution of $(*)$ as follows: Because $a_1 = \lambda a_2$ and $b_1 = \lambda b_2$, $(*)$ becomes

$\frac{dy}{dx} = f\left(\frac{\lambda(a_2x+b_2y)+c_1}{a_2x+b_2y+c_2}\right)$. (5)

We set $a_2x + b_2y = w$, where $w = w(x)$, and, by differentiating (5) with respect to x, we obtain $a_2 + b_2y' = w' \Leftrightarrow y' = \frac{1}{b_2}(w' - a_2)$, so that (5) becomes

$\frac{1}{b_2}\left(\frac{dw}{dx} - a_2\right) = f\left(\frac{\lambda w+c_1}{w+c_1}\right)$. (6)

The differential equation (6) can be solved by the method of separation of variables. When we find the general solution of (6), we set $w = a_2x + b_2y$.

First-Order Linear Differential Equations: The general form of these equations is

$\frac{dy}{dx} + Ay = B$, $(*)$

where A and B are functions of x, that is, $A = A(x)$ and $B = B(x)$. The general solution of $(*)$ is:

$y = e^{-\int Adx}\left(c + \int Be^{\int Adx}\,dx\right)$,

where c is an arbitrary constant.

Proof: If $B = 0$, then $(*)$ becomes $\frac{dy}{dx} + Ay = 0$, and it is said to be a homogeneous linear differential equation, which can be solved by separation of variables: $\frac{dy}{y} = -Adx \Rightarrow \int \frac{dy}{y} = -\int Adx \Rightarrow lny = -\int Adx + c \Rightarrow y = e^{-\int Adx+c} = e^ce^{-\int Adx} = ce^{-\int Adx}$, which is the general solution of the aforementioned homogeneous linear differential equation; and, if $c = 1$, then we obtain its partial solution $y_1 = e^{-\int Adx}$.

In order to find the general solution of $(*)$, we consider a new unknown function z of x such that

$y = y_1 z.$ (1)

By differentiating (1) with respect to x, we obtain

$y' = y_1' z + y_1 z'.$ (2)

Hence, by (1) and (2), the differential equation $(*)$ becomes

$y_1' z + y_1 z' + A y_1' z = B \Leftrightarrow (y_1' + A y_1) z + y_1 z' = B.$

But $y_1' + A y_1 = 0$, since y_1 is a partial solution of $\frac{dy}{dx} + Ay = 0$, and, therefore,

$y_1 z' = B \Rightarrow e^{-\int A dx} z' = B \Rightarrow z' = B e^{\int A dx} \Rightarrow z = \int B e^{\int A dx} dx + c.$

Because $y_1 = e^{-\int A dx}$ and $z = \int B e^{\int A dx} dx + c$, equation (1) gives the general solution of $(*)$, which is $y = e^{-\int A dx}\left(c + \int B e^{\int A dx}\, dx\right)$.■

Linearization of Nonlinear Differential Equations: Problems of nonlinear analysis started to exist ever since the creation of the universe. Some of them were solved by ancient Greek mathematicians, but many new nonlinear problems were created, both in pure mathematics and in other sciences, such as biology, physics, astronomy, economics, etc. The distinction between linear and nonlinear analysis is not quite clear, because a considerable part of information about a nonlinear system can be extracted from a linear approximation of the corresponding nonlinear problem. Moreover, it is often possible to extract information about the solution of a linear system from a relevant nonlinear one. The term "linearization" of a nonlinear differential equation refers to the reduction of a nonlinear differential equation to a linear differential equation that is either equivalent or almost equivalent to the given nonlinear differential equation, that is, the solution of the linear differential equation may give the solution of the nonlinear differential equation either exactly or approximately within an acceptable error. Two well-known examples of linearization of nonlinear differential equations are the following:

i. The Bernoulli equation:
$\frac{dy}{dx} + Ay = By^n,$ (1)
where A and B are functions of x, and $n \in \mathbb{R} - \{0,1\}$ (if $n = 0$, then the equation is linear; if $n = 1$, then the equation can be solved by separation of variables). Multiplying both sides of (1) by y^{-n}, we obtain
$y^{-n} \frac{dy}{dx} + A y^{1-n} = B.$ (2)
Let $y^{1-n} = w,$ (3)
where $w = w(x)$. By differentiating (3) with respect to x, we obtain

$$(1-n)y^{-n}\frac{dy}{dx}=\frac{dw}{dx}\Leftrightarrow y^{-n}y'=\frac{w'}{1-n}. \quad (4)$$

Hence, (2), due to (3) and (4), yields

$$\frac{w'}{1-n}+Aw=B\Rightarrow\frac{dw}{dx}+(1-n)Aw=(1-n)B, \quad (5)$$

which is a linear differential equation (whose dependent variable is w), and it can be solved according to the aforementioned method of solving linear differential equations. When we find the general solution of (5), we set $w=y^{1-n}$, according to (3), and, thus, we obtain the general solution of (1).

ii. The Riccati equation:

$$\frac{dy}{dx}=A+By+Cy^2, \quad (1)$$

where A, B, and C are functions of x. We can find the general solution of the Riccati equation only if we know one of its partial solutions. Suppose that $y=y_1$ is a partial solution of (1), so that

$$\frac{dy_1}{dx}+A+By_1+Cy_1^2=0. \quad (2)$$

Then, using the transformation

$$y=y_1+w, \quad (3)$$

where $w=w(x)$, and differentiating (3) with respect to x, we obtain

$$\frac{dy}{dx}=\frac{dy_1}{dx_1}+\frac{dw}{dx}. \quad (4)$$

Hence, (1), due to (3) and (4), yields

$$\frac{dy_1}{dx}+A+By_1+Cy_1^2+\frac{dw}{dx}+(B+2Cy_1)w+Cw^2=0.$$

But, due to (2), $\frac{dy_1}{dx}+A+By_1+Cy_1^2=0$, so that we obtain

$$\frac{dw}{dx}+(B+2Cy_1)w=-Cw^2, \quad (5)$$

which is a Bernoulli equation (where w is the dependent variable), and it can be solved according to the aforementioned method of solving the Bernoulli equation: multiplying both sides of (5) by w^{-2}, we obtain

$$w^{-2}\frac{dw}{dx}+(B+2Cy_1)w^{-1}=-C. \quad (6)$$

If we set

$$w^{-1}=z, \quad (7)$$

where $z=z(x)$, and if we differentiate (6) with respect to x, then we obtain

$$-w^{-2}w'=z'. \quad (8)$$

Therefore, (6), due to (7) and (8), yields

$$-z'+(B+2Cy_1)z=-C\Leftrightarrow z'-(B+2Cy_1)z=-C, \quad (9)$$

which is a linear differential equation (where z is the dependent variable), and its general solution is

$$z = e^{-\int -(B+2Cy_1)dx}\left(c + \int Ce^{\int -(B+2Cy_1)dx}\,dx\right).$$

By substituting this value of z into (7), we find w, and, by substituting the so found value of w into (3), we find the general solution of the Riccati equation (1).

An example of the application of differential equations in the discipline of economics: By the term "Gross Domestic Product," we mean the total monetary value of all final goods and services produced (and sold on the market) within a country during a period of time (typically one year). The formula for calculating Gross Domestic Product (GDP) is the following:

$$GDP = private\ consumption + gross\ private\ investment + government\ investment + government\ spending + (exports - imports)$$

and the term "gross" indicates that products are counted regardless of their subsequent use (a product can be used for consumption, for investment, or to replace an asset). Nominal GDP uses current prices in its measure. Real GDP is an inflation-adjusted measure of the total monetary value of all final goods and services produced (and sold on the market) within a country during a period of time (typically one year):

$$Real\ GDP = \frac{Nominal\ GDP}{GDP\ Deflator}$$

(for instance, if an economy's prices have increased by 1% since the base year that is used in order to calculate the Real GDP, then the GDP Deflator is equal to 1.01).

