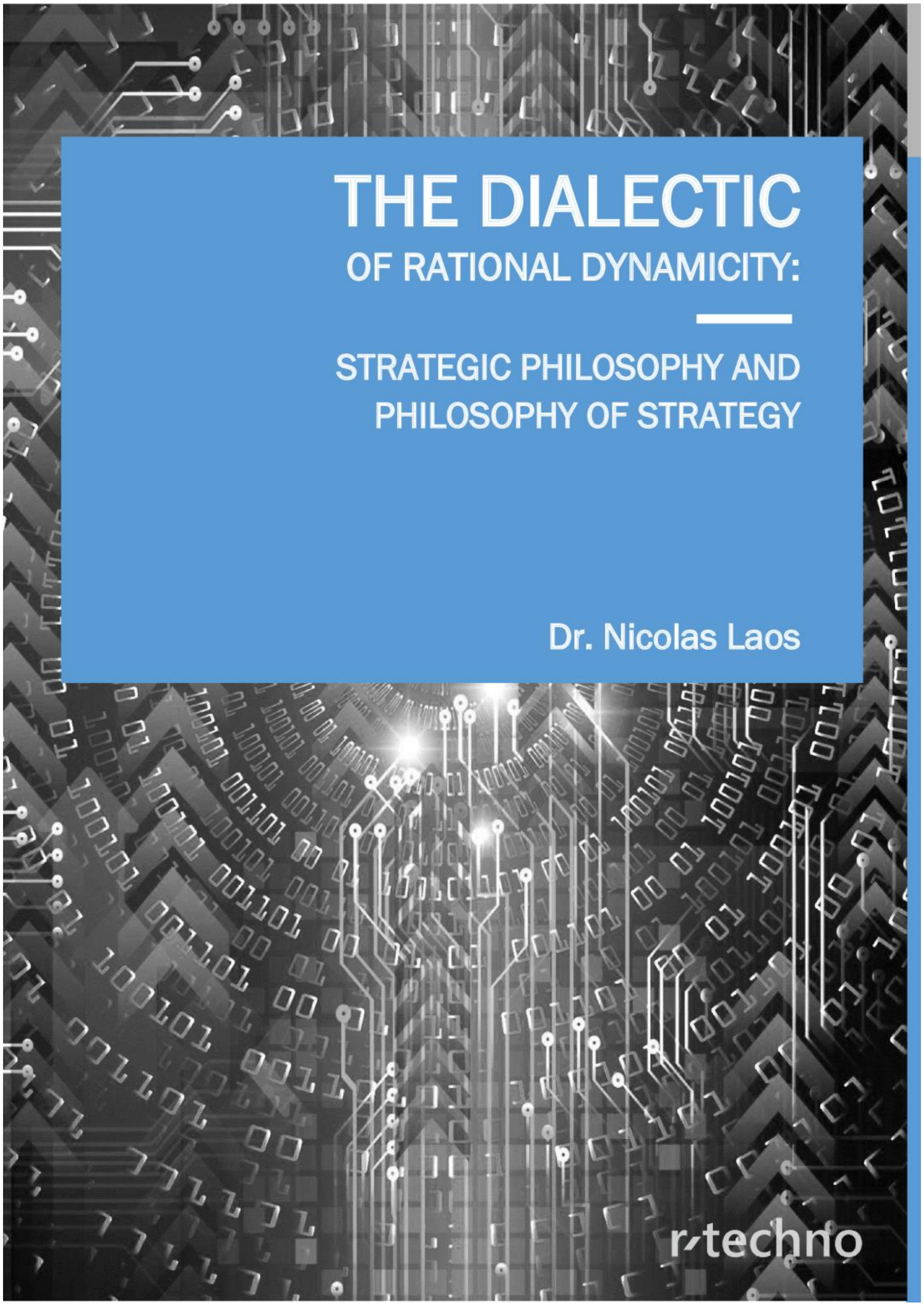
THE DIALECTIC
OF RATIONAL DYNAMICITY:
STRATEGIC PHILOSOPHY AND
PHILOSOPHY OF STRATEGY
Dr. Nicolas Laos

ABOUT THE AUTHOR AND THIS PROJECT

Dr. Nicolas Laos (born in Athens, Greece, on 2 July 1974) is a mathematician, philosopher, and noopolitics expert. His research interests and works span the entire spectrum of philosophy, the foundations of mathematics, mathematical analysis, cybernetics, as well as the interplay between philosophy, the social sciences, and the natural sciences.

The major part of the research work and the dissertation that he completed at the Department of Mathematics at the University of La Verne (La Verne, California), under the supervision of Academician Dr. Themistocles M. Rassias (Fellow of the Royal Astronomical Society of London and Accademico Ordinario of the Accademia Tiberina in Rome), on the foundations of mathematical analysis and differential geometry was published in 1998 as the volume no. 24 of the scientifically advanced Series in Pure Mathematics of the World Scientific Publishing Company. At the University of La Verne, in addition to completing his studies in mathematics, he completed a series of courses in humanities and the social sciences.

The results of Dr. Laos's research work in the fields of politology, political economy, philosophy of history, and the philosophy of the social sciences are contained in his book *Taking the Bull by the Horns: Causes, Consequences and Perspectives in Politology and Political Economy*, originally published in Greek by the Greek scholarly publisher ΚΨΜ (https://kapsimi.gr/). Having received a Doctoral Degree in Christian Philosophy (Summa cum Laude) from the Academia Teológica de San Andrés, Veracruz, Mexico, he has conducted advanced and systematic studies in philosophy and theology, specializing in ontology, philosophical psychology, philosophical anthropology, political philosophy, and modern philosophy. In fact, during his doctoral studies at the Academia Teológica de San Andrés, he studied a series of courses in philosophy (covering the entire history of philosophy) and a series of courses in "patristic theology" (and he realized that the study of the thought of the church fathers and of the manner in which they formulated church doctrines familiarizes one with the study of different languages and cultures, such as ancient Greek, Latin, and Hebrew (as well as with the languages and the cultures on which the secondary basic sources of "patristics" are based, namely, French and German), it includes the systematic study of classical and medieval philosophy, it requires the study of important aspects of the history of law and the history of science, and, in general, it cultivates an interdisciplinary mentality).

He is an adjunct member of the Faculty of Philosophy of the Academia Teológica de San Andrés, he has lectured at both undergraduate and postgraduate levels at several universities (including the University of Indianapolis Athens Campus, where he taught major undergraduate and postgraduate courses in epistemology and methodology) and research and business institutions, he has assumed analytical and/or executive roles in investment-services, construction, high-technology, and shipping companies, and he has published several investigative articles in authoritative political newspapers, magazines, and portals.

Moreover, Dr. Nicolas Laos has written several business essays, policy papers, and specialized reports: having worked as the Financial Derivatives and Fixed Income Assets Strategist at Eurofinance Investment Services S.A. (one of the first licensed Greek derivatives traders since the establishment of the Athens Derivatives Exchange in 1999), as the Director of Development of Ternica Group–Athens, Greece (land development, high technology, and construction), and as a financial analyst and broker, he has written business essays and reports on the mathematical modelling and the foundations of finance and fund management; having consulted with institutes of strategic studies and private intelligence companies, he has written essays, policy papers, and reports on the mathematical modelling and the analysis of arms races, arms acquisition patterns, competitive political and economic situations, international conflicts, etc.; and, having consulted with research-and-development and high-technology companies and institutes, he has written essays and reports on epistemology, methodology, and the technologization of the unified field theory.

Dr. Nicolas Laos is a Partner of the R-Techno International Ltd (Private Intelligence Company) and a consultant in noopolitics and mathematical modelling, specializing in cybernetics.

Noopolitics

The conduct of politics on the earth (geographical space) is called geopolitics, whereas the conduct of politics in the information field that is created by the communication between conscious entities is called noopolitics. The levels at which noopolitics can be conducted are the following:

- *Cyberspace:* this is the global system of the Internet-connected computers, communications, infrastructures, online conferencing entities, databases, and information utilities. However, the most important characteristic of the cyberspace is the communication between conscious entities and the social interactions involved rather than its technical implementation (i.e., the computational medium).

- *Infosphere:* it encompasses the cyberspace and information systems that may not be part of the Internet, such as the "mediasphere" (broadcast, print, and other media), libraries, military information infrastructures (Command, Control, Computer, Communications, Intelligence, Surveillance, and Reconnaissance Systems), etc. Intimately related to the conduct of noopolitics at the level of the infosphere are operations whose objective is the exercise of control over the mass media and the movies industry.

- *Noosphere:* this term, from the Greek word *nous* (mind), was coined by the Jesuit priest and philosopher Pierre Teilhard de Chardin in 1925, and, according to Teilhard de Chardin, it describes a globe-circling realm of the mind, or a "thinking circuit." Hence, at the level of the noosphere, noopolitics can be defined and practised as the systematic study and management of personal and social life in the context of the information field that is created by the communication between conscious entities, spanning philosophy of anthropology, of psychology, of sociology, of politics, and of economics.

Cybernetics

Cybernetics is a transdisciplinary (and, indeed, "antidisciplinary") systematic study of regulatory and purposive systems (their structures, constraints, and possibilities). Hence, cybernetics has been defined as "the art of governing or the science of government" (André-Marie Ampère), "the art of steersmanship" (Ross Ashby), "the study of systems of any nature which are capable of receiving, storing, and processing information so as to use it for control" (Andrey Kolmogorov), "the science and art of the understanding of understanding" (Rodney E. Donaldson), as well as "a branch of mathematics dealing with problems of control, recursiveness, and information, focuses on forms and the patterns that connect" (Gregory Bateson). Cybernetics includes noopolitics, and its most notable applications are the following:

1. Autopoiesis (it refers to a system capable of reproducing and maintaining itself by creating its own parts and further components),
2. Biological cybernetics,
3. Conversation theory (it explains how interactions lead to "construction of knowledge," or "knowing," and it is intimately related to noopolitics),

4. Engineering cybernetics,
5. Management cybernetics,
6. Medical cybernetics,
7. Perceptual control theory,
8. Second-order cybernetics (i.e., cybernetics of cybernetics: the recursive application of cybernetics to itself), and
9. Sociocybernetics (it includes noopolitics).

The purpose of this project

This project is a thorough and systematic study of the (philosophical, logical, and mathematical) foundations of noopolitics and cybernetics, and it proposes the “dialectic of rational dynamicity” as a method for the operation of consciousness and as a model of the operation of reality in general.

R-TECHNO INTERNATIONAL LTD WORKING PAPER

THE DIALECTIC OF RATIONAL DYNAMICITY: STRATEGIC PHILOSOPHY AND PHILOSOPHY OF STRATEGY

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Dedicated to the memory of
the Russian-Soviet philosopher and scientist Alexander Bogdanov (1873–1928)
and to the illuminating legacy of the Soviet Cybernetics.

"A little learning is a dangerous thing;
Drink deep, or taste not the Pierian spring:
There shallow draughts intoxicate the brain,
And drinking largely sobers us again."
Alexander Pope (1688–1744), poem *A Little Learning*.

CONTENTS

Chapter 1

REALITY, KNOWLEDGE, AND ACTION

1.1. THE MEANING OF PHILOSOPHY AND PRELIMINARY CONCEPTS

The word "philosophy" derives from the Greek word "philosophīa" ("φιλοσοφία"). The ancient Greek word "philosophīa" is composed of two other ancient Greek words, namely: "phileīn" ("φιλεῖν") and "sophīa" ("σοφία"). The word "phileīn" means "to love," "to endorse," and "to be wont to do (something)," and the word "sophīa" means "wisdom." Thus, according to the etymology of the ancient Greek word "philosophīa," philosophy means love for, pursuit of, and devotion to wisdom. By the term "wisdom," we may mean a set of dispositions, skills, and policies on the basis of which one can deliberate about the relationship between consciousness and the objects to which consciousness refers as well as about what matters and has value in life, and act accordingly.

The verb "philosopheīn" ("φιλοσοφεῖν"), which means "to philosophize," was used by the ancient Greek historian Herodotus, who, in his *Histories*, *1*:30:9–12, writes that Croesus, the King of Libya, entertained the Athenian philosopher, lawmaker, and poet Solon in the palace, and he addressed Solon as follows: "My Athenian guest, we have heard a lot about you because of your wisdom and of your wanderings, how as one who philosophizes [loves learning] you have travelled much of the world for the sake of understanding it."

The sixth-century B.C. Ionian Greek philosopher and mathematician Pythagoras was, arguably, the first person who invented the term "philosophy," and who called himself a "philosopher." In particular, Diogenes Laertius, in his *Lives of Eminent Philosophers* (Book VIII, Chapter 1: Pythagoras, 8) writes the following:

> Sosicrates in his *Successions of Philosophers* says that, when Leon the tyrant of Phlius asked him [namely, Pythagoras] who he was, he said, "A philosopher," and that he compared life to the Great Games, where some went to compete for the prize and others went with wares to sell, but the best as spectators; for similarly, in life, some grow up with servile natures, greedy for fame and gain, but the philosopher seeks truth.

Moreover, Diogenes Laertius, in his *Lives of Eminent Philosophers* (Book V, Chapter 1: Aristotle, 20) writes that, when Aristotle was asked what advantage he had ever gained from philosophy, Aristotle's response was the following: "This, that I do without being ordered what some are constrained to do by their fear of the law." From Aristotle's perspective, philosophy—expressing a continuous quest for knowledge, which is dialectically directed toward the ultimate knowledge—enables one to understand the underlying order and harmony of the world and, thus, to act rationally without coercion.

Plato, in his *Symposium*, 203e–204a, argues as follows:

> […] no gods ensue wisdom or desire to be made wise; such they are already; nor does anyone else that is wise ensue it. Neither do the ignorant ensue wisdom, nor desire to be made wise: in this very point is ignorance distressing, when a person who is not comely or worthy or intelligent is satisfied with himself. The man who does not feel himself defective has no desire for that whereof he feels no defect.

From the aforementioned Platonic perspective, philosophy is the pursuit of that part of wisdom which one has not yet acquired. Therefore, according to Plato, humanity's progress in philosophy is equivalent to humanity's progress toward its ontological integration and completion. Those who do not philosophize are either totally accomplished divine beings, already possessing the entire wisdom, or ignorant persons, who are characterized by either unconscious ignorance (that is, they are unaware of what they do not know) or complacent ignorance (that is, they are intellectually idle).

In general, philosophers are preoccupied with methodic and systematic investigations of the problems that originate from the reference of consciousness to the world and to itself. In other words, philosophers are preoccupied with the problems that originate from humanity's attempt to articulate a qualitative interpretation of the integration of the consciousness of existence into the reality of the world. The aforementioned problems pertain to the world itself, to consciousness, and to the relation between consciousness and the world.

It goes without saying that scientists are also preoccupied with similar problems. However, there are two important differences between philosophy and science. First, from the perspective of science, it suffices to find and formulate relations and laws (generalizations) that—under certain conditions and to some extent—can interpret the objects of scientific research, whereas philosophy moves beyond these findings and formulations in order to evaluate the objects of philosophical research, and, thus, ultimately, to articulate a *general method* and a *general criterion* for the explanation of every object of philosophical research. Whereas sciences consist of images and explanations of these images, philosophies are formulated by referring to wholes and by inducing wholes from parts. Hence, for instance, a philosopher will ask what is "scientific" about science, namely, what is the true meaning of science? Therefore, philosophy and science differ from each other with regard to the level of generality that characterizes their endeavors. Second, as the French philosopher Pierre Hadot has pointed out, philosophy—unlike the various scientific disciplines—is not merely a science, but it is a "way of life," and, specifically, philosophy signifies a conscious being's free and deliberate decision to seek truth for the sake of knowledge itself, since a philosopher is aware that knowledge is inextricably linked to the existential freedom and the ontological integration and completion of the human being.[1]

As the great Prussian philosopher and educationalist Wilhelm von Humboldt (1767–1835) has pointedly argued, the core principle and requirement of a fulfilled human being is the ability to inquire and create in a free and rational way. Thus, Humboldt promoted the concept of "holistic" academic education ("Bildung"), he identified knowledge with power, and he identified education with liberty.[2]

Beyond the similarities between philosophy and science, philosophy is an impetus for the creation of a world of meanings (in Greek, "noēmata") that express human creativity. Moving beyond those approaches that understand "meaning" as a constituent element of language, Edmund Husserl used the Greek term "nōema" (plural: "noēmata") to designate the intentional object, namely, that element due to which an intention of the human being—such as one's intention to say something, to move one's hand, etc.—acquires content and becomes significant. In particular, in his book *Ideas: General Introduction to Pure Phenomenology*, where he introduced the Greek term "nōema" (meaning "thought," or "what is thought about"), Husserl argued that any conscious experience is directed toward an object, and that, corresponding to all points in "the manifold data of the real (*reelle*) noetic content, there is a variety of data displayable in really pure (*wirklicher reiner*) intuition, and in a correlative 'noematic content,' or briefly 'noema.'"[3] Every intentional act has noematic content, or briefly "nōema," which is a mental act-process (such as an act of judging, meaning, liking, etc.) that is directed toward the intentionally held object (such as, the judged as judged, the meant as meant, the liked as liked, etc.).[4] In other words, every intentional act has, as part of its formation, a correlative "nōema," which is the object of the act.[5]

In view of the foregoing analysis of philosophy, the human mind is the foundation and the major focal point of philosophy. In the present book, by the term "mind," I mean a system of faculties or powers that constitutes an ontological attribute of a living organism. In particular, the major mental faculties (i.e., functional aspects of the mind) are the following[6]:

i. *Perception:* it is a process whereby a living organism organizes and interprets sensory-sensuous data by relating them to the results of previous experiences. In other words, perception is not a static, but a developing attribute of living organisms; it is active, in the sense that it affects the raw material of scattered and crude sensory-sensuous data in order to organize and interpret them; and it is completed with the reconstruction of the present (i.e., of present sensory-sensuous data) by means of the past (i.e., by means of data originating from previous experiences). Therefore, perception is intimately related to memory and judgment. Furthermore, there are two

[1] Hadot, *Philosophy as a Way of Life*.
[2] See: Günther, "Profiles of Educators: Wilhelm von Humboldt."
[3] Husserl, *Ideas*, p. 238.
[4] Moran, *Edmund Husserl*, p. 133.
[5] Husserl, *Ideas*, p. 229.
[6] See: Bateman and Holmes, *Introduction to Psychoanalysis*; Coon, Mitterer, and Martini, *Introduction to Psychology*; Freud, *A General Introduction to Psychoanalysis*; Gelder, Gath, and Mayou, *Oxford Textbook of Psychiatry*; B. J. Sadock and V. A. Sadock, *Kaplan and Sadock's Synopsis of Psychiatry*; Yalom, *The Gift of Therapy*.

kinds of perception: "external perception," which is caused as a result of external (physical/social) stimuli, and "internal perception," which is caused as a result of internal stimuli related to the awareness of one's ideas, thoughts, and desires.

ii. *Memory:* it is one's ability to preserve the past within oneself, or, equivalently, the function whereby one retains and accordingly mobilizes preexisting impressions.

iii. *Consciousness:* from a rather elementary perspective, it can be construed as an existential state that allows one to develop the functions that are necessary in order to know both one's existential environment as well as the events that take place around oneself and within oneself. Furthermore, as I maintain in this book, consciousness has all the attributes of a being itself, and it can be considered as the synopsis of the human being and the core of the mind. Therefore, in this book, I use the term "consciousness" instead of the term "conscious mind" and often in order to refer to the mind in general (since an integral mind is identified with consciousness).

iv. *Orientation:* it is a specific sense that helps one to verify one's position in space and time.

v. *Attention:* it is a mental faculty that focuses conscious functions on particular stimuli in a selective way, and it operates as a link between perception and consciousness.

vi. *Emotion or affect:* it is the mental faculty that determines one's mood. In general, one's capability to feel joy or sorrow as well as the intensity, the duration, and the stability of one's feelings depend on the proper functioning of emotion. When an emotion is endowed with a judgment, namely, when consciousness judges emotions, then an emotion becomes a "sentiment."

vii. *Thinking:* it is a complex mental faculty characterized by the creation and the manipulation of symbols (which represent various objects and events), their meanings, and their mutual relations. In the context of the communication between conscious entities, symbols are forms that express commonly accepted intentions and actions, and they can be organized into particular systems that are called codes. These codes underpin the activity and the behavior of conscious entities, and, therefore, a society of conscious entities reduces to an inter-subjective and conscious "continuum." The elements of such a code are called signs. Each sign is associated with a meaning in relation to the entire code to which it belongs as well as in relation to its acceptance by each and every conscious entity that uses the corresponding code.

viii. *Volition or will:* it is one's ability to make decisions and implement them kinetically. Conscious free will, in particular, may not initiate human action (e.g., due to physical-biological factors, or due to unconscious factors, etc.), but it can decide whether to allow a voluntary process to reach its conclusion, since it determines motor actions.

ix. *Association:* it refers to a phenomenon in which an idea that is present in consciousness attracts other relevant ideas to it in a way that is automatic and independent of one's will.

x. *Judgment:* one's ability to compare and contrast ideas or events, to perceive their relations with other ideas or events, and to extract correct conclusions through comparison and contrast.

xi. *Imagination:* it is a mental faculty that enables one to form mental images (representations) that do not—at least directly—derive from the senses. Imagination is not subject to the principle of reality as the latter is formed by the established institutions, and it develops because consciousness cannot conceive the absolute in an objective way. Therefore, imagination endows the things that it conceives with new significances, and it reorganizes them into new historical forms, utilizing elements of its external existential conditions (e.g., latent social trends and changes) that have not already crystallized into formally established institutions. From the aforementioned perspective, imagination can be considered as a kind of visionary perception.

The "organs" associated with mental homeostasis (namely, the ability to maintain a relatively stable mental state that persists despite changes in the external world), the communication between conscious entities, and humanity's adaptation to environment are the following[7]:

i. *Personality:* it is the set of all psychosomatic properties and functions by means of which a human being interacts with oneself and with one's environment. Intimately related to the term "personality" is the term "soul," because soul is the personal way in which one manifests the force of life.
ii. *Character:* it is the expressive organ of personality.
iii. *Behavior:* it is the executive organ of personality, and it consists of impulses and learning. By the term "impulse," we mean a sudden and compelling urge or desire to act. By the term "learning," we mean a function that enables a person to utilize experience and training and to acquire new types of behavior in order to ultimately supplement and expand one's innate capacity for adaptation and creativity.

The characteristics of the personality of a "normal person" can be summarized as follows[8]:

i. *A normal relation between optimism and pessimism:* A normal person is fundamentally optimistic. However, a normal person's optimism is rational, it is not in conflict with the principle of reality, and it does not give rise to irrational expectations.
ii. *A normal relation between a sense of dependence and a sense of independence:* Depending on the conditions and the needs that prevail in one's environment, a normal person can adapt to both the role of a leader (independent actor) and the role of a subordinate (dependent actor). Furthermore, a normal person does not spurn the others' offers of help, and willingly undertakes to help others.
iii. *Normal levels of organization and systematicness:* Normal persons have the tendency to be organized and neat and to tackle problems in a systematic way without, however, being "fixated" on (i.e., "obsessed" with) these properties, and, therefore, they do not allow these properties to clash with other desires, especially with those which underpin creativity.

[7] Ibid.
[8] Ibid.

iv. *A normal sense of curiosity:* A normal person utilizes curiosity to facilitate creative adaptation without harming others.
v. *Normal sexual identity:* Normal persons are reconciled and satisfied with their sexual identity, and they are free from fears and complexes pertaining to sex.
vi. *A normal relation between competitiveness and cooperation:* Normal persons can act autonomously and self-reliantly in order to achieve one's goals, but they are also capable of willingly and creatively cooperating with others.
vii. *A normal attitude toward authority:* Normal persons do not make *a priori* assumptions about authority, and, therefore, they neither *a priori* reject authority, nor do they *a priori* submit themselves to the governing authorities, being aware of the consequences of their choices.
viii. *Normal ways of expressing and controlling emotions:* Normal persons do not repress their emotions, but they control their behavior.
ix. *Ability to make close and stable relationships.*
x. *Ability to establish a viable equilibrium between the pursuit of satisfaction and the pursuit of safety.*
xi. *Self-esteem combined with the awareness of one's own constraints and weaknesses and the ability to appreciate, admire, and trust others.*
xii. *Capacity for responsible decision-making.*

In the 1900s, the Austrian psychiatrist Sigmund Freud, who is the acknowledged founder of psychoanalysis, articulated a structural model of the mind, according to which the elements that structure the "mental apparatus" are ordered in the "mental space" and constitute three vertically superimposed apartments that he called the "id," the "ego," and the "superego," and he defined them as follows[9]:

i. *"Id":* it consists of impulses, instinctive urges, and everything connected with the major biological needs of the human being. Instinct is a highly formalized behavioral code that reflects the logic of organic nature. The "id" does not have discretion, namely, the capacity to distinguish between "right" and "wrong," and, instead, it is motivated by the pleasure principle, which wants to immediately gratify all impulses. It is innate, and it remains unaffected by experience, for which reason it is not subject to any moral or sociogenic constraint.
ii. *"Ego":* it is the administrative center of personality, and it is created and gradually develops under the influence of accumulated experience. It contains all functions of consciousness as well as unconscious functions, such as the defense mechanisms of the ego. In particular, the ego's consideration of reality is conscious, but the ego may also keep censored or forbidden desires hidden by unconsciously repressing them by means of the defense mechanisms of the ego. The ego is motivated by the principle of reality, it has discretion, and it develops rational thinking in order to weigh pleasure against its consequences.
iii. *"Superego":* It consists of two components: (i) the ego ideal, or ideal self, namely, the rules and standards one should adhere to, and (ii) moral consciousness, which is the consciousness of existence itself when it operates as a judge. The "superego" is

[9] Ibid.

> the moral compass of personality, and it develops under influences exerted on the human being by the parents, persons of authority, and one's microsocial environment.

A normal person has a strong "ego" and, thus, great resistance to (psychologically) stressful situations. When reality refuses to satisfy a normal person's needs and desires, that person finds substitutes through consciously controlled mechanisms. However, normal persons do not repress their original needs and desires, nor do they hasten to satisfy them in ways that are socially demeaning or biologically harmful, but they try to satisfy them within particularly suitable spatio-temporal settings.

When normal persons are faced with severe stress, they initially try to modify the conditions that cause it. If they are unsuccessful in their attempt to modify the conditions that cause severe stress, then normal persons try to modify their own attitude toward these conditions, or they try to escape from them. If they are also unsuccessful in this stage of defense, and they start realizing the possibility of developing symptoms of a mental disorder, or of incurring a major social failure, then they look for ways of obviating them (for instance, they seek advice from experts).

With regard to the defense mechanisms of the ego, it is important to mention that, in contrast to persons who are not normal, normal persons do not easily resort to the mechanism of repression in order to tackle reality's challenges and their non-conformist desires. Repression is the major defense mechanism of the ego, and it consists in the unaware exclusion of distressing thoughts, desires, impulses (especially aggressive and sexual ones), and unacceptable experiences from the field of consciousness. Repression can protect the ego only for a relatively short period of time, but, in the long run, it causes the accumulation of unconscious material that has a detrimental effect on behavior and the decision-making process.[10] In fact, repression plays the major role in the pathogenesis of neurosis, which is characterized by severe and chronic feelings of anxiety and fear. According to Freud, there are three kinds of anxiety: "real anxiety" derives from a real, external threat; "cultural anxiety" is a peculiar kind of anxiety manipulated by certain social regimes; and "neurotic anxiety" is either (i) superegotic, in which case the superego punishes the ego, or (ii) instinctive, in which case impulses explosively manifest themselves in an unsuitable environment.[11] In addition, according to Freud, each neurotic phobia has a symbolic character (for instance, cleistophobia is a symbolic manifestation of one's fear of being trapped; acrophobia is a symbolic manifestation of one's fear of falling morally, etc.).[12]

When normal persons are faced with a situation in which their (positive or negative) emotional charge affects their handling of a given situation, they do not easily decide to use the following defense mechanisms either[13]: (i) denial, which involves the refusal to accept traumatic experiences, and, thus, it may contribute to the development of psychosis; (ii) isolation, which involves the separation of the emotional charge from the corresponding underlying thought, and it may contribute to the development of obsessive–compulsive disorder (for instance, in the Nazi concentration camps, physicians were committing crimes while believing that they were conducting scientific research); (iii) introjection, which

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[10] Ibid.
[11] Ibid.
[12] Ibid.
[13] Ibid.

involves the internalization of the object of love (such as the parents, persons of authority, etc.) in a way that the properties of that object are integrated into the ego of the person who resorts to introjection, so that this mechanism eliminates the awareness of the differences between one's ego and the object of one's love, in order to protect the ego from the anxiety of separation and the fear of loss, and, thus, introjection may ultimately contribute to the development of depression; and (iv) projection, which involves projecting undesirable impulses, urges, and emotions onto someone else, and, thus, it indicates a weak ego trying to trick itself, and it may contribute to the development of psychosis (for instance, Freud has mentioned the case of a woman who had been unfaithful to her husband but who accused her husband of cheating on her). Instead of tricking themselves with defense mechanisms, normal persons use mechanisms that are directly controlled by their consciousness. Hence, as I have already mentioned, consciousness can be considered as the synopsis of the human being.

Conclusively, even though, in the context of the modern "academia," philosophy and scientific psychology, as academic disciplines, differ from each other, they are characterized by important interconnections between them.[14]In particular, philosophical psychology, which is an integral part of classical philosophy, and scientific psychology are intertwined with one another through the concept of "normality." Scientific psychology aims to elevate a person from a sub-normal existential state to the normal existential state, thus restoring the rule of consciousness over the unconscious, by analyzing the secondary, unconscious mind, which is formed by wishes and desires that are repressed by social norms and by reason and common sense, which are adaptation mechanisms to reality. Philosophical psychology inquires into the meaning of consciousness, reason, reality, truth, and morality themselves, on which the concept of "normality" is based; and more spiritually "exalted" philosophies aim to elevate the human being from the normal existential state to a super-normal existential state, usually summed up by the concept of the wonderful. In the context of modern psychology, the significant interplay between scientific psychology and philosophical psychology was originally addressed by "existential psychotherapy." In particular, the American psychiatrist and psychoanalyst Irvin D. Yalom has emphasized and systematically studied the interplay between psychotherapy and what he has called the four ultimate existential concerns, namely, death, freedom, isolation, and meaninglessness.[15]

1.2. THE ABSTRACT STUDY OF A BEING

Logic may be defined as a theory of true propositions, or, equivalently, as a theory of correct reasoning. Any relation between concepts is formulated by means of propositions.

[14] "Scientific psychology" emerged in the nineteenth century as an autonomous scientific discipline under the influence of the German physician Wilhelm Wundt (1832–1920), who was the founder of "experimental psychology." In 1893, Eduard Titchener, an English student of Wilhelm Wundt's, founded his own formal laboratory for psychological research at Cornell University, after Oxford University had rejected the creation of a distinct department of psychology. Titchener placed psychological structuralism in a more scientifically rigorous setting than that of Wundt's theory. Titchener's psychological structuralism consists in analyzing consciousness into its constitutive elements (particularly, its experiences) in order to ascertain its structure. In the twentieth century, "scientific psychology" was further developed by several great modern psychologists, such as Pierre Janet, William James, John B. Watson, Sigmund Freud, Carl Jung, Alfred Adler, Jean Piaget, Max Wertheimer, Abraham Maslow, etc. See: Hothersall, *History of Psychology*.

[15] Yalom, *Existential Psychotherapy*.

According to Aristotle's *Organon*, the backbone of any science is a set of propositions, so that, starting from the very primitive principles and causes, one can proceed to learn the rest. Aristotle's logic is focused on the notion of deduction (syllogism), which was defined by Aristotle, in his *Prior Analytics*, I:2:24b18–20, as follows: "A deduction is speech (*logos*) in which, certain things having been supposed, something different from those supposed results of necessity because of their being so"; each of the things "supposed" is a premise of the argument, and what "results of necessity" is the conclusion.

By the term "concept," we mean the set of all predicates of a thing (or of a set of conspecific things) that express the substance of the given thing (or of the given set of conspecific things). In the scholarly discipline of logic, the "intension" of a concept is the set of all predicates of the given concept, namely, the set of all those elements due to which and by means of which the given concept can be known and distinguished from every other concept; in other words, the intension of a concept is its formal definition. For instance, the properties of the three angles and the three sides of a geometric figure constitute the intension of the concept of a triangle. Moreover, in the scholarly discipline of logic, "extension" indicates a concept's range of applicability by naming the particular objects that it denotes; in other words, the extension of a concept encompasses all those things to which the given concept refers. For instance, the extension of the concept of a tree consists of all particular trees; the extension of the concept of a human being consists of all particular humans, etc.

By the term "genus" (plural: "genera"), we mean a concept whose extension includes other concepts, known as "species," or "kinds," which fall within it. In other words, "genera" are concepts whose extension is bigger than that of other concepts, whereas "species," or "kinds," are concepts whose extension is smaller than that of other concepts. For instance, the concept of a geometric figure is a genus with regard to the concept of a triangle, whereas the concept of a triangle, which appertains to the concept of a geometric figure, is a kind with regard to the concept of a geometric figure.

Through the process of "abstraction," we decrease the intension of concepts and increase their extension. Thus, due to abstraction, the concept of a human being can be gradually generalized into the following concepts: "vertebrate," "mammal," "animal," "living being," and "being"; "being" is the most general concept, in the sense that its intension is minimum and its extension is maximum. "Being," to which every other concept is reducible, cannot be further analyzed into other concepts. Concepts of such general type, which are not susceptible of further analysis into simpler concepts, and to which other concepts are reducible, are called "categories." Aristotle, in his book *Categories*, attempted to enumerate the most general species, or kinds, into which entities in the world divide. In particular, in *Categories*, 1b25, Aristotle lists the following as the ten highest categories of things "said without any combination": "substance" (e.g., man, horse), "quantity" (e.g., four-foot, five-foot), "quality" (e.g., white, grammatical), "relation" (e.g., double, half), "place" (e.g., in the Lyceum, in the market-place), "date" (e.g., yesterday, last year), "posture" (e.g., is lying, is sitting), "state" (e.g., has shoes on, has armor on), "action" (e.g., cutting, burning), and "passion" (e.g., being cut, being burned).

1.2.1. Epistemological Presuppositions

Consciousness aims to maintain and develop itself, that is, it aims to preserve and reinforce existence. In particular, consciousness aims to preserve existence on the best possible terms in order to ultimately shift from the act of "being" to the act of "being better." In this way, the intentionality of consciousness concretizes its identity both as a tendency to participate in the world by assimilating the world and as self-knowledge. The levels at which the aforementioned activities take place are the levels of instinct, experience, and rational under7standing (intellect).

At the level of instinct, conscious activity is minimized, and every instinctive action of existence overlays or ignores every conscious activity. Instinct is a condensed logical manifestation whose correctness has been confirmed by the application of the problem-solving method that is known as "trial and error" by an unlimited number of generations, and it reflects the logic of organic nature. Thus, instinctive action has the character of an *a priori* integrated process. No obstacle to the affirmation of instinctive behavior can change the intrinsic logic of instinct itself. However, an obstacle to the affirmation of instinctive behavior can modify the manner in which existence adapts to each situation. In fact, adaptation is based on the method of "trial and error," and this term was coined by the British psychologist Conwy Lloyd Morgan (1852–1936), who also used the terms "trial and failure" and "trial and practice."

At the level of experience, the intentionality of consciousness is expressed through the activity of the senses, which are oriented toward the world, with which they connect existence. Experience is a conscious state, which is part of the receptive aspects of existence. However, at the level of experience, consciousness is passive.

At the level of rational understanding (corresponding to the mental process used in thinking and perceiving), consciousness plays an active role, whose manifestation is reason. According to the German philosopher Immanuel Kant (1724–1804), who was arguably the most important representative of the European Enlightenment, "reason" is an *a priori* (pre-experiential) structure in the context of which various categories are interrelated, and, whenever they are adequately activated, they can connect isolated empirical data with each other, thus making possible the formulation of synthetic judgments, though which one can creatively transcend the level of experience and ascend to the level of rational understanding.[16]

Intimately related to the different levels at which the intentionality of consciousness is expressed are the different degrees and forms of knowledge. By the term "knowledge," we mean: (i) the mental action through which an object is recognized as an object of consciousness; (ii) the mental action through which consciousness conceives the substance of its object; (iii) the object whose image or idea is contained in consciousness; and (iv) that conscious content which is identified with the substance of the object of knowledge. Therefore, the term "knowledge" can be construed as a firm consideration of an object as something that corresponds to reality.

The four basic forms of knowledge, namely, the four basic relations between consciousness and any object of consciousness, are the following:

i. *Belief:* The term "belief" has two meanings: first, it means that one accepts something as real, even though the claim about its reality is not based on experience

[16] See: Guyer, ed., *The Cambridge Companion to Kant*.

or on logical proofs; second, it means that one has merely formed an opinion by remaining focused on the appearance of things. In the latter case, the term "belief" is synonymous with the term "dōxa," which, according to Plato, means the acquaintance with an object that can be provided by an unstable appearance, and it is contrasted with "epistēme," which, according to Plato, means a firm and stable intellectual grasp.[17]

ii. *Empirical knowledge:* Empirical knowledge is a form of knowledge that is provided by the senses. Its object may be inside or outside us, so that, depending on the position of its object, empirical knowledge is distinguished into internal experience and external experience. Experience is a form of conscious knowledge that is superior to belief and inferior to logical knowledge.

iii. *Logical knowledge:* Logical knowledge is a form of knowledge that derives from the rational faculty of consciousness, and it is characterized by indisputable and logically grounded truths, namely, judgments about the reality of things. There are two groups of logical knowledge: the group of philosophical truths and the group of mathematical truths, which, exactly because they are both groups of logical knowledge, give rise to a homomorphism between mathematics and philosophy (see also section 1.3.3). Philosophy and mathematics are structurally similar to each other. In general, a "homomorphism" is a concept used in abstract algebra in order to compare two groups for structural similarities, specifically, it is a function between two groups that preserves the group structure in each group (for a more rigorous explanation of these concepts, see Chapter 2).

iv. *Intuition:* Intuition, like instinct, manifests itself as a direct and condensed logical conception of objects, and, simultaneously, as a system of accumulated experiences whose origin tends to become unconscious. By the term "intuition," we mean that consciousness conceives a truth (that is, it formulates a judgment about the reality of an object) according to a process of conscious processing that begins with a minimum empirical or logical datum and goes up to a whole abstract system with which consciousness realizes that it is connected or of which consciousness realizes that is an integral part. Moreover, according to Donald J. Puchala, the purpose of intuition as a metaphysics is to "properly deal with the nature of unobservable reality."[18] There are three different varieties of intuition: (i) sensuous or psychological intuition, (ii) logical intuition, and (iii) metaphysical intuition. As I shall explain later in this chapter, a characteristic type of sensuous or psychological intuition is Bergson's conception of intuition, a characteristic type of logical intuition is Husserl's conception of intuition, and a characteristic type of metaphysical intuition is the Neoplatonic concept of ecstasy.

Intuition, experience, and reason are different from each other, but, in practice, they do not contradict each other, since they cooperate with each other both in the context of everyday life and in the context of philosophical and scientific inquiries. However, knowledge presupposes what the Italian Dominican priest, scholastic philosopher, and theologian Thomas Aquinas (1225–74) has called "the correspondence between the intellect [of the

[17] Plato, *Republic*, 479c and 534a.
[18] Puchala, "Woe to the Orphans of the Scientific Revolution," p. 70.

knower] and the thing [the known]" ("adaequatio rei et intellectus"), naming this correspondence "truth."[19] This truth may be either "formal," when it refers to the agreement between different logical terms, or "substantial," when it refers to the agreement between sensory-sensuous or conceptual objects and their meanings, which represent them within consciousness. In Chapter 3, I shall propose a refined version of the correspondence theory of truth.

1.2.2. The Significance and the Presence of a Being

The concept of a being is the central concept of philosophical inquiry. The reality of a being is the reality *par excellence*. The study of the history of philosophy leads to the conclusion that a "being" is a self-sufficient reality that is maintained either by being a closed system or by being an open system. When a being is a closed system, not only does it maintain its structure but also includes its boundary conditions, and, therefore, it is existentially buttressed by its frontiers. When a being is an open system, it maintains its structure, but it tends to transcend its nature and expand itself beyond its normal frontiers. In Chapter 2, I shall clarify the concepts of closedness and openness in a more rigorous way through set theory. At this point, it suffices to make the following two remarks: First, when one considers the essence of being as a closed system, one gives priority to and emphasizes the distinction between "inside" and "outside," specifically, those elements that are counted as belonging to the system ("being") under consideration in contradistinction to those that are not; whereas, when one considers the essence of being as an open system, one gives priority to and emphasizes the dynamicity and the activity of being as well as the way in which a being is related to other beings. Second, if a being is a closed system, then it exists in a static way; whereas, if a being is an open system, then it exists in a dynamic way.

The primitive formation of the basic image (mental representation) of a being by philosophizing consciousness is due to the presence of the human reality in the world, and, therefore, it is based on experience. However, gradually, the basic image of a being undergoes further processing by consciousness. As a result of its processing by consciousness, the basic image of a being discards its most specific traits and its accidental properties, and it is projected onto a conscious construct, so that it is replaced by the most abstract representation of the given being (for instance, the fundamental problem of perspective (in both art and mathematics) consists in correctly representing a three-dimensional picture or situation in a two-dimensional picture of it). In this way, consciousness facilitates the conception and the functional interconnection of the most abstract aspect of a being and the world, into whose functional presence the given being is integrated. In fact, as the American historian of sciences and mathematician Carl B. Boyer has pointedly written, ancient Greece discovered science and philosophy because it realized that human consciousness "is something different from the surrounding body of nature, and it is capable of discerning similarities in a multiplicity of events, of abstracting these from their settings, generalizing them, and deducing therefrorm other relationships consistent with further experience," and, in particular, "the establishment of mathematics as a deductive science" is ascribed to Thales.[20]

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[19] Aquinas, *Summa Theologiae*, Q. 16.
[20] Boyer, *The History of Calculus and Its Conceptual Development*, p. 16–17.

Ontology, known also as "metaphysics,"[21] like it or not, has a long history, and it is inextricably linked to the history of science and to the very essence of science. According to the terminology of modern philosophy, all ontological theories can be divided into two categories: "philosophical realism" and "idealism." In other words, according to modern philosophical terminology, there are two general models whereby philosophers interpret the world: one gives primacy to the reality of the world, and it is known as philosophical realism, whereas the other gives primacy to the reality of consciousness, and it is known as philosophical idealism. Another important way of categorizing philosophical theories is based on the distinction between "monism" and "dualism." Monism attributes "oneness" or "singleness" to a fundamental kind, category of things, or principle; for instance, "substance monism" asserts the unique reality of only one kind of stuff, such as spirit (according to the spiritualist type of monism) or matter (according to the materialist type of monism), and it maintains that many different things may be made up of this stuff. As the British philosophers James Opie Urmson and Jonathan Rée have put it, substance monism is "the view that the apparent plurality of substances is due to different states or appearances of a single substance."[22] Contrasting with monism, dualism maintains that, at least in some domains, there are two fundamental kinds, or categories of things, or principles. According to Urmson and Rée, dualism "is the name for any system of thought which divides everything in some way into two categories or elements, or else derives everything from two principles, or else refuses to admit more or less than two substances or two kinds of substance."[23]

As I shall argue later in this chapter and in Chapter 3, in each of the aforementioned "schools" of ontology, one can find important problems, which have been thoroughly analyzed by, among others, the Soviet-Russian philosopher and psychologist Alexander Spirkin[24]: The monist varieties of philosophical realism are prone to oversimplifications, because they fail to identify and analyze important elements and aspects of reality. The dualist varieties of philosophical realism give rise to contradictions and logical gaps. On the other hand, idealism is highly malleable, since—expressing and highlighting the complexity and the diversity of the output of the functions of consciousness—it gives rise to a philosophical framework in which various philosophical differentiations can take place. However, as I shall argue later in this chapter and in Chapter 3, in line with Spirkin, idealism tends to underestimate the ontological autonomy of the world, and it is rather oblivious of what Spirkin has called the "dialogical nature of consciousness."[25] Therefore, I propose a synthesis between realism and idealism.

Every philosophical activity is fundamentally concerned with the study of being, and, in the context of philosophy, the term "being" is almost always construed according to the aforementioned definition, namely, as "a self-sufficient reality that is maintained either by being a closed system or by being an open system." The study of the history of philosophy indicates that, on several occasions, philosophers are overwhelmed by the wonders of the

[21] The concept of "metaphysics" originates from the Greek words "after" ("metā") and "the physical [treatises]" ("tā physikā"). In fact, when the Greek philologist Andronicus of Rhodes (first century B.C.) published the complete works of Aristotle, he placed the book in which Aristotle studies the reality of being (namely, ontology) after Aristotle's physical treatises. Hence, gradually, the term "metaphysics" (literally meaning "after the physical [treatises]") became a synonym (or rather a sobriquet) of "ontology."