Suppose that GDP grows at some constant rate g over time t. Then, if y denotes the current level of GDP in the economy, we can express the relationship as the following differential equation:

$$\frac{dy}{dt} \equiv y' = g \cdot y(t) \Leftrightarrow \frac{dy}{dt} - g \cdot y(t) = 0.$$

This is a first-order linear differential equation, and, therefore, as shown before, its solution is

$$y(t) = ce^{gt}.$$

The business cycle model shows how a country's real GDP fluctuates over time, and the typical business cycle has four phases, which progress as follows: (i) expansion (when real GDP is increasing and unemployment is decreasing), (ii) peak (the turning point in the business cycle at which real GDP stops increasing and starts decreasing), (iii) recession (when real GDP is decreasing and unemployment is increasing), and (iv) trough (the

turning point in the business cycle at which real GDP stops decreasing and starts increasing). Obviously, these phases can be mathematically modelled using the method provided by differential calclulus to study the behavior of a function.

Regarding price fluctuations, the cobweb model is a model that explains why prices may be subjected to periodic fluctuations in certain types of market. Let $x_d(t)$ denote the demand for product x at time t, and let $p(t)$ denote the price of x at time t. We assume that

$$x_d(t) = f\big(p(t)\big). \tag{1}$$

Let $x_s(t)$ denote the supply of product x at time t. We assume that

$$x_s(t) = g\big(p(t-\theta)\big) + u(t) \tag{2}$$

where θ is positive, and $u(t)$ denotes some external force that independently influences $x_s(t)$. For instance, $u(t)$ may be be some influence from another economic sector that is independent of the one considered here, or it may be some weather or socio-political factor. For market clearance (equilibrium) at time t, $x_d(t)$ must be equal to $x_s(t)$. If $x(t)$ is the quantity of x at which the market clears at time t, equations (1) and (2) and the market-clearance (equilibrium) condition lead us to the following equation:

$$x(t) = G\big(x(t-\theta)\big) + u(t). \tag{3}$$

The demand curve, depicting demand as a function of price, typically slopes downward from left to right, indicating that, as the price of a good or a service rises, the demand for it falls. The supply curve, depicting supply as a function of price, typically slopes upward from left to right, indicating that, as the price of a good or a service rises, the supply of it increases. The equilibrium price is where the supply of goods/services matches demand. Given the typical shapes of the demand curve and the supply curve, the first derivative of the function G in equation (3) is negative. If $u(t)$ is a constant, independent of t, then equation (3) is the standard cobweb model, and, in this model, there could be business cycles of period 2θ.

An example of the application of differential equations in the discipline of biology: Let us consider the spread of a disease through the population. Suppose that we have a number of people, say N, who are infected with a disease. We want to know how N will change in time. Hence, N is a function of t, which denotes time. Each of the N people has a certain probability to spread the disease to other people during some period of time. Let us quantify this infectiousness by using a constant k, so that the change in the number of infected people with respect to time equals this constant k times the number of people who are already infected. In

general, the change of a function with respect to time is the derivative of that function with respect to time. Therefore, we obtain the following differential equation:
$\frac{dN(t)}{dt} = k \cdot N(t) \Leftrightarrow \frac{dN(t)}{dt} - k \cdot N(t) = 0,$
and, therefore,
$N(t) = N_0 e^{kt},$
where N_0 is the number of the infected people at the initial time ($t = 0$), and the probability of infecting someone appears in the exponent (kt). Thus, we understand why infectious diseases begin by speading exponentially (since the number of infected people is proportional to the number of people who are already infected). When a disease begins to spread, the constant k in the aforementioned exponent is

$$k = \frac{R_0 - 1}{\tau}$$

where τ is the time an infected person remains infectious, and R_0 denotes the average number of people someone infects.

An example of the application of differential equations in the study of species competition (the competitive Lotka–Volterra equations): The problem of the growth of two species competing for the same resources has signigant applications in biology, ecology, and economics. Consider two mixed populations of species that are mutually interdependent and compete for the same resources. Let N_1 and N_2 denote the number of individuals of species one and of species two, respectively. Both N_1 and N_2 are functions of time t. Then we obtain the following logistic equations for these two species:

$$\frac{dN_1}{dt} = a_1 N_1 \left(1 - \frac{N_1}{M_1}\right)$$

$$\frac{dN_2}{dt} = a_2 N_2 \left(1 - \frac{N_2}{M_2}\right)$$

which are uncoupled equations, and $N_1 \to M_1$ and $N_2 \to M_2$. However, we have to model the competition between these two populations. If N_1 is much smaller than M_1, and if N_2 is much smaller than M_2, then resources are plentiful, and these two populations, N_1 and N_2, grow exponentially with growth rates a_1 and a_2, respectively. If species one and species two compete, then the growth of species one reduces resources available to species two, and vice versa. Because we do not know the exact impact species one and species two have on each other, we introduce two additional parameters to model competition. In particular, let q_{12} and q_{21}

be dimensionless parameters (constants) that respectively model the consumption of species one's resources by species two, and vice versa (for instance, if both species eat exactly the same food, but species two consumes twice as much as species one, then $q_{12} = 2$ and $q_{21} = 0.5$). Then we can modify and couple the two aforementioned logistic equations as follows:

$$\frac{dN_1}{dt} = a_1 N_1 \left(1 - \frac{N_1 + q_{12} N_2}{M_1}\right)$$

$$\frac{dN_2}{dt} = a_2 N_2 \left(1 - \frac{q_{21} N_1 + N_2}{M_2}\right)$$

and, as time increases, the solution of this model (system of differential equations), which starts at (N_1^*, N_2^*), approaches a point (N_1^T, N_2^T), so that one of the following cases holds: (i) the point (N_1^T, N_2^T) lies in the fully positive quadrant of the x, y-plane, so that both N_1^T and N_2^T are positive, which means that the species co-exist; (ii) the point $(N_1^T, N_2^T) = (0,0)$, which indicates extinction of both species; or (iii) one of N_1^T and N_2^T may be zero and the other positive, indicating a situation of competitive exclusion.

An example of the application of differential equations in the discipline of strategic studies: In warfare problems, the calculation of a force ratio may be achieved by simple rules or may include complex assumptions and subjective judgments. For the quantitative study of a force ratio, the following three variables are difficult to handle: (i) the disparity in number and lethality of weapons between similar organizations; (ii) the variations in concepts of combat support; and (iii) the concentration of forces.

Courtney S. Coleman (Department of Mathematics, Harvey Mudd College, Claremont, CA), following the warfare modelling approach of the English polymath and engineer Frederick W. Lanchester, studied some model problems on a mixed conventional-guerilla combat, such as Vietnam.

Let $x(t)$ and $y(t)$ denote respective strengths of the forces at time t, where t is measured in days from the start of the combat. We shall identify the strengths with the numbers of combatants. We shall consider the ideal case where $x(t)$ and $y(t)$ are differentiable functions of time. Then Coleman states that, even though we may not have a specific formula for $x(t)$ as a function of time, we may have sufficient information about the operational loss rate (OLR) of the x-force (i.e., the loss rate due to inevitable diseases, desertions, and other non-combat mishaps), the

combat loss rate (CLR), due to encounters with the y-force, and the reinforcement rate (RR). Hence,
$\frac{dx(t)}{dt} = OLR + CLR + RR$,
and a similar equation applies to the y-force.
Lanchester assumed that the loss rate of a force is directly proportional to the enemy force strength. The following three Lanchesterian models are of great significance; $x(t)$ and $y(t)$ denote the strengths of the opposing forces at time t, and t denotes time from the start of the combat (you may add reinforcement rates $P(t)$ and $Q(t)$ per day if relevant).

Model I: Conventional Combat (CONCOM):

$$\begin{cases} \frac{dx(t)}{dt} = -by(t) \\ \frac{dy(t)}{dt} = -cx(t) \end{cases}$$

where the constants are non-negative loss rate constants, b denotes the effectiveness of y and c denotes the effectiveness of x. Solving these equations gives

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{cx(t)}{by(t)}$$

so that (given that we have a separable equation, meaning that we can separate variables and integrate each side) we obtain Lanchester's Square Law:

$$b\int_{y_0}^{y} ydy = c\int_{x_0}^{x} xdx \Rightarrow b[y(t)^2 - y_0(t)^2] = c[x(t)^2 - x_0(t)^2]$$
$$\Rightarrow by(t)^2 - cx(t)^2 = by_0(t)^2 - cx_0(t)^2 \equiv K$$

where K is a constant. Therefore, x wins if $K < 0$, y wins if $K > 0$, and a stalemate (equilibrium) occurs if $K = 0 \Leftrightarrow by_0(t)^2 = cx_0(t)^2 \Leftrightarrow \frac{c}{b} = \left(\frac{y_0(t)}{x_0(t)}\right)^2$.

Model II: Guerilla Combat (GUERCOM):

$$\begin{cases} \frac{dx(t)}{dt} = -ax(t)y(t) \\ \frac{dy(t)}{dt} = -bx(t)y(t) \end{cases}$$

and solving these equations gives

$$\frac{dy}{dx} = \frac{dy/_{dt}}{dx/_{dt}} = \frac{bx(t)y(t)}{cx(t)y(t)} = \frac{b}{a}$$

so that (given that we have a separable equation, meaning that we can separate variables and integrate each side) we obtain the following Linear Law:

$$y(t) - y(0) = \frac{b}{a}[x(t) - x(0)] \Rightarrow ay(t) - bx(t) = ay(0) - bx(0) \equiv K$$

where K is a constant. In this case, a stalemate (equilibrium) occurs if $K = 0 \Leftrightarrow \frac{b}{a} = \frac{y(0)}{x(0)}$.

Model III: Mixed Guerilla-Conventional Combat (e.g., Vietnam War):

$$\begin{cases} \dfrac{dx(t)}{dt} = -ax(t)y(t) \\ \dfrac{dy(t)}{dt} = -bx(t) \end{cases}$$

where y is the conventional force (out in the open), and x is the guerilla force (hard to find). Solving for $\frac{dy(t)}{dx(t)}$, we obtain

$$\frac{dy}{dx} = \frac{dy/_{dt}}{dx/_{dt}} = \frac{-bx(t)}{-ax(t)y(t)} = \frac{b}{ay(t)}$$

which is again a separable equation, meaning that we can separate variables and integrate each side, obtaining the following Parabolic Law:

$$a\int_{y_0}^{y} ydy = b\int_{x_0}^{x} dx \Rightarrow \frac{a}{2}(y^2 - y_0^2) = b(x - x_0) \Rightarrow ay^2 - 2bx = ay_0^2 - 2bx_0 \equiv K$$

(notice that these are parabolas). Therefore, y wins if $K > 0$, x wins if $K < 0$, and a stalemate (equilibrium) occurs if $K = 0$.

An example of the application of differential equations in optimal control and differential game models (Dagobert L. Brito's dynamic model of an armaments race): Each nation chooses optimal rates of new arms acquisition and consumption given a specific estimate of its opponent's military spending. Specifically, nation X determines an index of security S_X defined as follows:

$$S_X = S_X(M_X, M_Y)$$

where M_X and M_Y denote the arms stocks (or military budgets) of nations X and Y, respectively. Hence, this model relates X's perception of its security to the respective arms stocks of nations X and Y. We assume that

the index S_X increases (resp. decreases) with M_X (resp. M_Y) at a decreasing rate (there are marginally diminishing returns to a nation building up its arms relative to another).
The social utility to nation X at any point in time t is denoted by U_{Xt} and is defined as follows:

$$U_{Xt} = u(C_X, S_X)$$

where C_X denotes X's level of consumption, and S_X denotes X's index of security (defined as above). Once it has been postulated that the norm relating means to ends is that of "intrinsic rationality," the maximization of a utility index is implied. The objective for nation X is to maximize a welfare function (i.e., an integral) that gives the discounted present value of all future utility levels, namely:

$$W_X = \int_0^{\infty} e^{-rt}\, u(C_X, S_X)dt$$

so that the welfare of nation X, specifically, W_X, is obtained by adding (integrating) the contributions to welfare at each instant of time over all time periods from the present ($t = 0$). For a scientifically rigorous study of the concept of social welfare, see: Kenneth Arrow, *Social Choice and Individual Values*, second edition, New York: John Wiley and Sons, 1963.
The aforementioned welfare integral, W_X, is maximized by the choice of an additional investment in arms, Z_X, and the level of consumption, C_X, under the following constraints:

$$M_X' = Z_X - k_X M_X \quad (1)$$

$$P_X = C_X + Z_X \quad (2)$$

where M_X' represents the rate of change in the arms stock of nation X, constraint (1) implies that the increase in X's arms stock is equal to its gross investment in new arms, Z_X, minus the cost of operating its old ones ($k_X M_X$), and constraint (2) implies that X's gross national product (P_X) is divided between consumption (C_X) and the additional investment in arms (Z_X). For a systematic study of optimal control methods, see: Michael Athans and P. L. Falb, *Optimal Control*, New York: Dover, 2007.

An example of the application of differential equations in the discipline of psychophysics: The German physician Ernst Heinrich Weber (1795-1878) is considered one of the founders of experimental psychology. He asked people to make yes–no judgments about whether the perceived weights of two objects were different. In particular, he held the mass of the first object constant, and he gradually increased the mass of the second object until people said "yes" (i.e., "different"). Thus, he managed to define the "just noticeable difference," namely, the smallest increase in weight ΔI

that a person could detect, and he found that it was not a constant but instead a function of the weight I of the first object; symbolically:
$\Delta I = kI$.
Moreover, Weber found that the value of k, which determined the "just noticeable difference," was a constant for most values of I, thus establishing what psychologists now refer to as *Weber's law*, which holds for a wide range of intensities I and across different stimulus modalities.

The German physicist, philosopher, and experimental psychologist Gustav Theodor Fechner (1801-87), who is the acknowledged founder of the scientific field of psychophysics and the first true mathematical psychologist, employed Weber's law by assuming that the psychological experience of a "just noticeable difference" is the same for all values of I. Hence, if the change in the psychological effect $\Delta S = c$ is equal to the same constant c for every noticeable difference ΔI, then

$$\frac{\Delta S}{\Delta I} = \frac{c}{kI}$$

and, using differential calculus, we obtain the following differential equation:

$$dS = \frac{c}{kI}\,dI$$

(*Fechner's law*). Therefore, solving the aforementioned differential equation, we realize that, according to *Fechner's law*, psychological effects S are a logarithmic fuction of physical intensity I; symbolically:

$$S = K \cdot lnI$$

for some constant K. In this way, Fechner pioneered the mathematical study of perception, stimulus discrimination, and response selection.