[22] Rée and Urmson, eds., *The Concise Encyclopedia of Western Philosophy and Philosophers*, p. 297.

[23] Ibid, p. 115.

[24] Spirkin, *Dialectical Materialism*.

[25] Ibid, Chapter 3.

physical world, and they methodically try to investigate and interpret them. However, even in these cases, in which philosophizing consciousness is oriented toward the world, philosophical activity is indirectly preoccupied with the human being, because it refers to the world in order to ultimately explain the "exceptional" presence of the human being in the world.

It goes without saying that the manner in which and the extent to which humanity is related to the world (of which everything seems to be an outgrowth) vary, and they are understood and evaluated in different ways by different cultural communities and different researchers. Furthermore, the study of the history of civilization indicates that humanity persistently tries to become autonomous from the world and to be reintegrated into the world in terms of a new equilibrium underpinned and controlled by humanity itself. In any case, irrespective of whether humanity is considered as a being extended in and related to the word or as a separate reality, the human being tries to impose itself as the most magnificent manifestation of being, and it does so through philosophy, science, art, technology, politics, and even religion. Thus, the very first attempts to articulate realist philosophies both of the materialist type, such as Democritus's and Epicurus's atomism, and of the spiritualist type, such as Plato's theory of ideas and Gottfried Wilhelm von Leibniz's monadology, are founded on the argument that the human being is an independent and mostly free whole and an indivisible structural actuality.

The ancient Ionian school of philosophy—whose members (namely, such Greek philosophers as Thales, Anaximander, Anaximenes, Heraclitus, Anaxagoras, and Archelaus) were called "physiologoi" (meaning those who discoursed on nature) by Aristotle[26]—sought to find the primary material substance from which hypothetically both the cosmic reality, which surrounds the human presence, and the human being as a reflection of this reality originate.[27] According to Thales, this substance is the element of water; according to Anaximenes, this substance is the element of air; according to Heraclitus, this substance is the element of fire, and it is intimately related to the continuous change of reality; according to Anaximander, this substance is the principle of infinity; according to Archelaus, the primary cosmological principle is the principle of motion, and it is intimately related to the separation of hot from cold. According to Empedocles, a distinguished Greek pre-Socratic philosopher who lived in Sicily, the primary cosmological principle consists of the attractive and the repulsive forces by which the classical elements (namely, earth, water, air, and fire) are interrelated. However, the first philosopher who conceived being as a unique and dynamic whole was Parmenides of Elea, a Greek pre-Socratic philosopher from Elea in southern Italy. Parmenides was the founder of ontology as the branch of philosophy that inquires into reality itself.[28]

Parmenides studies being as a "whole," specifically, as a unique set that imposes itself by being and opposes everything that is not. According to Parmenides's poem *On Nature*, being and non-being are two totally distinct ontological categories, and they cannot be reduced to each other. This dualist argument underpins the original formulation of the classical Platonic perception of ideas as "beingly beings." However, in his dialogue *Sophist*, Plato maintains that "being" and "non-being" are the extreme terms of an ontological series whose

[26] Aristotle, *Metaphysics*, 986b.
[27] See: Fried and Hademenos, *Biology*.
[28] See: Curd, *The Legacy of Parmenides*; Palmer, *Parmenides and Presocratic Philosophy*.

intermediate terms are the non-being of being and the being of non-being, and that these intermediate terms underpin the explanation of the presence of the world. The four aforementioned Platonic ideas were utilized by Plotinus, the greatest Greek Neoplatonic philosopher, who, in his *Enneads*, identified four primary hypostases (i.e., underlying substances, or fundamental realities that underpin all else), namely: (i) the One: it is the source of all existence, and, hence, it is totally transcendent (beyond the categories of being and non-being), it encompasses thinker and object, and it is identified with the ideas of “good” and “beauty”; (ii) the Nous (Mind, or Intelligence): it is the highest being, and it is directly emanated by the One; this second hypostasis, in which the ideas (namely, archetypal forms, which are the energies of the One) reside, emanates a third hypostasis, which is called the World Soul; (iii) the World Soul: it is an intrinsic connection between all living beings, and, according to Plotinus, it is composed of a higher and a lower part (the higher part is unchangeable and divine, and it provides the lower part with life), so that the World Soul contemplates both the intelligible realm and Nature as it previews what it produces, and, therefore, time and the physical world proceed from the World Soul; (iv) Matter: the process of emanation ends when being tends to non-being so much that a limit is finally reached, and this lowest stage of emanation is matter, which exists only potentially.[29] Matter is not substantially evil, since it ultimately (even though indirectly) emanates from the One (and, thus, it is linked to goodness), and evil resides in matter’s state of privation, or in matter’s ontological weakness. Plotinus’s metaphysical type of intuition, known as ecstasy, refers to a conscious state in which consciousness leaves the material body and seeks to be absorbed into the absolute, the “One” (this is the type of intuition that underpins mysticism, in general).

In his *Metaphysics*, where he expounded his ontology, Aristotle articulated a philosophically rigorous interpretation of reality. In particular, in his *Metaphysics* (Books 7 and 9), Aristotle studied the distinction between potentiality (being potentially) and actuality (being actually). According to Aristotle, the matter of a being, namely, the stuff of which it is composed, is linked to potentiality, whereas the form of a being, namely, the way that stuff is put together so that the whole it constitutes can perform its characteristic functions, is linked to actuality. For instance, consider a piece of wood that can be carved or shaped into a bowl. In Aristotle’s terminology, the wood has at least one potentiality, since it is potentially a bowl. The piece of raw wood in the carpenter’s workshop can be considered a potential bowl (since it can be transformed into one), and the wood composing the completed bowl is also, in a sense, a potential bowl, but, when the bowl is used for the purpose intended, it exists actually, it is an actuality.

Aristotle’s distinction between potentiality and actuality presupposes a state of becoming in which a being is increasingly actualized and imposed according to an existential model that is originally contained in the given being. According to Aristotle, the aforementioned existential model is the “entelechy,” that is, the program of actualization, of a being, and it remains immutable regardless of the changes that a being may undergo. Moreover, according to Aristotle, a being is the simplest mental presence, but it is not absolutely simple, since it can be conceived as a resultant of categories (systems of general concepts); these categories, which correspond to the fundamental modes of being, can be summarized as follows: substance, form, structure (namely, the cohesive bond between substance and form), time, and space. The aforementioned five categories (specifically, modes of being) are qualities that

[29] See: Gerson, ed., *The Cambridge Companion to Plotinus*.

can be identified in and attributed to a being. Through his distinction between actuality and potentiality and through his study of the mode of being, Aristotle managed to transcend the antithesis between being and non-being, which was originally addressed by Parmenides in his poem *On Nature*.

Aristotle maintains that both the matter and the form of a being must pre-exist,[30] but the source of motion in both cases (what Aristotle calls the "moving cause" of the coming to be) is the form. In artistic production, the form is found in the soul of the artisan ("the art of building is the form of the house,"[31] and "the form is in the soul"[32]). For instance, the builder has in mind the plan for a house, he knows how to build, and, ultimately, he "enmatters" that plan by putting it into the materials out of which he builds the house. In natural production, the form is found in the parent ("the begetter is of the same species as the begotten, not one in number but one in form—for man begets man"[33]).

Plato's way of resolving the Parmenidean contradiction between being and non-being consists in his theory of ideas, according to which the degree to which a particular being participates and progresses in the corresponding idea, namely, in the corresponding beingly being (archetypal form), determines its degree of being. Platonic "ideas" are transcendent vis-à-vis the sensory, material world, but they are innate in consciousness, and they can come to the foreground of awareness through an epistemological and psychological method that Plato calls "anamnesis."[34] However, according to Aristotle, Plato's ideas are abstractions (concepts), and Aristotle's way of resolving the Parmenidean contradiction between being and non-being consists in his study of the entelechy of being, namely, in the study of the intrinsic program of ontological actualization of each being, which underpins the transition from being potentially to being actually. The history of medieval ontology is, in essence, a history of debates about Plato's and Aristotle's ways of resolving the Parmenidean contradiction between being and non-being.

A variety of "exaggerated" Platonic realism inspired and underpinned medieval philosophical realism, which was represented by such scholars as John Scottus Eriugena, Anselm of Canterbury, and Walter Burley, whereas a variety of Aristotelianism inspired and underpinned medieval nominalism, which was represented by such scholars as Roscelinus (Roscelin of Compiègne), Peter Abelard, and William of Ockham.[35] In particular, nominalism was a peculiar anti-realist interpretation of Aristotle's philosophy. According to medieval nominalists, "universals" (i.e., a class of mind-independent entities, which are the characteristics or qualities that particular things have in common, and therefore, they are contrasted with individuals) are just names, or words (hence, the term "nominalism"). However, there are two varieties of nominalism: "soft nominalism" rejects universals, but it affirms the existence of abstract objects (such as properties, propositions, numbers, and possible worlds), arguing that abstract objects are particular or concrete objects (mainly in the context of propositional discourse logic); whereas "hard nominalism" rejects both universals and abstract objects, and it maintains that only individuals exist (thus gradually giving rise to "anti-foundationalism," whose most radical representatives are Friedrich Nietzsche and the

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[30] Aristotle, *Categories*, 1034b12.
[31] Ibid, 1034a24.
[32] Ibid, 1032b23.
[33] Ibid, 1033b30–32.
[34] Plato, *Meno*; *Phaedo*; *Republic* (Book 7); and *Symposium*.
[35] See: McGrade, ed., *The Cambridge Companion to Medieval Philosophy*.

post-modernists, such as Michel Foucault, Jacques Derrida, and Richard Rorty). In the thirteenth century, Thomas Aquinas managed to articulate a robust system of philosophical theology based on a variety of "moderate" Aristotelian realism, without being trapped in disoriented and philosophically debasing controversies between Platonism and Aristotelianism, and, simultaneously, he managed to recognize and highlight the importance of consciousness.[36]

Transcendentalism and dualist realism underpin both Platonism and Aristotelianism: the basis, or the seat, of Plato's transcendentalism is the world of ideas itself, in which, according to Plato, the human mind participates (at least potentially), whereas the basis, or the seat, of Aristotle's transcendentalism is the human mind itself, which, according to Aristotle, conceives ideas as species and, hence, as abstractions; thus, Aristotle substitutes the notion of "entelechy" for Plato's notion of "methexis" (participation), without, however, negating Plato's metaphysical teleology (and, thus, internalizing Plato's transcendentalism). From the perspective of structuralism, Platonic realism corresponds to the *ante rem* structuralism ("before the thing"), in the sense that, according to Platonism, the ideational structure of mental life is a real but transcendent principle vis-à-vis the mind itself and the sensible world, and philosophical consciousness tries to partake of and progress in the world of ideas, while Aristotelian realism corresponds to the *in re* structuralism ("in the thing"), in the sense that, according to Aristotelianism, structures are held to exist inasmuch as they are exemplified by some concrete system, and the mind itself, not the world of ideas, is a real and transcendent principle vis-à-vis the sensible world, and it conceives ideas as abstractions. Despite the particular differences between Plato's and Aristotle's philosophies regarding forms, the ancient Greek philosophical community was aware that Platonism and Aristotelianism were not opposite to each other, since both Platonism and Aristotelianism are dualist realisms. Thus, as we read in Diogenes Laertius's *Lives of Eminent Philosophers*, IV, 67, Aristotle was the pioneer of the "peripatetic Platonists." However, in the Middle Ages, particular socio-cultural reasons gave rise to new interpretations of Platonism and Aristotelianism that highlighted the differences between them in a radical way.

The medieval social actors who adhered to the vertical (authoritarian) hierarchical system of feudal societies (at the top of which was the bishop (as a type and in place of Christ), and, below him, there were the sovereign, the nobility, the monks, the clergy, the knights, the so-called "boni hominess" (i.e., the bourgeoisie[37]), and the simple people ("popolo")) endorsed philosophical realism, and they interpreted Platonism as their major philosophical underpinning, emphasizing Plato's argument that ideas-as-beingly-beings are transcendent and govern the beings and the things of the sensible world "from above." In particular, the medieval adherents of the vertical hierarchical system of feudalism identified the term "generality" (signifying the highest level of abstraction and logical necessity) with the term "universality" (signifying a mind-independent "whole"), and then they equated the degree of reality with the degree of generality, and they instituted a system of social hierarchy founded on their notion of generality. Therefore, it is clear that the "exaggerated" medieval realism and, in general, the medieval social actors who endorsed realism as the major philosophical

[36] See: Conti, "Realism in the Later Middle Ages."

[37] The bourgeois are those who are neither plebeians (or "simple people") nor members of the nobility. Thus, the term bourgeoisie has often been identified with the term "middle class," and, in this case, the "bourgeoisie" has been subdivided into "petite (or small) bourgeoisie," "middle bourgeoisie," and "haute (or high) bourgeoisie."

underpinning of the vertical hierarchical system of feudal societies ignored or silenced the fact that, when Plato argued that ideas (i.e., "universals") are transcendent to the sensible, material world, he added that human beings can participate and progress in the world of ideas (i.e., in "universals") according to each person's degree of mental purification and development, thus pursuing humanity's *experiential access* to beingly beings, namely, to the energies of the good-in-itself.

According to Plato, the soul, like the body, is characterized by "that *sensation* which we know term 'seeing,'"[38] so that the knowledge of the good-in-itself depends on an internal, mental sensation (spiritual vision). Therefore, Plato emphasizes that the knowledge of the absolute good (the good-in-itself) presupposes not only the ability to give an account (i.e., discursive reasoning) but also a psychic cleansing or cure. In his *Republic*, 443d–e, Plato argues that one has cured his soul if he has "attained to self-mastery and beautiful order within himself, and . . . harmonized these three principles [the three parts of the soul: reason, the emotions, and the appetites] . . . linked and bound all three together and made himself a unit, one man instead of many, self-controlled and in unison." Since, as we read in Plato's *Republic*, 585b, the purpose of our existence is our experiential participation in the pure being (the good-in-itself) and our unification with the good-in-itself, psychic cleansing (spiritual psychotherapy) is a prerequisite to our transformation into the corresponding absolute principle; for, as Plato argues in *Phaedo*, 67b, "it cannot be that the impure attain the pure." In fact, even though Plato's philosophy clearly belongs to the "school" of realism, his aforementioned arguments regarding the *experiential participation* of the human soul in the transcendent world of ideas disclose an idealist aspect of Plato's philosophy.

On the other hand, the rising medieval bourgeoisie sought to replace the vertical hierarchical system of feudal societies with a horizontal model of social organization based on individualism, and, therefore, it realized that it had to fight against the philosophical underpinnings of the feudal establishment, namely, against medieval philosophical realism.[39] Hence, the intellectual elite of the medieval bourgeoisie endorsed nominalism, and it claimed that the fundamental arguments of nominalism were philosophically underpinned by

[38] Plato, *Timaeus*, 45d (emphasis mine).

[39] By the fourth century A.D., the major towns of the Western Roman Empire had been destroyed by the invasions of barbaric—primarily, Germanic—tribes. However, in the tenth century A.D., towns began to grow in Western Europe, and, within a short period of time, they gained autonomy. Autonomous towns were founded in the West as a reaction against the feudal regime. The townspeople started acting collectively. Initially, their communities were organized around a belfry: at the sound of the bell, all had to gather together since bells were ringing not only for religious purposes but also in order to announce a state of emergency or an imminent danger. Gradually, towns established a popular judicial system, their own system of policing, and their own treasury. Later, towns gained their independence, either by purchasing it or by using violent means. Thus, towns became free republics, and the growth of private property and commerce increased significantly. The townspeople—namely, liberated serfs, tradesmen, Jews seeking higher levels of safety and better economic opportunities, impoverished aristocrats, and various other opportunists and fugitives from the feudal system—built their own walls around their towns, and, thus, they became permanent residents of those towns and were called "bourgeois" or "burgenses," which literally means "of a walled town." Towns were attracting more and more people, not so much for the pursuit of financial gain as for the pursuit of freedom. Serfs could earn their living by cultivating the land, but they could not enjoy enough freedom. The quest for freedom was the strongest motive of the people who were leaving their agricultural jobs in order to live in a town (see: Pirenne, *Medieval Cities*). For instance, in the Middle Ages, many Germans used to say "Stadtluft macht frei" (i.e., "urban air makes you free"). Carlo M. Cipolla argues that, like the first European immigrants to America, the liberated serfs were moving to towns in order to have more opportunities for social and economic success than those supplied by the traditional and closed agricultural societies (Cipolla, *Before the Industrial Revolution*).

Aristotelianism. However, as I have already mentioned, nominalism articulated a peculiar anti-realist interpretation of Aristotle's philosophy, in the sense that nominalism ultimately shifted away from the aforementioned *in re* structuralism toward the *post rem* structuralism ("after the being"), according to which "to exist" merely means to be placed in a rational structure. In particular, the intellectual elite of the medieval bourgeoisie emphasized Aristotle's argument that "entelechy" (as a program of ontological actualization) is intrinsic to being, thus affirming the ontology of particularity as a philosophical underpinning of individualism, ignoring or silencing the fact that Aristotle, in line with Plato's transcendentalism, maintains that knowledge is a mental function, and that the mind proper (as the entelechy of the body) is transcendent to the body, and arises from the outside.[40] Indeed, by dismissing the aforementioned realist aspect of Aristotelianism, the nominalists' variety of "Aristotelianism" marks their shift from the *in re* structuralism to the *post rem* structuralism. Furthermore, the intellectual elite of the medieval bourgeoisie interpreted Aristotle's logic as a means of individual power (in terms of oratorical skills and exhibitions of macho intellectuality), whereas, for Aristotle, logic was a means of clear and accurate communication between conscious entities and, hence, an underpinning of correct social life in its broadest sense. Therefore, in the realm of theology, the realist scholastics, in one way or another (and irrespective of the mistakes that they perpetrated in the context of the feudal system and its underlying authoritarian mentalities), emphasize and seek to approach the wisdom and the harmony of the deity, whereas the nominalists, such as Ockham, discard such quests and highlight only the freedom of God's will and humanity's faith, and, in this way, they involuntarily sow the seeds of nihilism.[41]

In the seventeenth century, the Dutch-French philosopher and mathematician René Descartes (Latinized: Renatus Cartesius), initiating modern philosophy in a systematic way, sought to resolve medieval ontological controversies by highlighting the significance of consciousness.[42] In his *Meditations on First Philosophy* (which was originally published in Latin in 1641 under the title *Meditationes de Prima Philosophia*, and, in 1647, it was published in French under the title *Méditations Métaphysiques*), Descartes argued as follows:

> We say, for example, that we see the same wax when it is before us, and not that we judge it to be the same from its retaining the same color and figure: whence I should forthwith be disposed to conclude that the wax is known by the act of sight, and not by the intuition of the mind alone, were it not for the analogous instance of human beings passing on in the street below, as observed from a window. In this case I do not fail to say that I see the men themselves, just as I say that I see the wax; and yet what do I see from the window beyond hats and cloaks that might cover artificial machines, whose motions might be determined by springs? But I judge that there are human beings from these appearances, and thus I comprehend, by the faculty of judgment alone which is in the mind, what I believed I saw with my eyes.[43]

Thus, in his *Meditations on First Philosophy*, Descartes argued that a being is present both in itself, that is, independently of consciousness, and within consciousness, and that, in the latter case, consciousness is consciousness of a being, that is, it refers to a being, and it

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[40] Aristotle, *On the Generation of Animals*, II, 3.
[41] Cunningham, *Genealogy of Nihilism*.
[42] See: Cottingham, ed., *The Cambridge Companion to Descartes*.
[43] Descartes, *Meditations on First Philosophy*, Meditation II, paragraph 13.

underpins the given being's presence. It is exactly on this thesis that Descartes and subsequent members of his philosophical "school," known as Cartesianism, such as Nicolas Malebranche (1638–1715), Baruch de Spinoza (1632–77), and Gottfried Wilhelm von Leibniz (1646–1716), founded modern ontology.

Based on the aforementioned fundamental thesis of Cartesianism, the Serbian-German philosopher and mathematician Leibniz, in his *Monadology*, argued that the activity of the human mind corresponds to "monads," which are immaterial, unextended, self-determining, and purposive substances (forces).[44] According to Leibniz, every monad is a process of evolution, it animates matter, it has perception and appetition, and it realizes its nature with an internal necessity. In his theological essays, Leibniz argues that God created the monads, and He transcends all monads, but the human being, even though it is a limited monad, can maximize the qualities that are processed by each and every monad to a certain degree, and, in this way, the human being can achieve a partial knowledge of God, since God, Leibniz contends, is supra-rational but not contra-rational.

In Leibniz's philosophy, monads are united with regard to their existential end (i.e., in terms of their "teleology"), and, in this way, Leibniz sought to synthesize Descartes's ontology and biblical theology, but, according to Leibniz, monads are natural, distinct, and separate "infinitesimals," and, therefore, they are entities-in-themselves. An infinitesimal can be considered as the inverse of infinity, the smallest number possible (i.e., close to zero as possible) yet bigger than zero. However, the conception of an infinitesimal as an entity-in-itself is, first, mathematically uncomfortable, because, given any two numbers $a < b$, there is always a number c that can fit between ($a < c < b$), which, indeed, can be defined as $\frac{a+b}{2}$, and, if such a number c is an entity-in-itself, then the concept of a number that is the smallest number possible but bigger than zero (namely, a number k such that $0 < k < r$ for any other number r) is logically alarming. If infinitesimal monads are entities-in-themselves, then no computation of lengths, areas, and volumes is perfectly exact, in the sense that every computation of lengths, areas, and volumes contains a small, "infinitesimal," error. Therefore, as I shall explain in Chapter 2, in mathematics, the concept of the infinitesimal was ultimately replaced by the concept of the limit, which is a rule for reducing a quantity and making it get infinitely close to zero, and it underpins the rigorousness and the consistency of modern mathematical analysis. Furthermore, Leibniz's monadology is theologically uncomfortable, too, because, according to Leibniz's monadology, the knowledge of the "whole," or God, concerns each monad individually as a conscious entity, and, therefore, it tends to give rise to absolute particulars, or absolute "egos."