An example of the application of differential equations in the discipline of neuroscience: The brain (the central nervous system) contains nerve cells that are highly specialized in transmitting messages. Each nerve cell, called a neuron, consists of the soma (i.e., the central body of the cell), the (neur)axon, and the dendrites. At the end of the neural tube, there is a special structure called a synapse, through which the neurons communicate with each other. When a message created in one neuron is about to be transmitted to the next, the first neuron releases specialized chemicals called neurotransmitters. The released neurotransmitters are taken up by specially shaped regions, called receptors, on the cell membrane of the next neuron involved in the particular synapse
Neurons send signals along an axon to a dendrite through junctions called synapses. The standard Leaky Integrate and Fire (LIF) equation is a point

neuron model that helps us to represent and study the dynamics of the neuron, and it is given as follows:

$$V'(t) = \frac{1}{C}\left[I_e(t) - \frac{1}{R}(V(t) - E_L)\right]$$
$$V(t) \leftarrow V_r, if\ V(t) > \Theta$$

where:
$V(t)$ denotes the membrane potential (i.e., the difference in electric potential between the interior and the exterior of a biological cell; in other words, the difference in the energy required for electric charges to move from the internal to the exterior cellular environments and vice versa);
C denotes membrane capacitance (parameter) and is proportional to the cell surface area;
E_L denotes resting potential (parameter) and is the electrical potential difference across the cell membrane when the cell is in a non-excited state;
I_e denotes the trans-membrane current (an excitatory synaptic input initiates a current flow across the membrane and into the neuron, and this current consists of an ionic flow of positive ions (e.g., $Na+$) in addition to capacitive currents, and it is by convention a negative trans-membrane current; this changes the membrane potential at the location of the synaptic input, initiating axial currents, that is, currents inside the neuron);
V_r denotes reset membrane potential (transmission of a signal within a neuron (from dendrite to axon terminal) is carried by a brief reversal of the resting membrane potential called an "action potential," and, when neurotransmitter molecules bind to receptors located on a neuron's dendrites, ion channels open, so that, at excitatory synapses, this opening allows positive ions to enter the neuron and results in "depolarization" of the membrane, that is, a decrease in the difference in the voltage between the inside and the outside of the neurone; a stimulus from a sensory cell or another neuron depolarizes the target neuron to its threshold potential (e.g., $-55mV$), and $Na+$ channels in the axon hillock open, allowing positive ions to enter the cell, and, once depolarization is complete, the cell must now "reset" its membrane voltage back to the resting potential by closing the $Na+$ channels);
Θ denotes firing threshold (i.e., the level that a depolarization must reach for an action potential to occur, and, in most neurons, the threshold is around $-55mV$ to $-65mV$; if the neuron does not reach this critical threshold level, then no action potential will fire);
t denotes time.

Chapter 13
Integral Equations

Differential equations typically encode the local behavior of a system, whereas integral equations typically encode the global behavior of a system. By the term "integral equation," we mean an equation in which the unknown function $u(x)$ to be determined occurs under an integral sign. Hence, the standard form of an integral equation is the following:

$$u(x) = f(x) + \int_{a(x)}^{b(x)} k(x,t) \cdot u(t)\, dt$$

where $k(x,t)$ is called the kernel of the integral equation. We can solve them using the methods of infinitesimal calculus.

Suppose that we have to solve the equation $y' = f(x,y)$ satisfying the initial condition $y(x_0) = y_0$. This problem reduces to the solution of the integral equation

$$y(x) = \int_{x_0}^{x} f(x,y)\, dx + y_0$$

(it is worth pointing out that the conversion of a differential equation to an integral equation is very useful in approximation theory; see: Richard Bellman, *Perturbation Techniques in Mathematics, Engineering & Physics*, New York: Dover, 2003).

For instance, consider the following integral equation:

$y(x) = 2 + \int_2^x [t - ty(t)]\, dt.$ (1)

Using the Fundamental Theorem of Infinitesimal Calculus and differentiating (1) with respect to x, we obtain:

$\frac{dy(x)}{dx} = \frac{d2}{dx} + \frac{d}{dx}\int_2^x [t - ty(t)]\, dt = 0 + x - xy \Rightarrow \frac{dy(x)}{dx} = x - xy \Rightarrow \frac{dy(x)}{dx} = x(1-y) \Rightarrow \frac{1}{1-y}dy = xdx \Rightarrow \int \frac{1}{1-y}dy = \int xdx \Rightarrow -ln|1-y| + c_1 = \frac{x^2}{2} + c_2 \Rightarrow -ln|1-y| = \frac{x^2}{2} + C$, where $C = c_2 - c_1$. In order to find C, we work as follows: given that the lower limit of the integral in (1) is 2, let us set $x = 2$, so that (1) implies that $y(x) = 2$ (notice that, whenever the upper and the lower limits of an integral are the same, the integral is zero). Thus, $-ln|1-2| = \frac{2^2}{2} + C \Rightarrow -ln(1) = 2 + C \Rightarrow 0 = 2 + C \Rightarrow C = -2$. Therefore, we obtain:

$-ln|1-y| = \frac{x^2}{2} - 2 \Rightarrow ln|1-y| = 2 - \frac{x^2}{2} \Rightarrow |1-y| = e^{2-\frac{x^2}{2}} \Rightarrow 1 - y = \pm e^{2-\frac{x^2}{2}} \Rightarrow y = 1 \mp e^{2-\frac{x^2}{2}}$.

Chapter 14
Mathematical Physics: Mechanics, Relativity, Conservation of Energy, Thermodynamics, Electromagnetism, Light, and Quantum Physics

The subject of physics very much depends on mathematics. Whereas mathematicians are concerned only with the structure of reasoning, physicists endow mathematical expressions with physical meanings. Therefore, they create mathematical models of the physical world.

The structure and the properties of atoms, molecules and, in general, of all ordinary matter are due to primarily electrical interactions between electrically charged particles. The fundamental building blocks of ordinary matter are the negatively charged "electron," the positively charged "proton," and the uncharged "neutron." In a neutral atom, the number of electrons equals the number of protons that exist in the nucleus, and the net electrical charge is zero. If one or more electrons are removed (resp. added), then the remaining positively (resp. negatively) charged structure is called a "positive ion" (resp. a "negative ion").

From the perspective of quantum mechanics, particles are discrete packets, "quanta," of energy with wave-like properties. In other words, according to quantum mechanics, energy is not continuous, but it is always parceled up into some tiny discrete "lump" (which is what "quantum" literally means: a discrete thing). In essence, an electron is a circular standing wave.

The term "quantum" derives from the Latin language, and it means an amount of something. In the context of quantum mechanics, the term "quantum" means the smallest amount of energy that can be measured. The central concept of quantum physics is that of a wave, here meaning a disturbance or oscillation that travels through space-time accompanied by a transfer of energy. The basic properties of a wave are its amplitude (i.e., the distance from the center line, that is, the still position, to the top of a crest or the bottom of a trough), its frequency (i.e., the number of cycles occurring per second; specifically, it can be measured by counting the number of crests of waves that pass a fixed point in one second), and its length (i.e., the distance over which the wave's shape repeats; for instance, the distance between two adjacent crests). From the perspective of quantum mechanics, the concept of a physical system is equivalent to the concept of a state. This, in turn, is a vector in a multi-dimensional geometric space that allows length and angle to be rigorously measured, specifically, in a Hilbert space.

Continuing in the context of quantum mechanics, a molecule can be thought of like a mountain range (described by a wave-function) filled with infinitely many energy steps, where each energy step, representing a quantum of energy, is a quantum state. A molecule stands on one of these quantum states, and all the other infinitely many quantum states are empty, they are virtual states. Moreover, each quantum state is characterized by a wave form. When a system stands on one of these states, the other states also exist, but potentially. This means that they cannot be observed, and they actually look empty. Those virtual states are potential modes of being, by virtue of which a molecule can jump into other quantum states. Due to Heisenberg's uncertainty principle, we know that molecules can make "quantum jumps," because they have empty states into which they can jump.

In simple terms, to construct an atom, one needs some protons and neutrons for the construction of the nucleus, and then one has to put some electrons around the nucleus until the whole system is electrically neutral (in fact, once you have a positively charged nucleus, it attracts electrons, which automatically form shells around the nucleus). However, it should be mentioned that the construction of an atomic nucleus is a complex process, because protons, being positively charged, repel each other. As a result, they have to come very close to each other in order for the nuclear force to start operating and, thus, keep them together, given that there exist sufficiently many neutrons. This process requires extremely high temperatures (hundreds of millions of degrees Kelvin). Such high temperatures existed briefly after the Big Bang.

According to the "Bing-Bang" cosmological model, gravity underpinned and, actually, determined the transition from the "Bing-Bang" cosmological "soup" to the galactic structure that we observe today: gravity started from the initial conditions of the Big Bang and made the universe much more complex because, even though the density of the universe was almost uniform, there were density quantum-mechanical fluctuations. Put slightly differently, there were small differences in the density of the universe from one region to another. Thus, a region of the universe with density slightly greater than the mean density of the universe acted upon itself by its own gravity and, gradually, it made itself denser. Consequently, instead of expanding with the rest of the universe, it drew matter into the given region. Ultimately, this region collapsed upon itself and did not participate in the universal expansion. In this way, a physical object was made out of such a region. Gradually, the universe was filled with small density inhomogeneities resulting from inflation due to

quantum-mechanical fluctuations, which ultimately merged into the structures of the universe that we observe today.

Antimatter is exactly the same as matter, except that the particles that make up antimatter have exactly the opposite charge to the particles that make up matter. We could say that antimatter is the mirror image of matter, except that it has an opposite charge. For example, for every proton, there is an antiproton; for every electron, there is an antielectron, etc. Let us compare the electron with the antielectron, also called the positron. Then we find the following:

Electron (a negative one elementary electric charge): e^{-}.

Positron or antielectron (a positive one elementary electric charge): e^{+}.

The mass of the electron is the same as the mass of the positron: $m_{e^-} = m_{e^+}$.

The spin of the electron is the same as the spin of the positron: $Spin_{e^-} = Spin_{e^+}$.

The electron charge and the positron charge are exactly opposite to each other:

$q_{e^-} = -1.6 \times 10^{-19} C$ and $q_{e^+} = +1.6 \times 10^{-19} C$,

where C denotes coulomb (the standard unit of electric charge).

The discovery of antimatter is due to the British theoretical physicist Paul Dirac. In 1928, trying to bridge quantum theory with Albert Einstein's theory of relativity, he proposed the Dirac equation—namely, a relativistic equation of motion for the wave function of the electron, thus describing the behavior of an electron when it is moving at relativistic speeds. At this point, let us recall from mathematics that the equation $x^2 = 1$ has two solutions—namely: $+1$ and -1. Similarly, Dirac's equation has two solutions: according to the first solution, the energy of the electron is a positive number, but, according to the second solution, the energy of the electron is a negative number. However, this mathematical result cannot hold in nature, because, according to classical physics, energy is always a positive number. For this reason, Dirac interpreted the results of his famous equation as follows: the particle that appears to carry the positive energy is an electron with its negative charge, whereas the particle that, according to the solutions of the Dirac equation, appears to carry a negative amount of energy is the electron with the opposite charge—namely, the antielectron. Therefore, mathematics suggested the existence of antimatter. Just as the equation $x^2 = 1$ has two solutions (namely, $+1$ and -1), so the Dirac equation suggests the existence of a particle and an antiparticle. The observation of antimatter—that is, the practical confirmation of its existence—was achieved in 1932 by the American physicist Carl Anderson, who, for this reason, shared the 1936

Nobel Prize in Physics with Victor Hess. In particular, using a cloud chamber (that is, a particle detector used for visualizing the passage of ionizing radiation), Anderson observed and photographed a positron (specifically, he recorded a particle coming from the bottom and curving to the left, indicating a positive charge, and this particle had the mass of an electron, meaning that it was a new kind of particle, a positive electron, or a positron). What happens when matter comes into contact with antimatter? When a particle (e.g., an electron) comes into contact, "collides," with its respective antiparticle (e.g., a positron), they annihilate each other (that is, they cancel each other out), leaving behind an amount of energy (according to Albert Einstein's formula $m = \frac{E}{C^2}$, where m denotes the mass of the corresponding particle-antiparticle pair, E denotes energy, and c^2 denotes the speed of light squared). Hence, the creation of the universe was made possible due to a fundamental asymmetry between matter and antimatter—that is, the amount of matter exceeded the amount of antimatter, and, thus, the amount of matter that survived the interaction with antimatter continued to create the universe.

Energy, Force, and Work

By the term "energy," we mean the impetus that underpins all motion and all activity—more specifically, the capacity for doing work. In physics, we typically look at the work that a constant force, F, does when moving an object over a distance of s. In these cases, the work is

$W = Fs$;

the force is parallel to the displacement.

But most forces are not constant and depend upon where exactly the force is acting. Therefore, assuming that the force at any x is $F(x)$, the work done by the force in moving an object from $x = a$ to $x = b$ is given by

$W = \int_a^b F(x)dx$,

where the force is parallel to the displacement.

Mechanics is the branch of physics that studies the relationships between the following three physical concepts:

i. *Force:* an agent that changes or tends to change the state of motion (i.e., the state of rest or of uniform motion) of an object. The "velocity" of an object is the rate of change of its position with respect to a frame of reference, and it is a function of time (i.e., velocity is the first derivative of displacement with respect to time).

ii. *Mass:* the quantity of matter that is concentrated in an object. The product of the mass times the velocity of an object is the "momentum" of that object.
iii. *Motion:* a change in the position of an object with respect to time.

The part of mechanics that is concerned with the study of motion is called kinematics. Due to the rigorous study of classical mechanics by Isaac Newton, the SI (Système International) unit of force, newton (denoted by N), has been named in his honor. One newton is defined as the force needed in order to accelerate one kilogram (kg) of mass at the rate of one meter (m) per second (sec) squared in the direction of the applied force.
First Law of Motion: An object will remain at rest or in a uniform state of motion unless that state is changed by an external force.
Second Law of Motion: The vector sum of the forces on an object is equal to the mass of that object multiplied by the acceleration of that object ("acceleration" is the rate of change of the velocity of an object with respect to time. Thus, acceleration is the first derivative of velocity with respect to time or, equivalently, the second derivative of displacement with respect to time); symbolically:
$F = ma$,
where F denotes force, m denotes the mass of an object, and a denotes the acceleration of the given object (thus, for any force you put on an object, an object of small mass will accelerate a lot, and an object of large mass will accelerate just a little). In case of circular motion (i.e., a movement of an object along the circumference $C = 2\pi r$ of a circle of radius r), if the period for one rotation is T, then:
the angular velocity (i.e., the angular rate of rotation) is
$\omega = \frac{2\pi}{T} = \frac{d\varphi(t)}{dt}$,
where $\varphi(t)$ denotes the angular displacement from the x-axis and is measured in radians, and t denotes time (measured in seconds);
the speed of the object travelling the circle is
$v = \frac{2\pi r}{T} = \omega r$;
the angular acceleration of the particle is
$\alpha = \frac{d\omega}{dt}$,
and, in case of uniform circular motion, $\alpha = 0$;
the acceleration due to change in the direction is
$\alpha_c = \frac{v^2}{r} = \omega^2 r$;
and the centripetal and centrifugal force can be computed using acceleration as follows (the centripetal force and the centrifugal force are actually the same force, depending upon the frame of reference):

$F_c = m\alpha_c = \frac{mv^2}{r}$.