The German philosopher and mathematician Christian Wolff (1679–1754) redefined philosophy as the science of the possible, and, in a sense, his philosophical work is a common-sense adaptation of Leibniz's monadology.[45] According to Wolff's *Ontologia*, the task of the philosopher is to provide "the manner and reason" of every possible thing, since, according to Wolff, everything, whether possible or actual, has a "sufficient reason" for why it is rather than not. In section 56 of his *Ontologia*, Wolff defines "sufficient reason" as that from which it is understood why something is or can be.

In the eighteenth century, Immanuel Kant expressed his opposition to the ontological excesses of Leibniz and especially of Wolff by recognizing the necessity of the thing-in-itself

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[44] See: Jolley, ed., *The Cambridge Companion to Leibniz*.
[45] Ibid.

(namely, the object as it is independent of observation) while refusing to accept that the thing-in-itself is knowable and arguing that the thing-in-itself is transcendent.[46] Even though Kant's philosophy avoids and aptly criticizes the ontological excesses and the consequent rigid rationalism of Leibniz and Wolff, it entails a risk of confining consciousness to the logical form of experience, denying consciousness access to the real content of experience and, thus, giving rise to a superficial type of consciousness. Whereas Plato, like Kant after him, maintains that the thing-in-itself (in his case, the "idea") is transcendent, he specifies that the thing-in-itself can be participated, that is, experienced, by philosophizing consciousness in the context of a peculiar spiritual sense, which requires both the development of the power of the mind to think, understand, and form judgments logically and the completion of a process of psychic cleansing (what, in modern terms, we could call existential psychotherapy).

On the other hand, the German philosopher Georg Wilhelm Friedrich Hegel (1770–1831), the major representative of German romantic idealism, proposed an alternative solution to the persistent, Parmenidean-like, ontological controversy.[47] According to Hegel, the thing-in-itself, namely, being, is the idea, which, by giving rise to a contradiction to itself, moves away from itself in order to, ultimately, return to itself enriched by its adventure. Hegel's dialectical model—namely, the transition of the idea ("thesis") to an upgraded version of itself ("synthesis") through its contradiction ("antithesis")—synthesizes the perception of being and the perception of becoming. However, whereas Aristotle's conception of the transition from being potentially to being actually indicates a state of becoming that consists in the actualization of an ontological program, Hegel's conception of the transition from the "in-itself" to the "for-itself" through the "outside-itself" indicates a state of becoming that consists in a process of mutation and alteration, since Hegel argues that all life and movement are founded on contradiction, which rules the entire world, and it encompasses Hegel's secular theology. Thus, ultimately, in Hegel's philosophy, being is identified with the logic of historical becoming, which is independent of the human person, and history takes the place of God. In other words, according to Hegel, even though the world constitutes a historical creation of humanity, the world has become independent from human consciousness. Hence, the German philosopher Arthur Schopenhauer (1788–1860) has argued that Hegelianism paralyzes mental power and stifles real thinking, and the Austrian-British philosopher Karl R. Popper (1902–94) has argued that Hegelianism provides justifications for absolutist regimes, such as that of Friedrich Wilhelm III (king of Prussia from 1797 to 1840), and, in general, for statism.

The key point here is just this, that the essence of being (in Greek, "to on": "τό ὄν"; in Latin, "ens") and the act of being (in Greek: "to einai": "τό εἶναι"; in Latin, "esse") represent two different yet complementary aspects of the same reality. The distinction between the essence of being and the act of being has been emphasized and systematically studied by a philosophical "school" that is known as existentialism. The origins of existentialism can be traced to the theologian and philosopher Augustine of Hippo (who became the bishop of Hippo Regius in 395 A.D.) and to the philosopher and mathematician Blaise Pascal (1623–62). However, the most important representatives of existentialism are the Danish philosopher and theologian Søren Kierkegaard (1813–55) and the German philosopher Martin Heidegger

[46] See: Guyer, ed., *The Cambridge Companion to Kant*.
[47] See: Beiser, ed., *The Cambridge Companion to Hegel*.

(1889–1976).[48] Whereas Aristotelian ontology emphasizes the essence of being (namely, that of which a thing consists), the existentialists maintain that the most important ontological question is not the essence of being, but the presence of being, namely, the "existence" of a being. By the term "existence," existentialists refer to the event that a being is present before oneself, or independently of oneself, or united with oneself. In other words, from the perspective of existentialism, the most important ontological issue is that one is conscious of one's own existence and of that which exists outside one's consciousness. Moreover, according to existentialism, if one is conscious of one's own existence, then objects, including one's self, exist not only "in themselves" but also "for oneself" (thus, giving rise to "reflective cogito"). This distinction, which has been highlighted by the French philosopher and political activist Jean-Paul Sartre (1905–80), reflects the influence that Hegel's dialectic exerted on existentialism, since Hegel's principle of contradiction(s) underpins the methodology of existentialism. Thus, Sartre maintains that the deeper purpose of an ideological program, irrespective of its external appearance, is to change one's basic condition through the awareness of the antitheses to the given condition.

The existentialist thesis that essence and presence are not necessarily identical to each other follows from the fact that one can think of essence independently of its reality. For instance, let us consider the concept of infinity, which most of us encounter the first time when we learn to count, realizing that we can go on counting forever, since we can always add one and, thus, obtain an even larger number. Moreover, in Chapter 2, I shall explain that, odd as it may sound, there are different types of infinity, and I shall study rigorous definitions of each type of infinity. However, in the context of physics, an element of a theory of nature is said to exist only if it is necessary in order to describe observations, and, because infinity cannot be practically measured, natural scientists do not actually need it in order to describe what they observe. Thus, in the natural sciences, infinity can always be replaced by a suitably large but finite number. When natural scientists have to measure something *practically* infinite, they usually mean that it is indeterminately much larger than something finite that they have already measured. In other words, the difference between "practical infinity" and "mathematical infinity" is the following: something is *practically* infinite if it is indeterminately much larger than something finite that one has already measured, whereas something is *mathematically* infinite if it is larger than anything that one could possibly have measured, and there is no experiment that can verify such a claim. We can analyze infinity and talk about its properties in the context of mathematics, but infinity does not practically exist in nature. By defining infinity in the mathematical sense, we declare its essence without, however, imposing its existence in practice. Similarly, by defining the mythical creature chimaera (which, according to Greek mythology, was a monstrous fire-breathing hybrid creature usually depicted as a lion with the head of a goat arising from its back, and a tail that might end with a snake's head), we declare its essence, but we do not impose its existence in the natural world.

In general, for human consciousness, essence and presence are not necessarily identical to each other. Human consciousness may differentiate essence and presence from each other, and it may judge each of them differently from the other. However, in the Bible, precisely, in the book of Exodus 3:14, we read that one of God's names is "I am that I am," which implies the union between God's presence and God's essence. This is an exceptional case in which

[48] See: Crowell, ed., *The Cambridge Companion to Existentialism*; Earnshaw, *Existentialism*.

God, who is absolute, reveals Himself, and, therefore, philosophy cannot consider this case as a typical one. From the perspective of philosophy, the aforementioned narrative about God's self-revelation can be approached as a case of metaphysical intuition, which is often referred to as an experience of divine illumination.

According to existentialism, existence precedes essence, not so much in the temporal sense as in the sense of importance. The first priority of the philosophers of existence consists in the following dual task: first, they have to explain the manner in which human existence and human knowledge progress from one level of being and knowledge to another; second, they have to explain the manner in which consciousness evolves gradually by confronting its own antinomies, thus progressing from an immediate and unformed state to a condition of internal unity and integral self-experience. In particular, the German-Swiss psychiatrist and philosopher Karl Jaspers (1883–1969) ascribed central status to "limit situations" (*Grenzsituationen), which* are moments, usually accompanied by experiences of dread, guilt, and/or acute anxiety, in which the human mind confronts the restrictions and pathological narrowness of its existing forms, and it allows itself to abandon the security of its limitedness and so to enter a new realm of self-consciousness.[49] Additionally, Jaspers developed a theory of the "*unconditioned"*(*das Unbedingte*), arguing that human limitations are neither absolute nor fixed, and that, in general, human life is basically about growing and outgrowing our old, immature and less perfect ways.

Existentialism inquires into the event of the emergence of existence out of non-existence. In particular, existentialism assigns primary importance both to the process according to which existence emerges out of non-existence and to the reality of non-existence out of which existence emerges. Thus, existentialists are ultimately preoccupied with the "archeology" of existence (i.e., of the presence of being), and, more specifically, they seek to find the reason for the emergence of existence out of non-existence and to determine whether existent reality emerges of itself for the sake of existence, or if, as Jaspers has argued, it is thrown out of its original "encompassing" (*Umgreifende*), which is a transcendent and obscure reality (the absolute being) within which existence is formed and maintained before being "thrown into the world."[50] However, even after Jaspers's contribution to existentialism, the philosophical "school" of existentialism is not complete, because—apart from inquiring into the process according to which existence emerges out of non-existence and into the reality of non-existence out of which existence emerges—one must also study the process of the creation of existence in relation to the exact moment at which the transition from non-existence to existence takes place and to clarify the relation between that moment and the event of existence (for instance, as I shall explain later in this chapter, modern physics assigns primary importance to the inquiry into the initial conditions of the universe and especially to the moment of the "Bing Bang").

Finally, it should be mentioned that, even though Martin Heidegger played a key role in the development of existentialism, he lapsed into false and exaggerated assertions, especially regarding the extent to which his existentialist philosophy marks a radical departure from modern Western ontology (of which two of the most important representatives are Descartes and Kant) and could provide a complete substitute for the thinking subject of modern Western ontology. In fact, in his philosophy, Heidegger replaced the "ego" (specifically, the subject as

[49] See: Schilpp, ed., *The Philosophy of Karl Jaspers*.
[50] Jaspers, *Reason and Existenz*.

a syllogistic, or representational, certainty) with the act of being *per se*, specifically, in the statement "I am," he separated the "I" from the "am" (in Latin, "sum"), and he discarded the "I" while keeping only the "am." Thus, Heidegger attempted to remove every element associated with the consciousness of the external world from the ego, because, even indirectly, such elements connect the ego with a transcendent reality (a transcendental signified). According to Heidegger, "Dasein" (i.e., "being there" or "presence") should be understood as the structure of existence, and not as the consciousness of existence, and, furthermore, for him, Dasein is the event that underpins the understanding of the act of being. Heidegger claimed that, in the aforementioned way, he achieved to totally dismiss the thinking subject of modern Western ontology as a redundant and problematic element, but, contrary to Heidegger's expectations and assertions, the modern subject is, at least indirectly or subconsciously, still present in Heidegger's philosophy, due to the fact that the ego is, at least indirectly or subconsciously, present within the "am," and due to the fact that the ego as "otherness" underpins the manifestation of Heidegger's concept of Dasein.

1.2.3. The Knowledge of a Being

Based on the Aristotelian distinction between actuality and potentiality, I term "ontological situation" the degree to which a being has actualized its entelechy, or, in other words, the degree to which a being is. Hence, an "ontological situation" is a stage, or a particular moment, of a being's ontological development. The act of being is a situational reality, while the essence of being is a specific reality. The degree to which consciousness knows the act of being and the essence of being depends on one's way of experiencing them. The most traditional way of knowing a being is related to "methexis," or "methexiological perception," which is an ancient Greek term meaning "participation" and "group sharing."

The Swiss psychiatrist and psychoanalyst Carl Jung (1875–1961) has pointed out that the so-called "archaic mentality"—as exemplified by ancient mystery cults[51] (such as the Isis and Osiris Mysteries, the Orphic Mysteries, the Eleusinian Mysteries, Zoroastrianism, Moses's religious and legal system, etc.) and by ancient Greek tragic poetry—is inextricably related to the "relation of identity" with the object of consciousness ("participation mystique") and to the "fusion of psychological functions" (e.g., thinking is fused with feeling, feeling is fused with sensation and intuition, etc., and a part of a psychological function may be fused with its counterpart).[52] Therefore, in the context of the archaic mentality, the interpretation of ontological activity is based on methexis (i.e., the idea of an analogical participatory view of reality), which is based on the hypothesis that there is continuity between beings, ontological states, and conscious experiences. Moreover, in the context of the archaic mentality, methexis is underpinned and secured by a series of transcendent "first causes," that is, supernatural forces, which act according to a complex system of choices known only to a few "initiates," "magi," or "prophets" whose consciousness can, arguably, intervene in the functioning of the world by activating or de-activating the hidden underlying forces of the world at will. This mentality discloses an intention and an attempt to subjugate the world to the intentionality of consciousness, and it is expressed by taking a specific form in the context of myth, which

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[51] See: Burkert, *Ancient Mystery Cults*.
[52] Jung, *Civilization in Transition*.

corresponds to the spiritual core of things and operates as a magic formula.[53] In principle, magic is the traditional science of the secrets of nature and of the human being. It is the old name of the subject matter of the ancient occult initiates and intellectuals of India, Chaldea, Persia, Egypt, and Homeric Greece. The French occultist and alchemist François Jollivet-Castelot (1874–1937) has explained the meaning of magic as follows:

> Magic is by no means, as most outsiders imagine, the negation of Science. Quite on the contrary Magic is *Science*, but Science with syntheses, almost integral Science, its horizons being the Absolute, the Infinite in Unity . . . In truth *Magic is the knowledge of the action and the combination of the forces of the Universe . . . the study of their conduct, their involution, their evolution*.[54]

However, methexis may hold not only in the context of the archaic mentality but also in the context of the philosophical mentality. Plato's philosophy and Neoplatonism are the major representatives of methexis in the realm of philosophy. Intimately related to the philosophical concept of methexis is a dynamic perception of reality. From the perspective of methexis, a being is not a closed, inviolable, and self-centered system. In contrast to any static ontological consideration, the philosophical concept of methexis is based on the hypothesis that there is a continuous dynamic communication between beings, in general, as well as between conscious minds that undertake ontological endeavors, in particular. Hence, from the perspective of methexis, all beings and all situations are connected with each other and continuously open to each other, so that they participate in each other, and, through these relationships, they ultimately participate in the unique cosmic reality out of which they have emerged as particular manifestations of being.

The passive variety of methexis is focused on heredity, specifically, on those features that a being or a situation has inherited and continues to preserve, as it is mentioned, for instance, in Aristophanes's speech in Plato's *Symposium* (189c–193e).[55]With regard to philosophical anthropology, the major concept that is subject to the passive variety of methexiological perception is that humans are all sprung from the same stock, partake of the same nature, and share the same hope. On the other hand, the active variety of methexis is focused on an attempt to create a new situation by means of which a conscious community (whose members share common goals) seeks to transcend an already existing situation. Thus, the passive variety of methexis highlights the interdependence of beings as well as their dependence on their common nature, whereas the active variety of methexis highlights the manner in which beings indentify with each other in the context of a collective activity.

Apart from the event and the awareness of methexis, the knowledge of a being in the context of philosophy is necessarily dependent on the use of a specific method. Hence, we talk about methodology. In general, philosophical methods can be distinguished into two categories: *a priori* ("from the earlier") methods and *a posteriori* ("from the later") methods.

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[53] See: Lévi, *The Doctrine and Ritual of High Magic.*

[54] Quoted in Poinsot, *The Encyclopedia of Occult Sciences*, p. 305. Moreover, see: Versluis, *Magic and Mysticism*.

[55] Aristophanes's speech focuses on human nature, and it provides a mythical account of the ontological significance of love. He explains that the present form of human beings originated from ancient gods' decision to cut the primeval, powerful androgynous type of human being in half in order to control humanity more effectively, and, thus, through love, the two sexes of the human species tend to recompose their previous common form of existence.

The major attribute of the *a priori* methods is that they are based on primitive hypotheses usually intuitively conceived and axiomatically accepted,[56] which deductively give rise to series of syllogisms, which, in turn, lead to ultimate conclusions, which are related to the preceding propositions in a logically rigorous way, even though it is often the case that big hypothetico-deductive systems have flaws. As I shall explain in Chapter 3, logical paradoxes have played an important role in the development of mathematics. In summary, in a "hypothetico-deductive" (or "axiomatic") system, there are two requirements that must be met in order that we agree that a proof is correct: (i) acceptance of certain statements, called "axioms," without further justification; and (ii) agreement on how and when one statement "follows logically" from another, that is, agreement on certain rules of reasoning. Inextricably linked to the aforementioned two requirements is the requirement that every person who applies hypothetico-deductive reasoning to a particular discourse understands the meaning of the words and the symbols that are used in that discourse. The more consistent and the more complete a hypothetico-deductive system is, the more its imposition is safeguarded. By the term "consistency," we mean that the axioms of a hypothetico-deductive system neither contain nor produce contradictions. By the term "completeness," we mean that the truth value of any proposition that belongs to a hypothetico-deductive system can be determined within the given hypothetico-deductive system (that is, according to the terms and the rules of the given hypothetico-deductive system).

During Antiquity, the first *a priori* philosophical methods were developed by the pre-Socratic philosophers, whose model (as mentioned in section 1.2.2) focuses on the determination of a principle that was assumed to be the origin of the world and to give rise to every particular reality. Inherent in pre-Socratic philosophy is a form of dogmatic scientism, which was successfully refuted by Socrates and the sophists. The sophists (namely, such orators and professional educators as Protagoras, Gorgias, Antiphon, Hippias, Prodicus, and Thrasymachus) argued that it is reasonable to question the absolute validity of previous philosophical achievements.[57] Socrates—through the "maieutic method," which he himself developed—sought to find a reliable method for obtaining truth.[58]Indeed, in the context of carefully structured conversations or dialogues, Socrates would ask probing questions that cumulatively revealed his interlocutors' unsupported assumptions and misconceptions, and, thus, his method would "give birth" to truth by eliciting a clear and consistent formulation of a thesis that was supposedly implicitly known by all rational beings. Despite the philosophical controversies between Socrates and the sophists, the philosophies of both Socrates and the sophists mark a major shift away from philosophies focused on the world toward philosophies focused on the human being.

In his early dialogues, Plato delineated Socrates's maieutic method combined with the practice of "Socratic irony," which is often condensed into the paradoxical statement "I know that I know nothing," which is attributed to Socrates, paraphrasing Socrates's statements in *Apology*, 29b–c, and *Meno*, 80d1–3. In particular, "Socratic irony" is a method of argumentation according to which one pretends to be ignorant in order to expose the ignorance or the inconsistency of someone else through adequately posed questions. However, in his middle dialogues (for instance, in *Phaedo*, *Symposium*, *Republic*, and

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[56] By being "axiomatically accepted," we mean that certain hypotheses are accepted, without proof, on the basis of their intrinsic merit, or because they are regarded as self-evident.

[57] See: Kerferd, *The Sophistic Movement*.

[58] "Maieutic" is a Greek word literally meaning "of midwifery."

Phaedro), Plato developed his own method, which is known as Plato's "dialectic." Plato's dialectic consists of two mutually complementary, particular processes of inquiry: the "ascending" process of inquiry and the "descending" process of inquiry. According to the ascending process of inquiry, consciousness starts from sensible objects (where the source of belief is sense perception) and ascends to higher levels of conceptual knowledge, which is conversant with the ultimate realities. According to the descending process of inquiry, consciousness starts from the knowledge of the ultimate realities and descends to the different levels of application, or manifestation, of those ultimate realities in the sensible world. In other words, through the ascending process of inquiry, the philosopher's consciousness proceeds from the phenomena to the ideas, which are participated by the phenomena and of which the phenomena are imitations, whereas, according to the descending process of inquiry, the philosopher's consciousness proceeds from the knowledge of ideas to the interpretation of phenomena.

Aristotle's methodology is similar to Plato's dialectic, and it also belongs to the category of *a priori* methods. In particular, Aristotle's philosophical methodology consists in determining a science of the "whole" being and in using this science in order to interpret every particular reality. There is a significant similarity between Aristotle's method and the geometric method, which is a style of proof that was used by Euclid in order to prove geometric theorems. In the sixteenth century, the Italian Aristotelian philosopher and logician Giacomo (or Jacopo) Zabarella described the geometric method as involving two aspects: (i) the resolutive aspect, known also as the analytic side of the geometric method, and (ii) the compositive aspect, known also as the synthetic side of the geometric method.[59] In his *Posterior Analytics*, Aristotle combines rational primitivism and empirical primitivism: (i) Aristotle's rational primitivism (reflecting the mentality of the *a priori* methods) is expressed by his thesis that demonstrative understanding (namely, understanding based on the geometric method) necessarily proceeds from elements that are true, primitive, immediate, and more familiar than, prior to, and explanatory of the conclusions; (ii) Aristotle's empirical primitivism (reflecting the mentality of the *a posteriori* methods) is expressed by his thesis that we must know the primitives (namely, axioms) by induction, since, according to Aristotle, induction is the way in which perception instills universals, and definitions are some of the most important elements of an axiomatic system that will be grasped by consciousness as a result of induction. In modern science, Leibniz used the geometric method emphasizing rational primitivism, while Isaac Newton used the geometric method emphasizing empirical primitivism.

The ascending and the descending processes of inquiry that constitute Plato's dialectic were reversed by Plotinus and, generally, by Neoplatonism: when the human soul descends from the World Soul into a particular (material) body, ascent, namely, the reversal of descent, necessitates that the descended soul generate love of the World Soul and of the higher dimensions of Nous and the One; and the generation of love by the descended soul gives rise to and underpins philosophy, namely, the love of wisdom. In this way, Plotinus and, generally, Neoplatonism articulated an apophatic approach to a totally transcendent One, which accounts for the unity and the existence of both formal reality and the material instantiation of that reality.

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[59] Zabarella, *On Methods*.

Plotinus utilized Plato's method of inserting two intermediate ontological terms into the initial Parmenidean perception of the antithesis between being and non-being, and he argued that we can successively contemplate both the emanation and the dialectical return of the four primary hypostases (the One, the Nous, the World Soul, and Matter). According to Plotinus, it is only the One—which is the origin of every other hypostasis—that is not susceptible of any methodical approach. However, Plotinus maintains that even the One is susceptible of knowledge, yet in an apophatic way (that is, through negating concepts that might be applied to it). In fact, Plotinus's method underpins apophatic theology.

During the Middle Ages, both the Platonic-Neoplatonic methodology and the Aristotelian methodology were used, and the Aristotelian methodology was endorsed and adjusted to the intellectual needs of medieval Christendom by Thomas Aquinas, the major representative of scholastic philosophy. Moreover, in the context of modern philosophy, Neoplatonism continues to play an important role, both due to the fact that Neoplatonism is based on a robust Platonic ontology, which can be discarded only if one is ready to totally negate the reality of the world, and due to the fact that Neoplatonism has given rise to several methods of overcoming the antitheses that characterize Platonic ontology (e.g., the antithesis between beingly beings and beingly non-beings). Thus, Neoplatonism has played an important yet implicit role in the development of modern dialectical philosophies, which, in turn, underpin the development of infinitesimal calculus by Newton and Leibniz.