Third Law of Motion: For every action in nature, there is an equal and opposite reaction.

Newton's Law of Universal Gravitation: An object attracts another object with a force that is directly proportional to the product of the masses of the objects and inversely proportional to the square of the distance between them, symbolically:

$F_g = G\frac{m_1 m_2}{r^2}$,

where F_g is the magnitude of the gravitational force on either object, m_1 and m_2 are their masses, r is the distance between them, and G is the gravitational constant, whose value is found to be (in SI units) $6.673 \times 10^{-1}\ N \cdot m^2 \cdot kg^{-2}$ (thus, the "weight" of a body is the total gravitational force exerted on the body by all other bodies in the universe).

Coulomb's Law: The magnitude of the force of interaction between two point charges (i.e., electric charges) is directly proportional to the product of the charges and inversely proportional to the square of the distance between them, symbolically:

$F = k\frac{|q_1 q_2|}{r^2}$,

where F denotes the magnitude of the force that each of two point charges q_1 and q_2 a distance r apart exerts on the other, and k is a proportionality constant, whose value is (in SI units) approximately $8.988 \times 10^9 N \cdot m^2 \cdot C^{-2}$. Due to the rigorous description of the electrostatic force of attraction and repulsion by the French military engineer and physicist Charles-Augustin de Coulomb (1736–1806), the SI unit of electric charge, the coulomb (denoted by C), has been named in his honor; it is approximately equivalent to 6.24×10^{18} electrons. "Charge" is a property of matter (just like mass, volume, or density), and it can come in two types: positive $(+)$ or negative $(-)$. In particular, a positive charge occurs when the number of protons exceeds the number of electrons, and a negative charge occurs when the number of electrons exceeds the number of protons.

By the term "field," we mean an area in which forces are exerted on things in its midst. The modern concept of a physical field was originally formulated in the nineteenth century by the English physicist Michael Faraday. An electric charge creates an "electric field" in the region of space surrounding it, in the sense that the properties of space are modified by the presence of an electric charge. "Electric field" (sometimes called "electric intensity") is defined as the electric force per unit charge. Therefore (in SI unites), the unit of electric field magnitude is one newton per coulomb (i.e., $1N \cdot C^{-1}$).

By the term "flux," we mean the quantity of a substance passing through a given area. Thus, "electricity" is the flow of electric charge along a path provided by a conductor (conductors are materials with high electron mobility). The "electric flux" through a surface is proportional to the number of field lines crossing that surface. In other words, its magnitude is proportional to the portion of the field perpendicular to the area:

$Electric\ Flux = Electric\ Field \times Area \times cos\theta$,

where $cos\theta$ denotes the cosine of the angle θ between the electric field and the vector that is perpendicular to the area. A "field line" is an imaginary line drawn through a region of space in such a way that, at every point, it is tangent to the direction of the electric-field vector at that point. In particular, in an "electrostatic field," every field line is a continuous curve with a positive charge at one end and a negative charge at the other. The amount of work needed in order to move a unit of electric charge from a reference point to a specific point in an electric field without producing acceleration is called an "electric potential." In terms of SI units, it is represented by

$V = \frac{joule}{coulomb}$,

where joule is the unit for work done, and $1\,joule = (1\,newton)(1\,meter)$; coulomb is the unit for the charge; and V denotes "volt," the derived unit for electric potential (electromotive force), and is named after the Italian physicist Alessandro Volta (1745–1827). The key to the flow of electricity is making a continuous electric circuit: connecting a wire between a source of electrons and an attractor of electrons (for which reason, for instance, a battery has two poles: a source (a negative), and an attractor (a positive); similarly, an electric plug has at least two tongs, one for incoming electrons and one for outgoing electrons). Electrons do not cease to exist. Rather, being carriers of charge, they move from the negative (source) to the positive (attractor), and they are useful as they follow the path to their destination in the context of a continuous electric circuit. By contrast, connecting two poles of a power source directly can actually be very dangerous: this is what is called a "short circuit," because there is no electric device between the source and the destination of electrons to power, such as a PC or a TV set. In case of a short circuit, the electron flow does not encounter any resistance, therefore the release of energy is instant, often paired with the involved wire heating dangerously.

Total Mechanical Energy of a System: $E_m = K + U$,

where E_m denotes mechanical energy, K denotes kinetic energy, and U denotes potential energy.

By the term "potential energy," we mean the energy possessed by a body by virtue of its position relative to others, stresses within itself, its electric charge, or other factors. For instance, gravitational potential energy (e.g., in the case of a ball whose mass is m and is dropped from height h) can be computed using the following formula:

$U = mgh$,

where m denotes the mass of the object, g denotes the acceleration constant due to gravity ($\approx 9.8\, m/sec^2$), and h denotes the height (displacement) of the object as a function of time.

By the term "kinetic energy," we mean the energy possessed by a body by virtue of its motion. Let us consider a body of mass m moving along the x-axis under the action of a constant resultant force of magnitude F directed along the axis. The body's acceleration is constant, and, according to Newton's Second Law of Motion, it is given by $F = ma$. The kinetic energy of this body can be computed using the following formula:

$K = \frac{1}{2}mv^2$,

where v denotes the body's velocity (which is, by definition, a function of time), and m denotes the mass of the object. Thus, the work done by the resultant external force on a body is equal to the change in kinetic energy of the body.

Relativity

Simply put, the central meaning of "relativity" is that different people may experience the same situation differently. The theory of relativity explains how to convert from the point of view of one observer to the point of view of another observer. For instance, Galileo theorized that, when two frames of reference (e.g., two drivers) are moving in uniform motion together, it is impossible to determine if one of the frames is moving or stationary from the perspective of the other frame in this system; it is only possible to understand an object's movement through a different frame of reference. The idea works for any two frames of reference moving with respect to each other (by the term "frame of reference," we mean any place where some physical process occurs or could occur).

Observers moving at different speeds will disagree about the distance and the time between two events. The German mathematician Hermann Minkowski depicted time as a length by proposing the following definition:

$distance = speed\ of\ light \times time = ct$.

Hence, if the speed of light in vacuum, commonly denoted by the letter c, is approximately $300{,}000{,}000\ meters/second$ (according to Rosa and Dorsey), then we say that $1/300{,}000{,}000$ of a second is one meter. In

other words, one meter is the distance travelled by light in vacuum during a time interval of $1/300{,}000{,}000$ of a second.

Einstein's special theory of relativity implies that our perception of "now," of presence, is subjective; it does not exist in the fundamental laws of nature. According to Einstein's special theory of relativity, the speed of light is finite, it is the same for all observers, and nothing can go faster than light (in vacuum). Hence, by the time an observer A sees an object B standing in front of A, the object B has already moved into the future. In fact, A sees B as B was in the past (how much in the past depends on the distance between them and on their relative motion). This delay (which may be very tiny, but still non-zero) exists in everything we see. The main reason that we do not perceive relativistic effects (such as the visual delay and the relativity of simultaneity) in our everyday life is because life around us is moving very slowly compared to the speed of light.