In the seventeenth century, the British philosopher and statesman Francis Bacon systematized the empirical method (induction), which was originally developed by Italian scientists during the Renaissance. Bacon's method is based on a double empirical and rational standpoint. In his *Novum Organum Scientiarum*, *induction* implies ascending to axioms as well as a descending to works, so that, from axioms, new particulars are inferred, and, from these, new axioms. In fact, induction starts from sensory-sensuous data and moves, through natural history (providing sensory-sensuous data as guarantees), to lower axioms or propositions, which derive from the tables of presentation or from the abstraction of notions. By the term "experience," Bacon does not refer to everyday experience, but he presupposes that his empirical method corrects and extends sensory-sensuous data into facts, which go together with his setting up of tables (tables of presence and of absence as well as tables of comparison or of degrees, namely, degrees of absence or presence).[60] However, Bacon's empirical method does not end here, since Bacon assumes that, from lower axioms, more general ones can be inferred by induction. Moreover, from the more general axioms, Bacon strives to reach more fundamental laws of nature, which lead to practical deductions as new experiments or works.

Descartes understood the significance of Bacon's new scientific method, and he used it in order to criticize and overcome scholasticism, even though Descartes was, to a large extent, intellectually molded by scholasticism, and, thus, his intellectual weapons were mainly of Aristotelian origin. Descartes formulated the analytical geometric method, which I shall systematically explain in Chapter 2. In a famous passage in his replies to Marin Mersenne's objections to the *Meditations*, in discussing the distinction between analysis and synthesis, Descartes remarks that analysis is the best and truest method of instruction, and it was this method alone that he employed in his *Meditations*. In his *Discourse*, Descartes showed how the arithmetic operations of addition, subtraction, multiplication, division, and the extraction

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[60] See: Malherbe, "Bacon's Method of Science," p. 85.

of roots can be represented geometrically. In general, within the framework of analytic geometry, problems can be broken down into simpler problems involving the construction of individual straight lines, thus leading to an analytical approach to geometry. Hence, Descartes's *Geometry* is based on the use of algebra, which was called an "art of analysis."

The core of the study of structures in mathematics consists in taking numbers and putting them into equations in the form of "variables"; and the rules for manipulating these equations are contained in algebra. By reducing geometric problems (namely, problems about shapes and the manner in which they behave in spaces) to equivalent algebraic ones, Descartes made a major contribution to mathematics. Furthermore, Descartes's analytic geometry is of great philosophical significance, too, because, by reducing geometric problems to algebraic ones, Descartes managed to formulate a type of an *a priori* geometric philosophical method whose primary principle is not an object of the external world, but it is conscious experience itself. Descartes, intellectually, moves away from objects that are external to consciousness and turns his attention to conscious experience itself, and, through the algebraic representation of geometric problems, he throws light on the structure of problem-solving in general.[61]

However, the Dutch philosopher Baruch de Spinoza, who was one of the most important representatives of Cartesianism, attempted to apply the geometric method in a way that gives rise to an extreme variety of logical formalism and to a suffocating rationalist worldview. In Spinoza's totally rationally organized universe, the only ways in which the human being can manifest humanity's freedom are murder, suicide, and madness. Descartes was much more careful than Spinoza, because, in contrast to Spinoza's formalist excesses, Descartes highlighted the importance of internal experience (intuition).

Kant's philosophy was the major underpinning of the second, in turn, great philosophical shift away from the world (philosophical cosmology) toward the human being (philosophical anthropology). In his *Prolegomena to Any Future Metaphysics that Could Come Forth as Science*, 4:260, Kant famously admitted that he was influenced by the Scottish philosopher David Hume's empiricism, which was inextricably linked to skepticism (through which Hume attempted to deconstruct ordinary claims to knowledge), and, in general, it was formulated within a cultural milieu determined by British philosophers' elaborations of Bacon's method.[62] Kant adopted a "critical," yet, in reality, ambivalent, attitude toward an *a priori* method of philosophical research and an *a posteriori* one, and, thus, his philosophy gives rise to two different philosophical methods, both of which have played important roles in modern philosophy, namely: (i) an "idealist" method, which, according to the modern interpretation of the term "idealism," is founded on the principle that research can be proved only by internal experience (i.e., by the empirical cognition of mental states, such as sensory perception, thinking, memory, imagination, feeling will, and desire), which was exalted by Descartes, who did not, however, negate the objective extension of consciousness; and (ii) a "positivist" method, according to which research can be proved only by empirical means (not argumentations), research should be mostly deductive (i.e., deduction is used to develop statements that can be empirically tested), and knowledge should be judged by logic and ideally should be true for every segment of space-time, whereas every object that is directly related to a transcendent reality should be discarded, since Kant argues that humans cannot have theoretical knowledge of things-in-themselves (however, Kant maintains that humans

[61]See: Gaukroger, *Cartesian Logic*.
[62] See: Guyer, ed., *The Cambridge Companion to Kant*.

can have practical knowledge of things-in-themselves). As the British philosopher Peter Strawson has explained, Kant "distinguishes between the *receptive* faculty of *sensibility*, through which we have intuitions, and the *active* faculty of *understanding*, which is the source of concepts."[63] Through the receptive faculty of sensibility, the objects are "given," whereas, through the active faculty of understanding, the objects become objects of "thought."[64]

In the second edition of his *Critique of Pure Reason* (B, ix–x), Kant maintains that, "so far as reason is to be in these sciences," something within them must be a kind of *a priori* knowledge, and this *a priori* knowledge must be related to its object in two ways: either merely to determine its object and its concept (which must be given from elsewhere), or also to make it actual; the former is "theoretical knowledge of reason," and the latter is "practical knowledge of reason." According to Kant, the goal of theoretical reason is to assess how things are, whereas practical reason decides how things ought to be and what persons should do. However, while practical reason decides what to do, it cannot remake reality in an arbitrary manner; instead, the successful practical agent must take account of truths about the world.

In his *Transcendental Aesthetic*, Kant refers to the followers of Newton's position as the "mathematical investigators" of nature, who contend that space and time "subsist" on their own, and to the followers of Leibniz's position as the "metaphysicians of nature," who think that space and time "inhere" in objects and their relations. At the ontological level, Kant's position is that space and time do not exist independently of human experience, but they are "forms of intuition" (i.e., conditions of perception imposed by human consciousness). In this way, he managed to reconcile Newton's and Leibniz's arguments: he agrees with Newton that space is an irrefutable reality for objects in experience (i.e., for the elements of the phenomenal world, which are the objects of scientific inquiry), but also he agrees with Leibniz that space is not an irrefutable reality in terms of things-in-themselves. At the epistemological level, unlike David Hume, Kant argues that the axioms of Euclidean geometry are not self-evident or true in any logically necessary way. For Kant, the axioms of Euclidean geometry are logically synthetic, that is, they may be denied without contradiction, and, therefore, consistent non-Euclidean geometries are possible (as Nikolai Ivanovich Lobachevski and Bertrand Riemann actually accomplished). However, Kant argues that the axioms of Euclidean geometry are known *a priori*, specifically, they depend on our intuition of space, that is, space as we can imaginatively visualize it. After the publication of Kant's philosophical works, numerous attempts have been made to articulate methods of philosophical research that synthesize idealism and positivism, or that at least combine aspects of idealism and positivism with each other.

Hegel's dialectic is both a method of philosophical research and a model of the process according to which reality develops and tends to its ontological integration. This dual nature of the term "dialectic" undermines the accuracy of the given term, and it induces ambiguity, which characterizes both Hegelianism itself and those philosophies which are inspired by Hegelianism and, under the influence of Hegelian prophetism, tend to understand dialectic as an oracle. For instance, let us consider the case of Karl Marx's political and economic theory. Influenced by Hegel's dialectical thought and by Wilhelm Weitling's theory of revolutionary

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[63] Strawson, *The Bounds of Sense*, p. 86.
[64] Ibid, p. 48.

communism, Marx based his conception of communism on a contrast between alienation of labor under capitalism and a communist society in which human beings could freely develop their nature by controlling the sum total of the relations of production in a way that expresses human freedom and creativity as well as social justice. However, Marx has not clarified whether "scientific materialism" (both as "dialectical materialism" and as "historical materialism"[65]) is a general method or a model of particular objective processes that he seeks to interpret and evaluate, and, therefore, from a rigorous philosophical perspective, scientific materialism is inherently ambiguous.

Dialectical materialism is the world outlook of Marxism–Leninism, and historical materialism is the extension of the principles of dialectical materialism to the study of social life. In fact, Karl Marx (1818–1883) articulated the reversal of Hegel's dialectic in where Marx argues that, with Hegel, the dialectic "is standing on its head," and that "it must be inverted, in order to discover the rational kernel within the mystical shell."[66] In addition, Marx argues that his dialectic is the direct opposite of Hegel's dialectic, in the sense that Hegel transformed the process of thinking, called "the Idea," into an independent subject, to which he attributed the creation of the real world, whereas, according to Marx, the ideal is an intellectual reflection of the material world "translated into forms of thought."[67] Moreover, Vladimir Lenin read Hegel through Marx's *Capital*, and, therefore, like Marx, he read Hegel by reversing Hegel's dialectic.[68] According to Lenin, "matter is a philosophical category" that denotes "the objective reality which is given to man by his sensations, and which is copied, photographed and reflected by our sensations, while existing independently of them."[69]

The strength of Marx's account of history and politics is his analysis of capitalism and of the conditioning of social, political, and intellectual life by the way in which people produce their means of subsistence and, particularly, by the classes yielded by the different relationships of social groups to the factors of production, but the predictive and prescriptive aspects of Marx's theoretical works are less satisfactory, since he has not articulated a clear and consistent analysis of the relationship between the objective and the subjective forces of history. The Italian communist philosopher, journalist, and politician Antonio Gramsci (1891–1937) identified the aforementioned ambiguity of scientific materialism, and he attempted to overcome it through his theory of "cultural hegemony" and by articulating a humanistic interpretation of Marx's thought in the context of a "philosophy of praxis" that transcends both traditional materialism and traditional idealism.[70] From the perspective of the philosophy of rational dynamicity, which I propose in this book, unless humans attain a high level of rationality (such as that envisaged by Immanuel Kant), and unless Marx's prescriptive arguments and ideas are interpreted according to Gramsci's humanism, Marxism degrades into an intellectual shelter for people who are imbued with class envy, and, simultaneously, they are unable to achieve their selfish goals through capitalism, for which reason they ostensibly resort to socialism. Hence, the aforementioned ambiguities that characterize the revolutionary theories of Karl Marx and of other social theorists inspired by Hegelianism pose a serious risk of political teratogeneses, specifically, fanatical and extremist

[65] Marx, *Capital*, vol. 1.
[66] Ibid, p. 103.
[67] Ibid, p. 102.
[68]See: Tabak, *Dialectics of Human Nature in Marx's Philosophy*.
[69] Lenin, *Materialism and Empirio-Criticism*, p. 130.
[70] Gramsci, *The Prison Notebooks*.

movements, whether social, national, or religious, in the context of which, as the philosopher Eric Hoffer has pointed out, frustrated people are attracted to revolutionary visions not because they genuinely seek their "self-advancement," but because they express their "passion for self-renunciation," namely, a desire for an escape from the self and one's personal responsibility, and, therefore, they ultimately give rise to totalitarian regimes.[71]

Furthermore, various positivist philosophies and especially the French philosopher Auguste Comte's positivism are also characterized by an inherent ambiguity, because, on the one hand, they seek to follow exclusively an *a posteriori* method, but, on the other hand, their philosophical activity depends on an *a priori* (axiomatically accepted) model of human progress in accordance with Comte's "law of the three stages,"[72] whose origins can be traced to the beliefs of the thirteenth-century Italian scholar Gerardo di Borgo San Donnino. Comte's positivism has managed to influence epistemology, but it has failed to stand as a general method of philosophical research (arguably, being able to offer only a general method of mystical scientism).

Auguste Comte (1798–1857) is one of the acknowledged founders of sociology, but he is also the father of the "Religion of Humanity." According to Comte's law of the three stages, in its development, humanity passes through three successive stages, namely: (i) the theological stage (during this phase, people believed whatever they were taught by tradition, and fetishism played a significant role); (ii) the metaphysical stage (it was a transitory phase that involved the justification of universal rights on the basis of the sacred, and, during this phase, people started reasoning and questioning, although no solid evidence was laid); and (iii) the positive stage (the phase of questioning authority and religion and of following science). In fact, Comte attempted to transform "positive science" into a form of "positive religion," a non-theistic religion of humanity and society, with its own calendar of saints (such as Adam Smith, Frederick the Great, Dante, Shakespeare, etc.).

In the twentieth century, positivism gave rise to neo-positivism, which was expounded and systematically promoted by a group of early twentieth-century philosophers (chaired by the German philosopher and physicist Moritz Schlick) who became collectively known as the "Vienna Circle."[73] In the context of neo-positivism, a "scientific theory" is defined to be any consistent set of sentences of a logic (formal language) L closed with respect to logical deductions (i.e., deductive inferences can be established), and theories may be articulated either as pure deductive systems or as applied (empirical) deductive systems. Pure sciences consist in pure deductive systems, and, therefore, they are tautological in character, that is, theorems derive from postulates through entailment or logical implication (see also Chapter 3). Hence, in pure sciences, theorems merely reassert what was already implied in the postulates. Yet, these theorems bring to light truths that, although they were implicitly contained in the adopted set of postulates, were not explicitly known to the scientists who have adopted the given set of postulates. In particular, the German philosopher Carl Gustav Hempel, who was also associated with the Vienna Circle, argues that a theorem's "content may well be *psychologically* know in the sense that we were not aware of its being implicitly contained in the postulates."[74] However, one should not get the impression that pure sciences cannot be transformed into empirical ones. There are certain conditions under which a pure

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[71] Hoffer, *The True Believer*.
[72] See: Gane, *Auguste Comte*.
[73] See: Richardson and Uebel, eds., *The Cambridge Companion to Logical Empiricism*.
[74] Hempel, "Geometry and Empirical Science," p. 241.

science can be transformed into an empirical one (the transformation of Riemannian geometry into physical geometry by Albert Einstein is a case in point). According to neo-positivism, the transformation of a pure science into an empirical one entails two steps that must be taken: (i) The first step consists in the epistemological correlation of the primitives (i.e., the concepts that are not defined in the given axiomatic system) to operationally defined concepts with empirical content, so that the postulates take on a truth value. (ii) Once the first step has been taken, the second step consists in the confirmation of the postulates: in fact, what one has to do in this step is to derive operationally meaningful theorems from the postulates and test them against the facts. In case the observations do not contradict the operationally meaningful hypotheses, the theory is provisionally acceptable. Otherwise, the theory is disconfirmed; if this is the case, then one has to look for different postulates that will give rise to a theory consistent with the observations.

During an important part of his life, the Austrian-British philosopher Ludwig Wittgenstein (1889–1951) was in close contact with the Vienna Circle. Wittgenstein was one of the founders of analytic philosophy, which has played a decisive role in the development of particular methods of identifying and investigating linguistic forms that express mental processes.[75] Nevertheless, analytic philosophy may lead to an impasse, because it urges one to repeat the distinction between cognition and the object of cognition *ad infinitum* (forever). Inherent in analytic philosophy is a more technical restatement of Kant's abortive attempt to define the presuppositions of the presuppositions of philosophy, which can continue *ad infinitum*.

It should be clear by now that the articulation of *a posteriori* methods is an arduous task, always undertaken at the risk of failure as a result of a single contradictory instance or an intrinsic inconsistency. However, as I shall explain in Chapter 3, a philosopher or a scientist should not discard one's theoretical construction for the sake of such a contradictory instance or an inconsistency, but one should test one's model in a particular context where its constituent statements are confirmed and will claim that the given model was meant for that context and not for the one in which it has been disconfirmed. Thus, as I shall explain in Chapter 3, it is necessary to introduce the concept of a "context" in which a theory is applicable, and, in particular, in order to avoid tautologies, the context in which a model is applicable must be characterized independently of the information contained in the postulates of the given model. Two other philosophies that were confronted with important ontological and/or epistemological obstacles in their own attempts to articulate *a posteriori* methods are pragmatism and Bergsonism.

Pragmatism—whose major representative is the American philosopher and psychologist William James (1842–1910)—maintains that truth—namely, the agreement between reality and its image within consciousness—is not a given, but it is "made"[76] in the course of human experience due to the activity of consciousness, so that consciousness can induce change in reality due to the reference of consciousness to reality. This perception is shared by every philosophy of action. For instance, let us recall Marx's eleventh thesis on Feuerbach: "Philosophers have hitherto only interpreted the world in various ways; the point is to change it."[77] Pragmatism has thrown light on particular psychological aspects of the philosophical

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[75] See: Martinich and Sosa, eds., *A Companion to Analytic Philosophy*.
[76] James, *Pragmatism*, p. 104.
[77] See: Liangjian, "It's Time to Change the World, So Interpret It!," p. 153.

work, but it cannot stand as a general *a posteriori* method. As a consequence of William James's argument that truth should be defined in terms of utility, philosophy ceases to be a scientific activity and a purpose itself, and it becomes self-contradictory and self-defeating, since, according to pragmatism, the adoption of the conclusions of philosophy lacks logical and scientific justification, and, therefore, the conclusions of philosophy become meaningless. Furthermore, just as relativism leads to a contradiction by adhering to at least one absolute proposition (that all propositions are relative), so too pragmatism is pragmatically self-defeating, because, by viewing truth merely as a function of the practices in which people engage, and, thus, by depending on and embracing the established cultural practices and mentalities in each segment of space-time, pragmatism cannot operate as a genuinely progressive force.

The philosophical work of the French philosopher Henri-Louis Bergson (1859–1941), which exerted a significant influence on the final formation of pragmatism, has faced important challenges, too. According to Bergson, the only reality is duration, in which there is no juxtaposition of events, and, hence, there is no mechanical causality.[78] Thus, for Bergson, duration offers the experience of freedom. Bergson argues that duration can be conceived through intuition. Intuition, in Bergson's sense, is an *a posteriori* method, but it has only one difference from the *a priori* methods: its object, namely, internal experience, is identified with consciousness itself. At this point, we can see that Bergson's method of intuition is a form of reversed Cartesianism: Bergson formulates his anti-rationalist and anti-Cartesian theses in a rationalist and Cartesian manner. This is the primal contradiction of Bergsonism.

Moreover, in Chapter 3 of his *Creative Evolution*, Bergson argues that "physics is simply reversed psychology," but Maurice Merleau-Ponty has pointedly observed the following: in the first two chapters of *Creative Evolution*, Bergson adopted a monist attitude, and he endorsed a dialectical view of the relation between life and matter, but, in Chapter 3, he adopted a dualist attitude, and he endorsed emanationism (i.e., a cosmological theory asserting that all things emanate from an underlying principle or reality), even though emanationism is in principle the negation of pure dualism.[79] For Bergson, matter is issued from the primal cause by the slackening of the latter, and life is that which dynamizes, within and beyond itself, matter by suffusing actuality, specifically, the material present, with the virtuality of memory.

Bergson's most important contribution to philosophy is his argument that the real object of philosophy transcends comprehensive analytic knowledge and that—in contrast to Kant's argument that the noumena (i.e., the posited objects or events that exist independently of human sense and/or perception) are unknowable—the real object of philosophy is accessible to consciousness. In his book *The Creative Mind*, Bergson argues that philosophy does not consist in choosing between concepts and in taking sides, since these antinomies of concepts and positions result from the habitual way in which our intelligence works.[80] Endorsing a pragmatic approach to human intelligence, Bergson argues that the habitual way in which our intelligence works is guided by needs, and, therefore, the knowledge that it gathers is relative, since it is not disinterested. In his book *Matter and Memory*, Bergson contrasts his method of intuition with habitual intelligence.[81] Habitual intelligence gathers knowledge through what

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[78] See: Pearson and Mullarkey, eds., *Henri Bergson*; Bachelard, *The Dialectic of Duration*.
[79] Hamrick and van der Veken, *Nature and Logos*, p. 157.
[80] Bergson, *The Creative Mind*, Chapter 6.
[81] Bergson, *Matter and Memory*, Chapter 4.

Bergson calls "analysis," that is, the dividing of things according to perspectives taken, and, thus, comprehensive analytic knowledge consists in the re-composition of a thing through the synthesis of various perspectives of it. But, even though this synthesis helps us to satisfy needs, it never gives us the thing itself; it only gives us concepts of things. In other words, according to the habitual working of intelligence, synthesis is merely a development of analysis. On the other hand, Bergson's method of intuition reverses the habitual working of intelligence. In his book *The Creative Mind*, Bergson calls intuition "sympathy,"[82] and, in his book *Time and Free Will*, Bergson explains that sympathy consists in putting ourselves in the place of others.[83]Furthermore, sympathy signifies the breaking down of the gap between subject and object, leaving a field of internal experiential content.[84]In other words, Bergsonian intuition consists in entering into the being rather than going around it from the outside. It is exactly this "entering into" which, according to Bergson, gives us absolute knowledge.

There are significant similarities between Bergson's intuitive *a posteriori* method and Edmund Husserl's phenomenological *a posteriori* method. The concept of phenomenology was coined by Hegel, and, in that case, it consisted in a method of conceiving the itinerary of spirit. According to the German philosopher Edmund Husserl (1859–1938),[85] whose philosophical underpinnings consist in a form of Cartesianism combined with scholastic views, phenomenology is a method according to which the researcher focuses on the essential structures that allow the objects that are taken for granted in the "natural attitude" (which is characteristic of both our everyday life and ordinary science) to "constitute themselves" in consciousness. Husserl's logical type of intuition consists in what he has described as the process of "seeing essences," which refers to a gradually formed conscious state that is due to the methodic, successive ascent of consciousness from phenomenality to substantiality.

Phenomenology is characterized by subjectivism, in the sense that phenomenological inquiries are initially directed, in Cartesian fashion, toward consciousness and its presentations. On the other hand, phenomenology is not characterized by any psychological forms of subjectivism, since the object of phenomenology is not the realm of psychological ideas affirmed by empiricism but rather the ideal meanings and universal relations with which consciousness is confronted in its experience. Husserl explicitly opposed the attempts made by Carl Stumpf and Theodor Lipps to reduce logic to psychology. Husserl's phenomenology does not preclude legitimate psychological investigation, and its opposition to "psychologism" is a polemic only against the presumptuous claims of psychology to supersede logic and phenomenology.