In the context of Einstein's theories of special and general relativity, "time" is a dimension of physical reality. In particular, time is a dimension similar to the three dimensions of physical space, but with a very important difference: we can stand still in physical space, but we cannot stand still in time. For simplicity, consider an xy-coordinate system where the vertical axis (i.e., the y-axis) represents time, and the horizontal axis (i.e., the x-axis) represents physical space. If you are standing still and then begin to walk, not only your position in space changes, but also your direction in physical space-time; since you are now moving into a direction that is a combination of both physical space and (physical) time. Such a change of velocity is called a "boost" (imparting linear momentum to a system), and the larger the boost the larger the angle you turn from time to physical space. But the speed of light is an upper limit, meaning that you cannot turn from a state in which you are moving through time and standing still in physical space to a state in which you are moving only in physical space and standing still in time. In the aforementioned xy-coordinate system, where the vertical axis (i.e., the y-axis) represents time, and the horizontal axis (i.e., the x-axis) represents physical space, there is a maximal angle you can turn in physical space-time by increasing your velocity, and that maximal angle is conventionally set to 45^o (but it is definitely some angle smaller than 90^o). Given that, in the aforementioned model, time is a dimension, it follows that time passes more slowly when you are moving than when you are not moving ("time dilation").

Conservation of Energy and Thermodynamics

The eighteenth-century French mathematician and natural philosopher Émilie du Châtelet proposed and tested the law of "conservation of

energy," according to which the total energy of an "isolated system" (i.e., one that does not interact with other systems) remains constant.

In order to clarify the meaning of the principle of the conservation of energy, let us consider the following example: setting fire to coal. The chemical bonds of the coal molecules store great amounts of energy. If we set fire to coal, then fire causes a chain reaction between the coal and oxygen in the air. In this reaction, energy from the chemical bonds is converted into kinetic energy of air molecules. Hence, the air becomes warm, and, for this reason, it will rise. This rising air can be used in order to drive a turbine and, for instance, move a vehicle, or in order to create electricity (by feeding it into the grid).

Alternatively, we can just burn coal without doing anything with the produced energy. This does not change the total energy in the system, because the total energy in the system is conserved. The chemical energy of the coal is converted into kinetic energy of air molecules, which are distributed in the atmosphere. Even though, in this case, the energy is useless, the total energy in the system remains the same. The difference between the aforementioned cases is entropy, or the measure of the molecular disorder, or randomness, of the system under consideration. Initially, the energy was packed into the coal, and the level of entropy was low. By setting fire to coal, the energy was distributed in the motion of air molecules, and the level of entropy became high.

When a system has energy in a state of low entropy, its energy can be used in order to create macroscopic change (e.g., drive a turbine), and this useful energy is called "free energy." Free energy is a type of energy that does "work." But, if the energy in the system is in a state of high entropy, then the energy is useless, and it is called "heat." Heat is a type of energy that does not do "work." Even though *total* energy is conserved, *free* energy is not conserved.

The conversion of mechanical energy into heat (thermal energy) is called "dry friction." Similarly, the conversion of electrical energy in a conductor into heat is called "resistance." In general, by the term "friction," we refer to a force that resists the relative motion between two objects; and its causes are molecular adhesion, surface roughness, and deformations. Friction converts its work into heat.

The first law of thermodynamics is a formulation of the law of conservation of energy, adapted to thermodynamic processes: the energy of the universe remains the same, in the sense that energy can neither be created nor destroyed, but it can be altered in form. The second law of thermodynamics states that the changes in the entropy in the universe can never be negative. In particular, according to the second law of

thermodynamics, no process can have as its sole result the transfer of heat from a cooler to a hotter body. Intimately related to the second law of thermodynamics is the concept of entropy, which provides a quantitative measure of disorder. Entropy counts the number of different microscopic configurations that have the same macroscopic appearance (or, in other words, how much information one could stuff into a macroscopic object if one kept track of the microscopic details). The second law of thermodynamics is equivalent to the "maximum entropy principle" and the "minimum energy principle": the maximum entropy principle states that, for a closed system with fixed internal energy (an isolated system), the entropy is maximized at stable equilibrium. The minimum energy principle states that, for a closed system with fixed entropy, the total energy is minimized at stable equilibrium. Hence, the second law of thermodynamics states that the quality of the energy of the entire universe, as an isolated system, is degraded irreversibly. In fact, physical, chemical, and electrical energy can be changed into heat, but the reverse (e.g., the transformation of heat into physical energy) cannot be fully accomplished without outside help or without an inevitable loss of energy in the form of irretrievable heat. This does not mean that the energy is destroyed, but it means that it becomes unavailable for producing work. This very degradation of energy is entropy. William James Sidis, in his book *The Animate and the Inanimate* (published in 1920), argues that life, which demonstrates perseverance and is characterized by evolution (according to Darwin's theory), implies a reversal of the second law of thermodynamics. Hence, the argument of classical mechanism (mechanical philosophy), according to which any living organism is merely a set of physical-chemical phenomena, contradicts the second law of thermodynamics.

Electromagnetism and Light

The electric field at a point can be calculated by using Coulomb's law in order to find the total force F on a test charge q' placed at the point, and then we divide F by q' to obtain the electric field E. If q' is positive, then the direction of E is the direction of F. The force on a negative charge, such as an electron, is opposite to the direction of E.

In order to analyze the motion of a particle with charge q in an electric field, we need to use Newton's Second Law of Motion, $F = ma$, with F caused by the electric field E, so that the magnitude of the electric force F is given by

$$F = qE.$$

If the field is uniform, then the acceleration is constant.

In simple terms, electrical interactions can be described as follows: a charge distribution sets up an electric field E, and the field exerts a force $F = qE$ on any charge q that is present. The same pattern can be followed in order to describe magnetic interactions (phenomena of attraction or repulsion that arise between electrically charged particles because of their motion). A moving charge, or a current, sets up a magnetic field in the space around it, and this field exerts a force F on a moving charge. Like electric field, magnetic field is a vector field (a vector quantity associated with each point in space). The symbol for magnetic field is B.

Whereas the electric-field force is the same whether the charge is moving or not, the magnetic force is proportional to the particle's speed. Thus, a particle at rest experiences no magnetic force at all. Furthermore, the magnetic force F acting on a charge q moving with velocity v does not have the same direction as the corresponding magnetic field B, but it is perpendicular to both the magnetic field B and v. Hence, the magnitude of the magnetic force F is given by

$F = |q|vBsin\varphi,$

where $|q|$ is the magnitude of the charge, and φ is the angle measured from the direction of v to the direction of B. The SI unit of B is $1N \cdot s \cdot C^{-1} \cdot m^{-1}$, where N stands for newton, s stands for second, C stands for coulomb, and m stands for meter. This unit is called 1 tesla ($1T$), in honor of the prominent Serbian-American scientist and inventor Nikola Tesla (1857–1943).

Using vector notation, the force that a magnetic field $\vec{B}$ exerts on a charge q with velocity $\vec{v}$ is given by

$\vec{F} = q\vec{v} \times \vec{B},$

where $\vec{v} \times \vec{B}$ denotes the cross product of the velocity and the magnetic field.