The phenomenological method comes from a position prior to reflexive thought, called pre-reflexive thought, which consists of a turn to the very things. At that moment, the phenomenologist holds a phenomenological stance that enables one to keep oneself open

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[82] Bergson, *The Creative Mind*, Chapter 6.

[83]Bergson, *Time and Free Will*, Chapter 1.

[84] According to Bergson, our experience of sympathy begins with our putting ourselves in the place of others. Moreover, Bergson argues that intuition enables us to transcend the divisions of the different "schools" of philosophy like rationalism and empiricism or idealism and realism. Bergsonism and pragmatism maintain that the antinomies of philosophical concepts and positions result from the habitual way in which human intelligence works. According to Bergson, intuition reverses the habitual working of intelligence, which is analytic (synthesis being only a development of analysis), and this reversal of habitual intelligence is called "the turn of experience"; Bergson, *Matter and Memory*, pp. 184–85.

[85]See: Spiegelberg, *The Phenomenological Movement*; Ströker, *Husserl's Transcendental Phenomenology*.

enough to live that experience in its wholeness, preventing any judgment from interfering with one's openness to the description. The phenomenologist is not concerned with the particular elements of the object under investigation, but with the given object's ideal essence, which is hidden by and shines through the particulars. Husserl used the Greek term "epoché" (i.e., suspension of judgment) in order to refer to the purification of experience of its factuality. The phenomenological method involves an initial suspension of judgment regarding the factuality (whether physical or psychical) of the mind's representations of phenomena. Epoché, namely, the phenomenological bracketing of the factual aspects of our experiences, is a methodological attitude that allows consciousness to investigate the essential constitution of experience. For instance, pure mathematics systematically brackets the factual aspects of our experience of space and quantity and focuses attention on ideal relations.86 In his preface to *Ideas Pertaining to a Pure Phenomenology*, Husserl argues that phenomenology, like mathematics, is "the science of pure possibilities," which "must everywhere precede the science of real facts." By bracketing factuality, phenomenology exerted an important influence on existentialism, and, in fact, it became the method of existentialism.

The phenomenologist is focused on the ideal entities with which one is confronted after one has bracketed factuality. Husserl argues that these ideal objects are not Platonic universals, and he refuses to assign to them any ontological status beyond the mere fact that they are envisaged. Like the Austrian philosopher and psychologist Alexius Meinong (1853–1920), Husserl invokes the theory of intentionality in his interpretation of the objects of phenomenological inquiry. Moreover, Husserl distinguishes between intentional and non-intentional units of consciousness: the former have intentional content (i.e., they always represent something as something), whereas the latter have not (e.g., pain). Thus, according to phenomenology, intentionality is an intrinsic trait of the subjective processes of consciousness, and the subjective processes of consciousness refer to objects by means of intentionality. The objects of phenomenology are intentional objects. The important thing for phenomenology is not the ontological status of ideal objects but the fact that such objects may be investigated in their interrelations and that the conclusions of such descriptive analysis are coercive and communicable. Then ideal objects possess the only kind of objectivity that is necessary or desirable in order for the phenomenologist to gather genuine knowledge.

It should be clear by now that the original purpose of the phenomenological method was to explain the mind's representations of phenomena. The next major step in the development of the phenomenological method took place when this method was applied to the elements that constitute the structure of reality, since these elements are the most abstract and most basic elements of reality, and their knowledge precedes the knowledge of the essence of reality. By the term "structure," we mean an internal reality that is governed by each own order, which it creates and recreates by itself. In other words, a structure consists of the fundamental rules that govern the behavior and the relations of the members of a system (a "system" being a set endowed with a structure).

The first social theorists who applied structuralism to sociology and social anthropology in a rigorous way were the French sociologists Émile Durkheim (1858–1917) and Marcel Mauss (1872–1950). In particular, Mauss has argued that "social anthropology" means, first, positing the unity of the human species and, second, constructing a scientific table by

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[86]See: Centrone, *Logic and Philosophy of Mathematics in the Early Husserl*.

examining the differences between human communities and, hence, by articulating a sociological method.[87] Moreover, according to the French social-anthropologist and ethnologist Claude Lévi-Strauss (1908–2009), a structure consists of a model that must conform to the following four basic requirements:

> First, the structure exhibits the characteristics of a system. It is made up of several elements, none of which can undergo a change without effecting changes in all the other elements.
>
> Second, for any given model there should be a possibility of ordering a series of transformations resulting in a group of models of the same type.
>
> Third, the above properties make it possible to predict how the model will react if one or more of its elements are submitted to certain modifications.
>
> Finally, the model should be constituted so as to make immediately intelligible all the observed facts.[88]

Structuralism provides a conceptual and analytical setting within which one can study the three fundamental principles of Orphic cosmology—namely, "Chaos," "Gaia," and "Eros"—in a synthetic and creative way. One of the scientists who have proposed such an approach is the distinguished American mathematician Ralph H. Abraham (founder of the Visual Math Institute at Santa Cruz in 1975), who has pointed out that "Chaos" does not mean disorder, but it means the "creative void" that is the "source of all form," "Gaia" means "the physical existence and the living spirit of the created world," and "Eros" means "the spiritual medium connecting Chaos and Gaia; the creative impulse."[89]Before the development of structuralism in the context of modern philosophy, the term "structure" was originally used in physics, biology, and linguistics.

Structuralism in Physics

If we summarize the history of physics from the pre-Socratic philosophers until the beginning of the twenty-first century, then we shall realize that the laws of nature can be distilled into the following four fundamental forces[90]:

i. *gravity:* a natural phenomenon by which all things with mass or energy are brought toward each other (it helps us to calculate the motions of celestial bodies);
ii. *electromagnetism:* a type of physical interaction that occurs between electrically charged particles (it has given us the wonders of the electric age);
iii. *weak nuclear force:* the mechanism of interaction between subatomic particles (it is responsible for the radioactive decay of the subatomic particles, and, thus, it plays an essential role in nuclear fission, which is a form of nuclear transmutation); and
iv. *strong nuclear force:* the mechanism that binds the component particles of an atom's nucleus. An energy field that permeates the entire universe is known as the "Higgs field" (the smallest bit of which is called the "Higgs boson"), and it explains why some subatomic particles have a great deal of mass, while others have little, and

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[87] See: Dumont, *Essays on Individualism*.
[88] Lévi-Strauss, *Structural Anthropology*, pp. 279–80.
[89] Abraham, *Chaos, Gaia, Eros*, chapters 11 and 12.
[90] See: Clegg, *Dark Matter and Dark Energy*; Gamow, *Thirty Years that Shook Physics*; Gubser, *The Little Book of String Theory*; Heilbron, ed., *The Oxford Guide to the History of Physics and Astronomy*.

> others have none at all: the Higgs field interacts with the subatomic particles and determines their mass (very massive particles interact a lot with the Higgs field, while massless particles do not interact at all).

Physical structuralism is expressed in terms of natural laws. This methodology is based on the following fundamental laws and definitions[91]:

Newton's Three Laws of Kinematics

Mechanics is the branch of physics that studies the relationships between the following three physical concepts:

i. *Force:* an agent that changes or tends to change the state of motion (i.e., the state of rest or of uniform motion) of an object. The "velocity" of an object is the rate of change of its position with respect to a frame of reference, and it is a function of time.
ii. *Mass:* the quantity of matter that is concentrated in an object. The product of the mass times the velocity of an object is the "momentum" of that object.
iii. *Motion:* a change in the position of an object with respect to time.

The part of mechanics that is concerned with the study of motion is called kinematics. Due to the rigorous study of classical mechanics by the English physicist and mathematician Sir Isaac Newton (1643–1727), the SI (Système International) unit of force, newton (denoted by N), has been named in his honor. One newton is defined as the force needed in order to accelerate one kilogram (kg) of mass at the rate of one meter (m) per second (sec) squared in the direction of the applied force.

First Law of Motion: An object will remain at rest or in a uniform state of motion unless that state is changed by an external force.

Second Law of Motion: The vector sum of the forces on an object is equal to the mass of that object multiplied by the acceleration of that object ("acceleration" is the rate of change of the velocity of an object with respect to time); symbolically:

$$F = ma,$$

where F denotes force, m denotes the mass of an object, and a denotes the acceleration of the given object.

Third Law of Motion: For every action in nature, there is an equal and opposite reaction.

When a physical body undergoes a displacement with magnitude s along a straight line as a consequence of the fact that a constant force with magnitude F, directed along the same line, acts on it, the "work" W done by the force is defined as follows:

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[91] See: Sears, Zemansky, and Young, *College Physics*, pp. 77–86, 130–31, 147–49, 186, 406–53, 535–74, and 923–45.

$W = Fs.$

In general, when the force F is constant, and the angle between the force and the displacement is θ, the work done is given by

$W = Fscos\theta,$

where $cos\theta$ denotes the cosine of the corresponding angle (for the study of trigonometric concepts, see Chapter 2). The SI unit of work is the joule (denoted by J), which is named after the nineteenth-century English physicist James Prescott Joule, and it is defined as the work required in order to exert a force of one newton through a displacement of one meter.

The rotational equivalent of a linear force is "torque." In other words, "torque" is the measure of the force that can cause an object to rotate about an axis. The point where the object rotates is called the "axis of rotation." In order to find a linear force, we need to know a mass and an acceleration, but, in order to find a torque, we need to know not only a mass and an acceleration, but also how far that force is from the axis of rotation. Therefore,

$T = Frsin\theta,$

where T denotes torque, F denotes the linear force, r denotes the distance measured from the axis of rotation to where F is applied, and $sin\theta$ denotes the sine of the angle θ between F and r (obviously, the unit of torque is a newton-meter, denoted by Nm).

Newton's Law of Universal Gravitation

An object attracts another object with a force that is directly proportional to the product of the masses of the objects and inversely proportional to the square of the distance between them, symbolically:

$F_g = G\frac{m_1 m_2}{r^2},$

where F_g is the magnitude of the gravitational force on either object, m_1 and m_2 are their masses, r is the distance between them, and G is the gravitational constant, whose value is found to be (in SI units) $6.673 \times 10^{-11} N \cdot m^2 \cdot kg^{-2}$.

Around 1907, Albert Einstein set himself the goal of understanding the force of gravity. Until 1907, most physicists believed that gravity was pretty much understood from the work of Isaac Newton. However, Einstein asked the following very simple and basic question: how does gravity really work? In other words, how is it possible that one object in the universe, like the Sun, can somehow exert a pull on another object, like the Earth, even though there is nothing connecting them since there is effectively empty space between them? How does gravity operate? In fact, when Einstein read Newton's seminal book Philosophiae Naturalis Principia Mathematica, in which Newton had written down the law of gravity, he noticed that Newton had also written that he leaves the question of the mechanism by which gravity operates to the consideration of the reader. Thus, Newton formulated the equation that governs the influence of the force of gravity, but he could not actually explain how gravity works. Einstein spent about ten years trying to explain how gravity actually works, and, finally, he articulated an answer with his general theory of relativity: "The general theory of relativity is the theory of the gravitational field; the description of its language and concepts."[92]

Einstein's general theory of relativity set the stage for the development of the idea that there are possibly more than three dimensions in space. A very simple way in which one can present Einstein's general theory of relativity is the following metaphor: Imagine a big rubber sheet stretched nice and taut before your eyes. If you watch a little marble as it rolls across the surface of this rubber sheet, then you will realize that it follows a simple straight-line trajectory. But if you watch the movement of a heavy rock on this rubber sheet, then you will realize that now the rubber sheet is deformed, warped, curved. In contrast to the previous marble, this rock does not follow a straight-line trajectory, but it follows a curved trajectory along the curved surface of the rubber sheet. Einstein took this idea and applied it to the study of the universe, the fabric of space. Thus, originally, the fabric of space may look nice and flat, like the rubber sheet in the previous example, but, if the Sun appears, the fabric of space curves. Similarly, in the vicinity of the Earth, the fabric of space curves, and the Moon is kept in orbit around the Earth because it rolls along a valley in the curved environment that is created by the Earth's mass. This is the manner in which, according to Einstein, gravity is communicated from place to place, namely, through warps and curves in the fabric of the space, specifically, through warps and curves in space-time; for instance, the Earth is kept in orbit around the Sun because it rolls along a valley in the curved environment that is created by the Sun's mass, and, similarly, as I mentioned before, the Moon is kept in orbit around the Earth because it rolls along a valley in the curved environment that is created by the Earth's mass.

According to the "Bing-Bang" cosmological model, gravity underpinned and, actually, determined the transition from the "Bing-Bang" cosmological "soup" to the galactic structure that we observe today: gravity started from the initial conditions of the Big Bang and made the universe much more complex, because, even though the density of the universe was almost uniform, there were density quantum-mechanical fluctuations, namely, small differences in the density of the universe from one region to another. Thus, a region of the universe whose density was slightly greater than the mean density of the universe acted upon itself by its own gravity, and, gradually, it made itself denser, so that, instead of expanding with the rest of the universe, it drew matter into the given region, and, ultimately, this region collapsed upon itself and did not participate in the universal expansion. In this way, a physical object was made out of such a region. Gradually, the universe was filled with small density inhomogeneities resulting from inflation due to quantum-mechanical fluctuations, which, ultimately, merged into the structures of the universe that we observe today.[93]

Conservation of Mass and Energy

By the term "energy," we mean the impetus that underpins all motion and all activity, and, more specifically, the capacity for doing work. The eighteenth-century French mathematician and natural philosopher Émilie du Châtelet proposed and tested the law of "conservation of energy," according to which the total energy of an "isolated system" (i.e., one that does not interact with other systems) remains constant. In order to clarify the meaning of the principle of the conservation of energy, let us consider the following example:

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[92] See: Stephani, *General Relativity*, p. 2.
[93] See: Hawking, *The Universe in a Nutshell*.

setting fire to coal. The chemical bonds of the coal molecules store great amounts of energy. If we set fire to coal, then fire causes a chain reaction between the coal and oxygen in the air. In this reaction, energy from the chemical bonds is converted into kinetic energy of air molecules. Hence, the air becomes warm, and, for this reason, it will rise. This rising air can be used in order to drive a turbine and, thus, for instance, move a vehicle, or in order to create electricity (by feeding it into the grid). Alternatively, we can just burn coal without doing anything with the produced energy. This does not change the total energy in the system, because the total energy in the system is conserved. The chemical energy of the coal is converted into kinetic energy of air molecules, which are distributed in the atmosphere. Even though, in this case, the energy is useless, the total energy in the system remains the same. The difference between the aforementioned cases is entropy, namely, the measure of the molecular disorder, or randomness, of the system under consideration: initially, the energy was packed into the coal, and the level of entropy was low, but, by setting fire to coal, the energy was distributed in the motion of air molecules, and the level of entropy became high. When a system has energy in a state of low entropy, its energy can be used in order to create macroscopic change (e.g., drive a turbine), and this useful energy is called "free energy." Free energy is a type of energy that does "work." But, if the energy in the system is in a state of high entropy, then the energy is useless, and it is called "heat." Heat is a type of energy that does not do "work." Even though *total* energy is conserved, *free* energy is not conserved.

In 1905, Albert Einstein published his seminal research paper "On the Electromagnetics of Moving Bodies," in which he introduced his famous equation that governs the relationship between energy and mass under certain conditions:

$$E = mc^2,$$

where: E denotes energy, specifically, the energy of a moving particle; m denotes mass; c denotes the speed of light in vacuum, and its value is (in SI units) approximately $300{,}000\ km/sec$; and mc^2 denotes the energy of a particle at rest.[94] Hence, a more accurate, or rather more general, way of formulating the aforementioned equation is

$$E = \gamma mc^2,$$

where $\gamma = \frac{1}{\sqrt{1-(\frac{v}{c})^2}}$, v is the object's velocity relative to the observer, and c is the speed of light (so that the equation $E = mc^2$ holds when $v = 0$, namely, when $\gamma = 1$, and, hence, when an object does not move with respect to the observer). In the aforementioned research paper, Einstein presented his special theory of relativity, based on the following two axioms:

Principle of Relativity: The laws of physics are the same in all inertial reference frames.

Principle of Constancy of the Speed of Light: Light always propagates in a vacuum at a definite velocity that is independent of the state of motion of the emitting body.

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[94] See: Sears, Zemansky, and Young, *College Physics*, pp. 923–45.

Remark: The association between energy and matter implies that, in order to measure small structures, we need to compress more energy into small volumes of space (in fact, this is what high-energy particle colliders, such as the Large Hadron Collider, do). Higher energy allows us to find out what happens when distances become very small.

Laws of Thermodynamics

Thermodynamics is the study of energy relationships that involve heat, mechanical work, as well as other aspects of energy and energy transfer, and it was pioneered by the German scientist Otto von Guericke (1602–86) and the British scientists Robert Boyle (1627–91) and Robert Hooke (1635–1703). By a "thermodynamic system," we mean "a system that can interact with its surroundings in at least two ways, one of which must be heat transfer."[95]

The Zeroth Law of Thermodynamics: If two thermodynamic systems are each in thermal equilibrium with a third one, then they are in thermal equilibrium with each other.

The First Law of Thermodynamics: In case of a thermodynamic process that does not allow any transfer of matter,

$$\varDelta U = Q - W,$$

where $\varDelta U$ denotes the change in the internal energy of a closed system, Q denotes the quantity of energy supplied to the system as heat, and W denotes the amount of thermodynamic work done by the system on its surroundings. In other words, the first law of thermodynamics is an adaptation of the law of conservation of energy to thermodynamic processes.

The Second Law of Thermodynamics: "It is impossible for any process to have as its sole result the transfer of heat from a cooler to a hotter body."[96] In other words, "no heat engine can have a thermal efficiency of 100%."[97] Intimately related to the second law of thermodynamics is "entropy," S, which provides a quantitative measure of disorder. In particular, entropy counts the number of different microscopic configurations that have the same macroscopic appearance (or, in other words, how much information one could stuff into a macroscopic object if one kept track of the microscopic details). The entropy change $\varDelta S$ during a reversible isothermal process is defined as

$$\varDelta S = \frac{Q}{T},$$

where Q denotes the quantity of heat (notice that "heat transfer is energy transfer brought about solely by a temperature difference"[98]), and T is the absolute (Kelvin) temperature of the substance. According to the second law of thermodynamics, the total entropy (disorder) of an isolated system can never decrease over time, and it is constant if and only if all processes are

[95] Ibid, p. 407.
[96] Ibid, p. 439.
[97] Ibid, p. 440.
[98] Ibid, p. 353.

reversible.[99] The second law of thermodynamics is equivalent to the "maximum entropy principle" and the "minimum energy principle":

The maximum entropy principle: For a closed system with fixed internal energy (namely, an isolated system), the entropy is maximized at stable equilibrium. For instance, consider a marble on the edge of a bowl (i.e., the marble is in a state of unstable equilibrium). Assume that the marble and the bowl constitute an isolated system. Then, when the marble drops, the potential energy will be converted to the kinetic energy of the motion of the marble. At any instant time, t, the marble has potential energy given by

$$E_{potential} = mgh,$$

where m denotes the mass of the marble, g denotes the acceleration constant due to gravity ($\approx 9.8\, m/sec^2$), and h denotes the height of the marble as a function of time; and, at any instant time, t, the kinetic energy of the marble is given by

$$E_{kinetic} = \frac{1}{2}mv^2,$$

where v denotes its velocity (which is, by definition, a function of time). Furthermore, friction (which is a stabilizing force) will convert this kinetic energy to heat (which is an energy transfer process based on a temperature difference between the system and its surroundings), and, at stable equilibrium (i.e., when it stops rolling), the marble will be at rest at the bottom of the bowl, and the marble and the bowl will be at a slightly higher temperature. The total energy of the given system that consists of the marble and the bowl will be unchanged. The potential energy that previously existed in the marble will now reside in the increased heat of the marble–bowl system. In other words, due to the heating effects, the entropy has increased to the maximum value possible given the fixed energy of the system.

The minimum energy principle: For a closed system with fixed entropy, the total energy is minimized at stable equilibrium. For instance, in the previous example of the marble–bowl system, assume that, using a suitable apparatus, the marble is lowered very slowly to the bottom of the bowl, so that no heating effects occur (and, thus, this process is reversible). Then the entropy of the marble and the bowl will remain constant, and the potential energy of the marble will be transferred as work (which is another energy transfer process) to the apparatus that is lowering the marble. The potential energy is now at a minimum with no increase in energy due to the heat of either the marble or the bowl, and, thus, the total energy of the system is at a minimum.

The Third Law of Thermodynamics: A system's entropy approaches a constant value as the temperature approaches absolute zero (the coldest possible temperature).[100] In other words, at absolute zero, the entropy of a perfect crystal is equal to zero.

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[99] Ibid, p. 446.
[100] See: Wilks, *The Third Law of Thermodynamics*.

Electrostatic Laws

The structure and the properties of atoms and molecules and, in general, of all ordinary matter are due to primarily electrical interactions between electrically charged particles. The fundamental building blocks of ordinary matter are the negatively charged "electron," the positively charged "proton," and the uncharged "neutron." In a neutral atom, the number of electrons equals the number of protons that exist in the nucleus, and the net electrical charge is zero. If one or more electrons are removed (resp. added), then the remaining positively (resp. negatively) charged structure is called a "positive ion" (resp. a "negative ion").

In simple terms, to construct an atom, one needs some protons and neutrons for the construction of the nucleus, and then one has to put some electrons around the nucleus until the whole system is electrically neutral (in fact, once you have a positively charged nucleus, it attracts electrons, which automatically form shells around the nucleus). However, it should be mentioned that the construction of an atomic nucleus is a complex process, because protons, being positively charged, repel each other, and, therefore, they have to come very close to each other in order for the nuclear force to start operating and, thus, keep them together, given that there exist sufficiently many neutrons; and this process requires extremely high temperatures (hundreds of millions of degrees Kelvin). Such high temperatures existed briefly after the Big Bang.

Coulomb's Law: The magnitude of the force of interaction between two point charges is directly proportional to the product of the charges and inversely proportional to the square of the distance between them; symbolically:

$$F = k\frac{|q_1 q_2|}{r^2},$$

where F denotes the magnitude of the force that each of two point charges q_1 and q_2 a distance r apart exerts on the other, and k is a proportionality constant, whose value is (in SI units) approximately $8.988 \times 10^9\ N \cdot m^2 \cdot C^{-2}$. Due to the rigorous description of the electrostatic force of attraction and repulsion by the French military engineer and physicist Charles-Augustin de Coulomb (1736–1806), the SI unit of electric charge, the coulomb (denoted by C), has been named in his honor; it is approximately equivalent to 6.24×10^{18} electrons.

In general, by the term "field," we mean an area in which forces are exerted on things in its midst. The modern concept of a physical field was originally formulated in the nineteenth century by the English physicist Michael Faraday. An electric charge creates an "electric field" in the region of space surrounding it, in the sense that "the properties of space itself are modified by the presence of an electric charge."[101] "Electric field" (sometimes called "electric intensity") is defined as the electric force per unit charge, and, therefore (in SI unites), the unit of electric field magnitude is one newton per coulomb (i.e., $1N \cdot C^{-1}$).

In general, by the term "flux," we mean the quantity of a substance passing through a given area. The "electric flux" through a surface is proportional to the number of field lines crossing that surface. In other words, its magnitude is proportional to the portion of the field perpendicular to the area:

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[101] Sears, Zemansky, and Young, *College Physics*, p. 551.

$$Electric\ Flux = Electric\ Field \times Area \times cos\theta,$$

where $cos\theta$ denotes the cosine of the angle θ between the electric field and vector that is perpendicular to the area.A "field line" is an imaginary line drawn through a region of space in such a way that, at every point, it is tangent to the direction of the electric-field vector at that point. In particular, in an "electrostatic field," every field line is a continuous curve with a positive charge at one end and a negative charge at the other. The amount of work needed in order to move a unit of electric charge from a reference point to a specific point in an electric field without producing acceleration is called an "electric potential," and, in terms of SI units, it is represented by

$$V = \frac{joule}{coulomb},$$

joule being the unit for work done, and coulomb being the unit for the charge; V denotes "volt," namely, the derived unit for electric potential (electromotive force), and it is named after the Italian physicist Alessandro Volta (1745–1827).

Gauss's Law: The flux of the electric field through an arbitrary closed surface is equal to the net charge enclosed divided by the permittivity of free space.

Moreover, the German mathematician and physicist Johann Carl Friedrich Gauss (1777–1855) proposed similar laws relating to magnetism and electromagnetism ("magnetism" refers to physical phenomena arising from the force that is caused by magnets, namely, by objects that produce fields that attract or repel other objects). In fact, the fundamental nature of magnetism can be found in interactions involving electric charges in motion. In 1831, Michael Faraday discovered electromagnetic induction: he placed a stationary magnet inside or outside a coil, and he observed no deflection in the galvanometer, but, at the moment that he moved the magnet toward (into/above/below) the coil, he saw the pointer deflecting in one direction, and, at the moment that he moved the magnet way from the coil, he saw the pointer deflecting in the opposite direction. This was a really amazing discovery, because one could make something move without ever touching it, only by using the field. Indeed, we can affect things far away and develop telecommunications using electromagnetic fields. Moreover, Faraday was the first to understand that waves of the electromagnetic field are what we call light. In simple terms, electromagnetic radiation consists in electric and magnetic fields oscillating around each other creating a freely propagating wave that can travel from one place to another, and this event explains light, the operation of radio stations, the operation of microwave ovens, etc. (these are electromagnetic phenomena, and they differ from each other only with respect to the wavelength of the corresponding oscillation, so that we use different names for electromagnetic radiation depending on the corresponding wavelength; for instance, if we can see electromagnetic radiation, then we call it light, light with large wavelengths is red, light with larger wavelengths that is invisible is called infrared, while, at even larger wavelengths, electromagnetic radiations are called microwaves, and if the wavelengths are even larger, then electromagnetic radiations are called radio-waves).[102]

[102] By the term "radiation," in general, we mean energy transferred by waves or particles. For instance, radiation may take the form of electromagnetic waves, which, however, are made of particles, specifically, photons.

As I have already mentioned, electromagnetism is the study of the interaction between electrically charged particles. For instance, if a large object has a negative charge, this means that it has more electrons than protons. If we have a static object with a charge, it will affect only other charges. If we have a static magnet, it will affect only other magnets, not other charges. But if we have a moving charge, it will affect a magnet, and, if we have a moving magnet, it will affect a charge. These are the four principles of James Clerk Maxwell's theory of electromagnetism.

In the 1910s, the German mathematician and physicist Theodor Franz Eduard Kaluza thought that, since Einstein was successful in describing the mechanism of the force of gravity in terms of warps and curves in space-time, it might be possible to describe the mechanism of another natural force, namely, electromagnetism, in a similar language. If the mechanism of gravity consists of warps and curves, then the mechanism of the electromagnetic force could possibly consist of warps and curves, too. However, the following question emerged: warps and curves in what? Einstein had used up space-time as the substratum of gravity's warps and curves, and there seemed to be nothing else to warp and curve in order to explain the mechanism of electromagnetism. Therefore, Kaluza asked himself the following question: what if there are more dimensions in space than we know about? For instance, what if there is one more hidden spatial dimension, so that, instead of having three spatial dimensions and one temporal dimension, we actually have four spatial dimensions and one temporal dimension? In fact, Kaluza took Einstein's equations that were derived and formulated in a context where there are three dimensions of space and one dimension of time, and he reformulated them in a context where there are four dimensions of space and one dimension of time. Working in this way, Kaluza came up with new equations some of which were essentially the same as the equations that Einstein had derived in the context of three dimensions of space and one dimension of time, but, since Kaluza had postulated the existence of an additional dimension of space, he came up with an additional equation that was identical to the equation that James Clerk Maxwell had formulated a few years ago in order to describe the influence of the electromagnetic force. Kaluza's excitement was great, because he had managed to put gravity and electromagnetism together by imagining and postulating that there is one more spatial dimension that, for some reason, we do not see.

As a consequence of Kaluza's research work, the following question emerged: if there is an additional dimension of space, then where is it, and why can't we see it? In 1926, the Swedish theoretical physicist Oscar Benjamin Klein suggested the following possibility: maybe, there are two varieties of dimensions, namely, a variety of "big" dimensions that we can easily see (left–right, back–forth, and up– down), and "tiny" dimensions that are too small to see, even though they are around us. In order to get a sense of that assumption, we can think of a cylindrical cable, which, from a distant vantage point, looks one-dimensional (namely, like a line) since we do not have the visual acuity to see that it is cylindrical, but, if we zoom in, then we can realize that it is a three-dimensional object. Small dimensions can be difficult to see compared to big dimensions that are far more obvious. By analogy, if we zoom

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Gravitational radiation is transferred in gravitational waves, which are, actually, periodic deformations ("wiggles") of space-time, and, according to rigorous physical assumptions, gravitational waves are made of a peculiar kind of particles called gravitons (a graviton is assumed to be a quantum of gravity, namely, an elementary particle mediating the force of gravity). The term "graviton" was coined in the 1930s by the Soviet physicists Dmitrii Blokhintsev and F. M. Galperin.

in more and more on the three-dimensional space itself, we can encounter hidden additional spatial dimensions, which are too small to be seen in the context of everyday life.

In the 1980s—string theory, which is based on the research work of the American physicists Richard Phillips Feynman and Edward Witten, the Austrian theoretical physicist Julius Erich Wess, and the Italian theoretical physicist Bruno Zumino—gave an important boost to Klein's aforementioned assumption about extra dimensions. String theory tries to explain what things are made of. From the perspective of string theory, if we zoom in sufficiently deep inside matter, beyond molecules, atoms, and subatomic particles, we shall ultimately see tiny vibrating strings, and different vibrational patterns of these tiny strings give rise to the different kinds of particles of the world around us. Strings are assumed to be one-dimensional extended bodies whose characteristic length scale is typically on the order of the Planck length (approximately 10^{-35}meter), whereas, on much larger length scales, such objects would appear to be zero-dimensional point particles. When physicists started analyzing the mathematics of string theory, they found certain equations that are meaningful and internally consistent only if the universe had more than three dimensions, thus vindicating Kaluza's pioneering research work. Not only does the mathematics of string theory force the possibility of extra spatial dimensions upon us, but it also shows that these extra spatial dimensions have a very interesting and rich geometry, and they are mathematically described as Calabi–Yau manifolds (this type of surfaces has been named in this way in honor of the Italian-American mathematician Eugenio Calabi, who first conjectured that such surfaces might exist, and the Chinese-American mathematician Shing-Tung Yau, who proved the Calabi conjecture). Even though string theory has encountered very serious theoretical difficulties,[103] and, on several occasions, it was replaced by calculations based on standard quantum chromodynamics, to which I shall refer shortly, it has played a positive role in helping us to understand the dynamic nature of the physical world.

Modern physics, as it was developed and formalized in the twentieth century, has managed to scientifically confirm the theories that explain gravity, electromagnetism, weak nuclear force, and strong nuclear force, but it has not discovered a fifth physical force. However, in the beginning of the twenty-first century, physicists discovered a new energy source larger than our galaxy itself: "dark energy." In particular, in the beginning of the twenty-first century, a new cosmological model prevailed according to which, in our universe, about seventy-two percent of the total energy is in the form of dark energy, known as the "energy of nothing," which blows the galaxies farther and farther apart from each other. Dark energy is the energy of the "Big Bang" itself, and, in fact, it was dark energy that made the universe (being originally a very hot, small, and dense mix of the four fundamental forces) "bang," according to the Big Bang theory.[104]

Intimately related to the notion of "dark energy" is the notion of "dark matter." Dark matter is a peculiar form of material that neither emits, reflects, nor absorbs electromagnetic radiation (light and all the different variations of light, like, for instance, radio waves and gamma rays). Thus, in this case, "dark" means "invisible." For instance, if there is a cloud of dark matter between a source of light and an eyeball (observer), the light that is emitted from the source just goes straight through the cloud of dark matter without bouncing off or interacting in any way with the dark matter, and, thus, it is seen by the eyeball (observer) that

[103] See: Smolin, *The Trouble with Physics*; Yau and Nadis, *The Shape of Inner Space*.
[104] See: Jepsen, "Four Things You Might Not Know About Dark Matter."

is on the other side of the dark matter. By contrast, normal matter (of which the common substances studied in physics, chemistry, and biology are made) appears "dark" to our eyes, but this is due to the fact that it absorbs or reflects light, namely, it interacts with light (and, hence, it is visible).

The reason why, in the twentieth century, physicists started believing that dark matter exists is that, whenever they look in the universe, there is evidence of something that they cannot directly see but that has gravitational effects on things that physicists can actually see. In fact, dark matter explains gravitational lensing (namely, the fact that gravity from matter between us and galaxies bends light: normal mass alone is not sufficient to explain the observations, in the sense that the strength with which mass focuses light presupposes the existence of dark matter) and the behavior of galaxy clusters (galaxy clusters are collections of hundreds of galaxies that are held together by their own gravitational pull, and the higher the total mass in the cluster the higher the average velocities of the galaxies in the cluster, but it has turned out that the observed mass is not high enough to explain the observed velocities, and, thus, it has been postulated that galaxy clusters contain large amounts of dark matter).

According to new cosmological models that prevailed in the beginning of the twenty-first century, about twenty-three percent of the universe is "dark matter," namely, a peculiar invisible form of matter (thought to be non-baryonic in nature, being composed of some mysterious subatomic particles).[105] Stars, made out of hydrogen and helium, make up about four percent of the universe. We, the higher elements of the universe, namely, humans, made out of hydrogen, oxygen, carbon, nitrogen, calcium, phosphorus, etc., make up only about zero point zero three (0.03) percent of the universe.

Quantum Mechanics

When scientists investigate physical-biological structures at the nanoscale (namely, the scale of nanometers; one nanometer being one billionth of a meter), they actually work at the edge of quantum mechanics (namely, on the boundary of the world in which the quantum rules start to take effect). Beyond that, scientists can investigate even smaller particles. In the second half of the twentieth century, it was understood that all matter is ultimately made of four building blocks: up and down quarks (which make up the protons and the neutrons, namely, the components of atomic nuclei), electrons (balancing the atomic nuclei), and neutrinos (a neutrino is an elementary particle that interacts only via the weak subatomic force and gravity). The term "quantum" derives from the Latin language, and it means an amount of something. In the context of quantum mechanics, the term "quantum" means the smallest amount of energy that can be measured. The central concept of quantum physics is that of a wave, namely, a disturbance or oscillation that travels through space-time, and it is accompanied by a transfer of energy. The basic properties of a wave are its amplitude (i.e., the distance from the center line, that is, the still position, to the top of a crest or the bottom of a trough), its frequency (i.e., the number of cycles occurring per second; specifically, it can be measured by counting the number of crests of waves that pass a fixed point in one second), and its length (i.e., the distance over which the wave's shape repeats; for instance, the distance between two adjacent crests). In mathematical terms, from the perspective of quantum mechanics, the concept of a physical system is equivalent to the concept of a state,

[105] Ibid.

which, in turn, is a vector in a Hilbert space (these mathematical concepts will be clarified in Chapter 2).

The theoretical nuclear physicist Nouredine Zettili has summarized the origins of quantum mechanics as follows:

> The first real breakthrough came in 1900 when Max Planck introduced the concept of the *quantum* of energy. In his efforts to explain the phenomenon of blackbody radiation, he succeeded in reproducing the experimental results only after postulating that the energy exchange between *radiation* and its surroundings takes place in *discrete*, or *quantized*, amounts. He argued that the energy exchange between an *electromagnetic wave* of frequency v and matter occurs *only in integer multiples* of hv, which he called the energy of a quantum, where h is a fundamental constant called *Planck's constant*.[106]

Notice that every object with a temperature above absolute zero (–273.15^{o}C) emits energy in the form of electromagnetic radiation, which travels through space as electric energy and magnetic energy. A "blackbody" is a model body that absorbs all incident electromagnetic radiation, regardless of frequency or angle incidence (the term "blackbody" is used because such a perfect absorber of energy will absorb incident visible light, instead of reflecting it, and, therefore the surface of such a body will appear black). Blackbody radiation is the theoretical maximum radiation expected for temperature-dependent thermal self-radiation. The hotter the emitter, the more energy emitted and the shorter the wavelength.

Quantum mechanics describes the building blocks of physical-biological reality, and it provides us with the rules that inform us about the way in which the subatomic world behaves (namely, about the ways in which atoms fit together to make molecules, the ways in which particles come together to make atoms, as well as the properties and the behavior of all these particles). Without quantum mechanics, most of modern technology that we rely on and take for granted today would be impossible because the whole realm of modern electronics (for instance, laptops, CD players, mobile telephones, etc.) ultimately relies on chips (integrated electronic circuits), which, in turn, rely on semiconductors, and we would not understand how semiconductors operate without an understanding of the rules of quantum mechanics.

In quantum mechanics, the state of an electron is described by four quantum numbers: (i) the principle quantum number n (it describes the energy and the distance from the nucleus, and it represents the shell), (ii) the angular momentum quantum number l (it describes the shape of the subshell and its orbitals), (iii) the magnetic quantum number m_l (it describes the orientation of the orbitals within the subshells), and (iv) the electron spin quantum number m_s (by the term "electron spin," we mean a form of angular momentum, and an electron can spin clockwise or anticlockwise, in two opposite directions). According to Pauli's Exclusion Principle, no two electrons in the same atom can have identical values for all four of their quantum numbers, and, therefore, no more than two electrons can occupy the same orbital, and two electrons in the same orbital must have opposite spins.

From the perspective of quantum mechanics, particles are discrete packets, "quanta," of energy with wave-like properties. In other words, according to quantum mechanics, energy is not continuous, but it is always parceled up into some tiny discrete "lump" (which is what "quantum" literally means: a discrete thing). In essence, an electron is a circular standing

[106] Zettili, *Quantum Mechanics*, p. 2.

wave. In the 1950s, physicists used the term “strong force” to mean the nuclear force. In the 1970s, it was proved that the protons and the neutrons at the center of atoms are made of quarks. Because protons and neutrons are found in the nuclei of atoms, they are called nucleons. Each nucleon consists of three quarks. A nucleon can be thought of as a tiny sphere with a radius of approximately one quadrillionth of a meter, and, inside that tiny sphere, quarks zoom around travelling at nearly the speed of light. Whenever particles move at that extremely high speed in an extremely small volume, an extremely strong force is needed in order to hold them together. The force governing the motion of quarks is a quantum force.

Quantum electrodynamics describes the manner in which electrically charged particles interact by shooting photons back and forth between each other. Electrons, being zero-dimensional, lack spatial extension (that is, they have practically zero volume), and, therefore, they interact with each other by exchanging photons. As two electrons move toward each other, a photon is passed from one to another, and it changes the momentum of both of them, thus pushing them off. Therefore, in contrast to the folk understanding of “touch,” when we say that electrons “touch” each other, we mean that they interact with each other by exchanging a photon. The photon is a quantum of light and the force carrier of the electromagnetic force (the electromagnetic force is the result of the fact that particles with an electric charge exchange photons with each other). Even though photons propagate magnetic fields, they cannot be seen, because they are “virtual particles,” namely, particles that cannot be directly detected and may not obey all the laws that physicists force all real physical particles to adhere to (for instance, virtual particles do not necessarily need to obey the Einstein energy–momentum relation). In fact, it is not just electrostatic repulsion that prevents atoms from getting close, but it is primarily the Pauli Exclusion Principle that forces the electrons and the quarks that make up the atom to arrange in shells instead of sitting on top of each other. In other words, since our atoms’ electrons repel objects when they are approximately 10^{-8} m (one eight-billionth of a meter) away from us, we technically never touch anything, but we can feel the force of the resistance. However, the quarks inside the nuclei of atoms work a little differently: first, the relevant charge is not the electric charge, but the strong force charge, which physicists call color (which is a different kind of charge, having nothing to do with the way in which the term “color” is used in everyday language); second, unlike electric charge, which exists in two varieties (plus and minus), the strong charge (“quantum color”) exists in three varieties, called red, blue, and green; third, the particles that colored quarks exchange are called gluons (just as the photon is the particle of the electromagnetic force, so the gluon is the particle of the strong force). Hence, the aforementioned model has been called quantum chromodynamics.

Intimately related to the study of matter is the division of matter into ever smaller parts, and, ultimately, we end up with something that one cannot divide any more; this is what ancient Greek scientists called “atom,” literally meaning “indivisible.” According to ancient Greek scientists, atoms are the smallest things in the universe and cannot be divided. Of course, modern physics has proved that atoms can be further analyzed into smaller particles and can be divided, and, by the end of the twentieth century, it was already clear that molecules are made of atoms, atomic nuclei are made of neutrons and protons, and the neutrons and the protons are made of quarks and gluons (as I have already mentioned, a gluon acts as the exchange particle for the strong force between quarks, and it is analogous to the exchange of photons in the electromagnetic force between two charged particles, and, according to string theory, quarks and gluons are made up of tiny vibrating strings). But the important thing regarding ancient Greek atomic theory is the idea that the analysis of matter leads us to an "ultimate," indivisible element of physical reality. Quantum physics has shown that, when we have a single atom, or, in general, a single particle, in a vacuum, it becomes a wave. According to the theory of wave mechanics, which was formulated in the 1920s by the Nobel Prize-winning Austrian-Irish physicist Erwin Schrödinger, a wave itself does not have units of matter or energy, but it is just form, specifically, a pattern of information. In other words, waves are just numbers, and, in a sense, their existence corroborates Pythagoras's argument that physical things are ultimately numbers.

When we isolate a single atom, or, in general, a single particle, so that it does not interact with anything else, it becomes a wave, and a wave spreads out in space. If we try to determine its position, then we realize that it is in a state of potentiality or probability, which transcends sensory perception. In fact, when we observe a wave, we actually destroy its state, in the sense that our observation changes the information structure of the given wave: "to detect a particle [which has become a wave], the detector must interact with it, and this interaction unavoidably changes the state of motion of the particle, introducing uncertainty about its original state."[107] From this perspective, waves are non-empirical, logical constituent components of the material world. According to Heisenberg's uncertainty principle, "neither the momentum nor the position of a particle can be predicted with arbitrary great precision, as classical physics would predict."[108] In quantum mechanics, particles do not have classical properties like "position" or "momentum," but they are described by a "wave-function," which is a complex-valued probability amplitude, usually denoted by the Greek letter psi, ψ. According to the Born rule,[109] the probability of a particle being observed at a particular location is given by the square of the amplitude of the wave-function at that location—symbolically:

$$Probability(x) = |amplitude(x)|^2$$

(regarding the mathematical underpinnings of these theories, see section 2.22).

In the context of quantum mechanics, a molecule can be thought of like a mountain range (described by a wave-function) filled with infinitely many energy steps, where each energy step, representing a quantum of energy, is a quantum state. A molecule stands on one of these quantum states, and all the other infinitely many quantum states are empty, they are virtual states. Moreover, each quantum state is characterized by a wave form. When a system stands on one of these states, the other states also exist potentially, but they cannot be observed, they actually look empty. However, those virtual states are potential modes of being, because, otherwise, a molecule could not jump into other quantum states, and, due to Heisenberg's uncertainty principle, we know that it can (molecules can make "quantum jumps," because they have empty states into which they can jump).[110]

Consequently, one of the most important problems in the foundations of physics is the quantization of gravity (namely, the unification of gravity and quantum mechanics into one

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[107] See: Sears, Zemansky, and Young, *College Physics*, p. 986.
[108] Ibid.
[109] See: See: Abbott, Davies, and Pati, eds., *Quantum Aspects of Life*. In classical physics, phenomena are described and explained by continuous functions, that is, they do not make jumps. Hence, in classical physics, we require continuity of curves.
[110] See: Schäfer, *Infinite Potential*.

consistent theory), which was originally studied in the 1960s by the American theoretical physicists Richard Feynman and Bryce DeWitt. As I mentioned earlier, according to Einstein's general theory of relativity, matter curves space-time in its vicinity, and this curvature, in turn, affects the motion of matter. General relativity predicts that light rays bend around massive objects, like the Sun; it predicts gravitational lensing (a gravitational lens is a distribution of matter between a distant light source and an observer, and it is capable of bending the light emitted by the source as the light travels toward the observer; in other words, as the light emitted by distant galaxies passes by massive objects in the universe, the gravitational pull from these objects can bend the light); it predicts that the universe should expand (especially after the correction of some of Einstein's original calculations by the Russian physicist Alexander A. Friedmann); it predicts that time runs more slowly in gravitational potentials; it predicts black holes (a black hole is a space-time "singularity," namely, a region of space-time where gravity is so strong that nothing can escape from it); and it predicts gravitational waves (disturbances, "ripples," in the curvature of space-time that are caused by accelerated masses and travel across the universe stretching and squeezing space-time as they move). All these predictions have been scientifically confirmed. However, Einstein's general theory of relativity does not fit well with quantum mechanics, as indicated, for instance, by the following case: Let us consider an electron going through a double slit. According to quantum mechanics, the electron goes through both slits at the same time simultaneously (according to the uncertainty principle, particles can be in two places at the same time). However, an electron has a mass, and masses generate a gravitational pull by bending space-time. Thus, the following question emerges: to which place does the gravitational pull go if the electron travels through both slits at the same time? One could expect that the gravitational pull would also go to two places at the same time, but this cannot be the case in the general theory of relativity, because the formalism of the general theory of relativity is not identical to the formalism of quantum mechanics. This problem calls for a specific interpretation of the quantum properties of gravity, and, because, according to Einstein, gravity refers to the curvature of space-time, we need a theory for the quantum properties of space-time.