In 1831, the English scientist Michael Faraday discovered electromagnetic induction: he placed a stationary magnet inside or outside a coil, and he observed no deflection in the galvanometer. However, at the moment that he moved the magnet towards (into/above/below) the coil, he saw the pointer deflecting in one direction, and, at the moment that he moved the magnet way from the coil, he saw the pointer deflecting in the opposite direction. Using the aforementioned notation, the entire electromagnetic force F on the charged particle is called the Lorentz force (after the Dutch physicist H. A. Lorentz), and its magnitude is given by

$$F = F_{electric} + F_{magnetic}.$$

Faraday's discovery was really amazing, because one could make something move without ever touching it, only by using the field. Indeed,

we can affect things far away and develop telecommunications using electromagnetic fields. Moreover, Faraday was the first to understand that waves of the electromagnetic field are what we call light. In simple terms, electromagnetic radiation consists of electric and magnetic fields oscillating around each other, creating a freely propagating wave that can travel from one place to another. This event explains light, the operation of radio stations, the operation of microwave ovens, etc. These are electromagnetic phenomena, and they differ from each other only with respect to the wavelength of the corresponding oscillation, so that we use different names for electromagnetic radiation depending on the corresponding wavelength; for instance, if we can see electromagnetic radiation, then we call it light, light with large wavelengths is red, light with larger wavelengths that is invisible is called infrared, while, at even larger wavelengths, electromagnetic radiations are called microwaves, and, if the wavelengths are even larger, then electromagnetic radiations are called radio-waves.

By the term "radiation," we generally mean energy transferred by waves or particles. For instance, radiation may take the form of electromagnetic waves—which, however, are made of particles, photons specifically. A photon is a type of elementary particle that serves as the quantum of the electromagnetic field and the force carrier for the electromagnetic force. In particular, quantum electrodynamics describes the manner in which electrically charged particles interact by shooting photons back and forth between each other. Electrons, being zero-dimensional, lack spatial extension (that is, they have practically zero volume). Therefore, they interact with each other by exchanging photons. As two electrons move towards each other, a photon is passed from one to another, and it changes the momentum of both of them, thus pushing them off.

Gravitational radiation is transferred in gravitational waves, which are actually periodic deformations ("wiggles") of space-time. According to rigorous physical hypotheses, gravitational waves are made of a peculiar kind of particles called gravitons. A graviton is assumed to be a quantum of gravity (an elementary particle mediating the force of gravity). The term "graviton" was coined in the 1930s by the Soviet physicists Dmitrii Blokhintsev and F. M. Galperin.

The four fundamental forces

The laws of nature can be distilled into the following four fundamental forces:

i. *gravity:* a natural phenomenon by which all things with mass or energy are brought towards each other (it helps us to calculate the motions of celestial bodies);
ii. *electromagnetism:* a type of physical interaction that occurs between electrically charged particles (it has given us the wonders of the electric age);
iii. *weak nuclear force:* the mechanism of interaction between subatomic particles (it is responsible for the radioactive decay of the subatomic particles, and thus plays an essential role in nuclear fission, which is a form of nuclear transmutation);
iv. *strong nuclear force:* the mechanism that binds the component particles of an atom's nucleus (in 1911, the New Zealand physicist Ernest Rutherford discovered the basic structure of the atom: it consists of a small and dense core of positive electric charge called the nucleus, surrounded by a cloud of negatively charged electrons). An energy field that permeates the entire universe is known as the "Higgs field" (the smallest bit of which is called the "Higgs boson"), and it explains why some subatomic particles have a great deal of mass, while others have little, and others have none at all: the Higgs field interacts with the subatomic particles and determines their mass (very massive particles interact a lot with the Higgs field, while massless particles do not interact at all).

It is worth noticing that mass is a type of energy, but not all energy is mass. For instance, photons do not have any mass, but they have energy. The formula $E = mc^2$ (meaning that the relativistic/effective mass m of a particle times the speed of light squared, c^2, is equal to the kinetic energy E of that particle) refers to the energy of a particle when it is sitting still with respect to some inertial observer (the same particle is seen by another observer moving at a speed v). The more general formula is

$$E^2 = m^2c^4 + p^2c^2$$

where p is the momentum of the particle.

Quantum Physics

In quantum physics, everything is described in terms of wave functions, a wave function is a vector in a complex Hilbert space, and the vector coefficients are complex numbers. According to Paul Dirac's notation, in quantum physics, vectors are symbolized in the following way, known as the bra-ket notation:

$$|\Psi\rangle = a_1\begin{pmatrix}1\\0\\0\end{pmatrix} + a_2\begin{pmatrix}0\\1\\0\end{pmatrix} + a_3\begin{pmatrix}0\\0\\1\end{pmatrix}, \text{ where } a_1, a_2, a_3 \in \mathbb{C}.$$

The aforementioned type of brackets helps us to keep track of whether a vector is a row vector or a column vector: $|\Psi\rangle$ is a column vector, whereas $\langle\Psi|$ is a row vector. In quantum mechanics, if we convert a row vector to a column vector, then we have to take the complex conjugate of each coefficient. In other words, for instance,

$|\Psi\rangle = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\langle\Psi| = (a_1^*, a_2^*, a_3^*)$, where a_1^*, a_2^*, a_3^* are, respectively, the complex conjugates of a_1, a_2, a_3.

In quantum mechanics, all vectors describe probabilities. Usually, we choose the basis of the space under consideration in such a way that the basis vectors correspond to possible measurement outcomes; for instance:

$|\Psi\rangle = a_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + a_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ corresponds to

$|\Psi\rangle = a_1|X\rangle + a_2|Y\rangle + a_3|Z\rangle$.

Hence, the probability of a particular measurement outcome is the absolute square of the scalar product with the basis vector that corresponds to the outcome; so that, for instance, the probability of measuring X is

$|\langle X|\Psi\rangle|^2 = a_1 a_1^*$,

and this is known as Born's Rule. In other words, the probability density of finding a particle at a given point, when measured, is proportional to the square of the amplitude of the particle's wave function at that point. In quantum physics, the gradient of a wave function is denoted as follows:

$\nabla|\Psi\rangle = \frac{\partial}{\partial x}|\Psi\rangle\hat{\imath} + \frac{\partial}{\partial y}|\Psi\rangle\hat{\jmath} + \frac{\partial}{\partial z}|\Psi\rangle\hat{k}$.

In order to understand quantum physics, we must understand the difference between the potential mode of being and the actual mode of being. Hence, we must never confuse the realm of potentiality with the realm of actuality—that is, we must never attribute actuality to probability. As the famous quantum physicists Alain Aspect, John Clauser, and Anton Zeilinger have experimentally shown, particles do not have definite values before they are measured, but they have definite values after they are measured.

As already mentioned, in quantum physics, every system is described by a wave function, usually denoted by the Greek letter Ψ, from which physicists calculate the probability of obtaining a specific measurement outcome. In other words, this wave function is a way of studying the realm of potentiality in a scientifically rigorous way. For instance, from this wave function, one can calculate that a particle that enters a beam-splitter has a 50% chance of going left and a 50% chance of going right. This is a way of analyzing that particle's *potential* mode of being. On the other

hand, we can analyze that particle's *actual* mode of being by measuring the given particle.

After measuring the particle, we know with 100% probability where it is. Therefore, we must update our probabilistic study of the particle under consideration accordingly and with it the wave function. This update is known as the "wave function collapse," and it is an observational requirement that stems from the fact that, by measuring the particle, we have achieved a transition from potentiality to actuality. At the level of potentiality, or when we study the potential mode of being of a particle, that particle may be 50% at point A and 50% at point B; at the level of actuality, or when we study the actual mode of being of a particle by managing to measure it, that particle is 100% in a particular position, and we never observe a particle that is 50% at point A and 50% at point B. If we observe a particle at all, then we find that it is either in a particular position or not.

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