It should be pointed out that the real problem is not the quantization of gravity itself. We can, indeed, quantize gravity in the same way that we can quantize other interactions, but the problem is that the theory with which one comes up breaks down at high energies, and, therefore, it cannot explain the manner in which nature works at the subatomic level. This naive quantization of gravity is known as "perturbatively quantized gravity," and it was proposed in the 1960s by the American theoretical physicists Richard Feynman and Bryce DeWitt. Perturbatively quantized gravity is an approximation of actual quantized gravity.

Furthermore, in the 1930s and in the early 1940s, quantum physicists and mathematicians, such as Erwin Schrödinger, Werner Heisenberg, and Ernst Pascual Jordan, highlighted the importance of quantum mechanics in understanding and explaining biology and, especially, the dynamics and the complexity of the phenomenon of life. Indeed, one of the most creative and thought-provoking scientific disciplines that helps one to understand the dynamics and the peculiarities of human life is quantum biology,[111] which encompasses physics, chemistry, and biology. The application of quantum mechanics to biological objects helps us to explain random mutations in DNA, the manner in which birds orient themselves

[111] See, for instance: Al-Khalili and McFadden, *Life on the Edge*.

while migrating, the manner in which photosynthesis works, and other complex biological phenomena. However, it is worth pointing out that, unfortunately, for a long period of time after the end of World War II, quantum biology became disreputable, and the progress of this scientific discipline was slow because the acknowledged pioneer in the study of quantum mechanics, namely, Ernst Pascual Jordan, was an advocate of the Nazi ideology.

Furthermore, one of the most important applications of quantum mechanics in medicine and material science is quantum metrology, which consists in a collection of techniques to improve measurements by help of quantum effects. Metrology is the scientific study of measurement. Quantum measurements can be achieved with very few particles, and, therefore, they cause minimal damage to the sample, for which reason quantum metrology plays a decisively important role in technology.

Structuralism in Biology

In the eighteenth and the nineteenth centuries, the most important representatives of structuralism in the scholarly discipline of biology were the French natural scientist Étienne Geoffroy Saint-Hilaire (1772–1844) and the Scottish biologist and mathematician Sir D'Arcy Wentworth Thompson (1860–1948). In the twentieth century, some of the most important representatives of structuralism in the scholarly discipline of biology were the German paleontologist Adolf Seilacher, the American paleontologist and evolutionary biologist Stephen Jay Gould, the American evolutionary biologist, mathematician, and geneticist Richard Charles Lewontin, the Canadian mathematician and biologist Brian Goodwin, and the British-Australian biochemist Michael John Denton. According to biological structuralism, a significant part of the order of the biological world arises from "laws of form," which are part of the overall order of the natural world. With regard to structure, biological forms can be explained in the same way as crystals, galaxies, and atoms. However, the origins of the idea of biological structuralism can be traced to Aristotle's theory of forms, according to which the biological world, at its base, consists of primal patterns, or basic forms, generated by laws of form in nature.

It goes without saying that biological structuralism does not explain the entire order of the biological world (for instance, it does not explain adaptation), but it is an attempt to explain the basic underlying patters of the world (for instance, why we have insects and vertebrates) and the structural stability of the world. As it has been pointed out by Michael Denton, one of the simplest examples of "structural order" is the cell membrane, which separates the interior of the cell from the outside environment, and organizes itself into a semi-permeable lipid bilayer due entirely to natural law (the hydrophobic character of its lipid components) irrespective of any functional end it may serve.[112] According to Denton, the laws of biological form limit the way in which organisms are built to a few basic types, just as the laws of chemical form or crystal form limit chemicals and crystals to certain sets of "legitimate" forms.[113] Biological structuralism accepts that organisms exhibit adaptations to serve external environmental conditions, but it maintains that adaptations are "adapted masks" grafted onto underlying "primal patterns."[114] Hence, the diverse vertebrate limbs (i.e., fins for swimming, hands for grasping, and wings for flying) are all modifications of the same

[112] Denton, *Evolution*.
[113] Ibid.
[114] Ibid.

underlying pattern, which serves no particular external environmental necessity. In addition, according to Denton, biological structuralism is compatible with the idea of intelligent design, because the laws of form are part of the laws of nature, and, according to cosmology, the laws of nature are clearly fine-tuned to an extraordinary degree for life on earth.

The renowned French mathematician and philosopher René Thom (1923–2002), who won the Fields Medal in 1958, has argued that "almost any natural process exhibits a kind of local regularity . . . which allows one to distinguish recurrent identifiable elements denominated by words," and that, "otherwise, the process would be entirely chaotic and there would be nothing to talk about."[115] These "recurrent identifiable elements" can be characteristic shapes (for instance, a snowflake or a butterfly) or characteristic stages of a dynamic process (for instance, the formation of snowflakes from water vapor or the metamorphosis that turns a caterpillar to a butterfly). In either case, according to Thom, they have the property of "structural stability," in the sense that they have recurrent qualitative features, irrespective of the quantitative complexity that characterizes the circumstances that give rise to those features.[116] Thus, for instance, "an apple seed may experience a wide range of temperature, moisture, soil acidity and so on, but if it grows at all it will grow into an apple tree, not a cactus or a cattail."[117]

Of all known complex systems that exist in the physical universe, the human brain is the most complex one. If we were to construct a computer that would model the human brain, then the volume of that computer would be several thousand cubic meters, it would have to be cooled down by a river, and it would need a nuclear power plant to energize it (whereas the human brain operates with just about 20 Watts). We can use dynamical systems in order to create a model of the operation of the brain and, in this way, to explain the relationship between the brain and primordial consciousness.

In mathematics, by the term "dynamical system," we refer to any system whose state evolves with time over a "phase space" according to a fixed rule. The "phase space" of a dynamical system is the set of all possible states of the system. Thus, each point in the phase space corresponds to a different state of the system. A state in which a system finally settles is said to be an "attractor." In other words, an attractor is a set of numerical values (system states) toward which a system tends to evolve for a wide variety of its starting conditions (initial data) after transient processes. A "strange attractor" represents a trajectory upon which a system runs from situation to situation without ever settling down. A strange attractor, then, is an orbital attractor determined by a function that has mathematical discontinuities. Thus, an attractor is said to be "strange" if it has a fractal structure, namely, a structure that is characterized with self-similarity.[118] In other words, a strange attractor is a dynamic kind of equilibrium, whereas an attractor is a static state of equilibrium.

A system in which the change of the output is not proportional to the change of the input, and, therefore, it cannot be arranged in a straight line, is called "nonlinear." Nonlinear systems may exhibit chaotic behavior. The best heuristic definition of chaos is that chaos means sensitive dependence on initial conditions. Scientists cannot forecast the precise state of a chaotic system, but chaotic systems are characterized by structural stability, in the sense that they trace repetitive patterns that often provide useful information. Hence, often scientists

[115] Quoted in: Woodcock and Davis, *Catastrophe Theory*, p. 17.
[116] Ibid, p. 18.
[117] Ibid, pp. 19–20.
[118] See: Peitgen, Jürgens, and Saupe, *Chaos and Fractals*.

use the term “deterministic chaos.” According to Michael J. Radzicki, deterministic chaos is characterized by self-sustained oscillations whose period and amplitude are non-repetitive and unpredictable, but they are generated by a non-random system.[119] For instance, we do not know exactly where or when tornadoes and hurricanes will strike, but we do know what conditions lead to their occurrence, when and where they are most frequent, and their likely paths. To give a second example, we know that the economy cycles through recessions and booms, but we cannot predict very well the depth or the duration of a particular recession.[120]

Neurons (specialized nerve cells) fire a signal when they are activated by incoming signals from other neurons. Each neuron can be considered to represent one variable, and, therefore, in the phase space that models the brain, each neuron is given one dimension. Hence, there are as many dimensions as are the neurons of the human brain (namely, there are billions of dimensions).[121] The brain is a reducing viber-filter that underpins primordial consciousness, and, therefore, to the extent that consciousness is related to the activity of these neurons (an issue to which I shall return later in this chapter), consciousness can be represented as a point moving in the aforementioned phase space. Regarding the behavior of this point (namely, consciousness), we can draw the following conclusions: (i) Its path is chaotic, in the sense that, even though the overall system is subject to particular laws, the behavior of the point is unpredictable (as a result, we can never totally predict human behavior). (ii) Even though the movement of the point is chaotic, it is not random, because it follows a strange attractor. In this case, the strange attractor is the phenomenon of “personality,” which is inextricably linked to culture, which, in turn, is a factor that transcends pure biology. (iii) This model is not algorithmic, in the sense that it is neither predictable nor sequential.

Structuralism in Linguistics

The acknowledged founder of structuralism in the scholarly discipline of linguistics is the nineteenth-century Swiss linguist and philosopher Ferdinand de Saussure. Before Saussure, linguists were preoccupied with the “diachronic” study of language, namely, with the study of the evolution of language over time. However, Saussure founded “synchronic linguistics,” which consists in the study of the manner in which a language operates at a given point in time. According to Saussure, signification, that is, the relationship between the signifier (i.e., a written or spoken word) and the signified (i.e., the thing to which it refers) is almost always arbitrary.[122] In other words, language is not directly related to the world, in the sense that, for instance, the only reason that the word “tree” should be used to describe a perennial plant with an elongated stem, or trunk, supporting branches and leaves in most species is that, over time, speakers of the English language have come to an agreement on this signification. Instead of operating as descriptors of certain objects or actions, words operate according to the principle of differentiation. Hence, Saussure analyzed language as a formal system of differential elements.

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[119] Radzicki, “Institutional Dynamics, Deterministic Chaos, and Self-Organizing Systems.”

[120] See: Butler, “A Methodological Approach to Chaos.”

[121] See, for instance: Luo, *Principles of Neurobiology*; Presti, *Foundational Concepts in Neuroscience*. There are several neuro-correlates of consciousness, but “correlation” is not identical with “causation,” and, for this reason, the brain is not the cause of the entire phenomenon of consciousness.

[122] Saussure, *Course in General Linguistics*, Chapter 1.

According to Saussure, the conceptual part of linguistic value depends only on relations (similarities and differences) with other signs in the language, and, therefore, language is a self-contained formal system of differential elements, and reason transcends language.[123] The knowledge that is conveyed by individual words or phrases is puny, indeed. Individual written phrases or verbal utterances ("parole") hold meaning due to their relations (similarities and differences) with other written phrases or verbal utterances in the wider linguistic structure which Saussure calls the "langue." Therefore, the analysis of the meaning of a written phrase or a verbal utterance necessarily takes place with reference to the "langue" of which it is part.

Saussure's structuralism is based on his argument that human language is not a function of the speaking subject, namely, it is not something owned by the speaker, but it is a social product (specifically, a social convention) assimilated by the speaker. In other words, the fact that human language does not originate in a particular person, namely, the fact that human language is not one's private language, implies that, whenever one speaks, one uses something that is not strictly one's own. In fact, Saussure argues, human language is conventional, and, therefore, it belongs in the public sphere, namely, to all of us. The fact that human language is not private, but belongs to all of us, allows it to be communicative and an object of scientific research.

Philosophical Structuralism and Hermeneutics

In philosophy, structuralism involves a systematic attempt to uncover and study underlying universal mental structures, which manifest themselves in social and cultural phenomena, and, in general, to study relations between competence and performance, relations between surface and deep structure, and relations between innate rules and experience.[124] The structuralist method is the last adaptation of phenomenology to the problems that stem from the philosophical inquiry into the deepest structures of reality. Indeed, if this method is applied carefully, then it can lead to the identification and the understanding of the most relevant views of reality. Intimately related to the attempt to conceive the deepest meaning of reality is the hermeneutic method, which was originally developed by the German philosopher Hans-Georg Gadamer (1900–2002), and it is aimed at a deep dialogue between consciousness and its object.

Gadamer has argued that language exists genuinely only in conversation, or dialogue, and, therefore, we have to study language not only as a system by means of which we exchange signs, but also as a system of "linguistic togetherness."[125] According to the hermeneutic method, the whole must be understood from the individual, and the individual must be understood from the whole. In other words, Gadamer proposes a circular model of understanding, in the sense that he argues that the movement of understanding is always from whole to part, and back to whole. In this way, the hermeneutic method aims to broaden, in concentric circles, the unity of the meaning that is understood by consciousness.

Gadamer maintains that "interpretation" is a peculiar immanent approach to being, in the sense that interpretation does not objectify, nor does it seek to determine something as a neutral observer, but it seeks to acquire what is actually to be understood in "a fabric of

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[123] Ibid.
[124] See: Mepham, "The Structuralist Sciences and Philosophy."
[125] Gadamer, *Truth and Method*, in conjunction with: Palmer, ed., *Gadamer in Conversation*, and Zimmermann, *Hermeneutics*.

meaning."[126] In addition, according to Gadamer, interpretation seeks to acquire what is actually to be understood in "a fabric of meaning," not by pursuing a mere objective determination of truth, but by making the object of consciousness "speak" and bring out what is in the structures of meaning that correspond to the given object of consciousness.[127] Therefore, influenced by Heidegger, Gadamer argues that, in a conversation, language does not only mean that someone speaks, but also speaks itself. However, the hermeneutic method is not focused on a particular means of communication, but it is focused on a particular basic stance of the human being in the world: this basic stance consists in being in conversation with one another. From this perspective, according to Gadamer himself, hermeneutics is "the art of being able to listen," and this art is one that must be taught methodically, because people should learn to take back, or discard, the prejudicial effects of their own will to understand and let someone oneself or something itself speak.[128] Hence, in hermeneutics, the actual subject is "understanding-in-the-world."[129]

A synthesis between aspects of Neoplatonism (especially, aspects of Plotinus's dialectical spirit and of Proclus's cosmology), modern structuralism, Kant's philosophy of critical reasoning, Marx's analysis of the material underpinnings of historical becoming (especially as it has been interpreted by Antonio Gramsci, Rosa Luxemburg, and Alexander Spirkin), and hermeneutics gives rise to the method of rational dynamicity, which I shall present and study in section 1.3. The purpose of the method of rational dynamicity is to interpret both the ontological reality and the intentionality of consciousness, which imposes its own structures on reality in order to ultimately reap the benefits of the dynamic action of consciousness in the world. As a philosophical method, rational dynamicity recognizes and analyzes both the reality of consciousness and the reality of the world, and, hence, it recognizes and analyzes both the objective and the subjective forces of history.

In his *Hamlet*, the great English dramatist William Shakespeare (1564–1616) uses the "Old Mole" to represent the ghost of Hamlet's father who keeps speaking from under the stage, despite Hamlet and Horatio shifting their ground seeking a suitable place to swear their oath.[130] Hegel, in his *Philosophy of History*, interpreted the ghost of Hamlet's father, namely, the "Old Mole," as a metaphor for the Spirit of history, while Karl Marx, in his book *The Eighteenth Brumaire of Louis Bonaparte*, interpreted the aforementioned "Old Mole" as a metaphor for the thoroughgoing revolution. However, the "Old Mole" is not only a metaphor for the objective forces and conditions of history, but it is also a metaphor for the subjective forces of history, namely, for humanity's own creativity, since the Ghost says: "Swear!", and Hamlet says: "Well said, old mole! Canst work i' th' earth so fast? A worthy pioneer! Once more remove, good friends!"[131] The aforementioned oath represents and expresses one's personal decision and personal commitment to methodically and critically act in order to transform a possibility provided by the objective historical conditions into an actuality. For instance, the German political theorist and activist Rosa Luxemburg (1871–1919) has highlighted the use of the dialectical term "or," which, as she has explained, implies that socialism is a possibility that is objectively offered to humanity and especially to the working

[126] Ibid.
[127] Ibid.
[128] Ibid.
[129] Ibid.
[130] Shakespeare, *The Complete Works of Shakespeare*, edited by David Bevington.
[131] Shakespeare, *Hamlet*, in David Bevington, ed., *The Complete Works of Shakespeare*, Act 1, Scene 5.

class, but it is by no means certain that humanity, in general, or the working class, in particular, will endorse this possibility and will decide to act in order to actualize and impose this possibility.[132] Moreover, in Shakespeare's *Midsummer Night's Dream*, an elf called Puck is a metaphor for the revolutionary subjective forces of history, and, thus, speaks as follows: "Up and down, up and down,/I will lead them up and down:/I am fear'd in field and town:/ Goblin, lead them up and down./Here comes one."[133]

Let us consider the case of a very controversial revolutionary leader and statesman, Joseph Stalin (born Iosif Vissarionovich Dzhugashvili), who was the second leader of the former Soviet Union (in particular, he was the general secretary of the Communist Party of the former Soviet Union from 1922 until 1952). Stalinism signifies the domination of an authoritarian bureaucratic regime, which cannot be considered as a logical, linear extension of the original Bolshevik revolutionary movement. However, Stalinism cannot be properly explained only in terms of Stalin's personality, ethos, and own political choices. In order to explain Stalinism in a philosophically and scientifically rigorous way, one has to take account of both the subjective forces and the objective forces that were at play during that historical period.

Apart from Stalin's own intellectual and moral qualities, the major subjective forces and the major objective forces that gave rise to Stalinism and to which Stalin fell prey were the following: (i) the socio-cultural underdevelopment of Russia (until the middle of the nineteenth century, the overriding majority of the Russians were slaves to an authoritarian tsarist-oligarchic regime and overwhelmed by ignorance and superstitions, and the Russian economy was significantly underdeveloped vis-à-vis the great Western industrial and commercial powers); (ii) particular aspects of traditional Russia's national character (e.g., excessive and eruptive emotionalism, a tendency to be self-absorbed to the degree of failing to be rationally integrated into historical becoming and of resorting to excessive dreaming and daydreaming, a geopolitically underpinned and motivated deep sense of threat and insecurity, a herd mentality resembling the behavior of the Eurasian wolf, etc.) that rendered a significant part of the Russian people intellectually and morally incapable of really and creatively understanding and implementing a socialist program for the liberation of the human being (as, for instance, Marx and Engels had envisaged it);[134] (iii) the defeat of revolutionary communist movements in the metropolitan capitalist countries; (iv) the decision of the Social Democratic Party of Germany (SPD) to discard the revolutionary communist ideology and strategy of the "Spartacus League" (founded by Karl Liebknecht, Rosa Luxemburg, and others) and to gradually acculturate to the German bourgeois establishment (the way to the notorious "Moscow Trials" was largely paved by the German social democrats who discarded the Spartacus League and by the assassins of Rosa Luxemburg);[135] and (v) the imposition of fascist-Nazi regimes in several European countries (e.g., in Germany under Adolf Hitler, in

[132] Luxemburg, *Socialism or Barbarism*.

[133] Shakespeare, *Midsummer Night's Dream*, in David Bevington, ed., *The Complete Works of Shakespeare*, Act 3, Scene 2.

[134] The Russian novelist Fyodor Dostoevsky has eloquently analyzed significant aspects of the traditional Russian national character. Moreover, see: Dicks, "Observations on Contemporary Russian Behavior"; Gorer and Rickman, *The People of Great Russia*.

[135] On 10 November 1918, the leader of the Social Democratic Party of Germany (SPD), Friedrich Ebert, at the time the Chancellor of Germany, and Wilhelm Groener, Quartermaster General of the German Army, signed the Ebert–Groener pact, with which Groener assured Ebert of the loyalty of the armed forces, and, in return, Ebert promised that his social democratic government would take prompt action against leftist uprisings and that the military would retain its "state within the state" status (see: Ruge, *Weimar: Republik auf Zeit*).

Italy under Benito Mussolini, in Spain under Miguel Cabanellas and Francisco Franco, in France under Philippe Pétain, in Greece under Ioannis Metaxas, etc.).[136]

Being aware of the aforementioned historical forces, trends, and conditions, in the early 1920s, Vladimir Lenin introduced the term "cultural revolution" into the Soviet political language in order to refer to the whole liberation of the people from all forms of political, economic, social, and spiritual despotism and backwardness (Mao Zedong's notion of a "cultural revolution" is something different). From the perspective of rational dynamicity, which I propose in this book, a necessary condition for the development of a worthy and meaningful model of socialism and for its successful implementation is that the members of the socialist movement must have assimilated the rational and liberal thought of the European Enlightenment, and they must understand their political task as an attempt to take the legacy of the European Enlightenment to its logical conclusion. On the other hand, in the twentieth-century Russia, the superstitious and fatalistic mentality of the Eurasian steppes and the intrinsic contradictions of the traditional Russian civilization proved to be stronger than Marxism–Leninism (which is part of the spiritual tradition of the European Enlightenment), and they actually spiritually conquered and subjugated Marxism–Leninism, predetermining its failure and the transformation of socialism-communism into a Soviet bureaucratic autocracy and, subsequently, especially during the presidencies of Boris Yeltsin and Vladimir Putin, into a post-Soviet Russian regime whose underpinning ideology is a mixture of oligarchic capitalism, *Realpolitik*, an updated version of the Russian "school" of nihilism (pioneered by Ivan S. Turgenev and Anton P. Chekhov), and elements of romanticism. Moreover, as the authoritative journalist Vladimir V. Pozner has pointed out, the raid that Western capitalist and military-bureaucratic elites launched against Russia in the 1990s and NATO's misguided belligerence played a major role in the development of authoritarianism and rigid bureaucratic structures in post-Soviet Russia under Vladimir Putin's presidency.[137]

Fyodor Dostoevsky has argued that, in the nineteenth century, the collective imaginary of the Russian society was largely bipolar, in the sense that the emotional aspect of the Russians' collective soul was largely determined by non-modern and Eurasian elements (specifically, by dreams of a highly emotional religious and magical nature), whereas the intellectual aspect of the Russians' collective soul, especially, among the most modernized and most educated members of the Russian society, was increasingly assimilating modern Western elements, which call for a decisive affirmation of reason and history. Thus, for instance, in the beginning of the twentieth century, faced with the failure and the discontents of the traditional tsarist society, Russia resorted to and endorsed a rational, modern Western ideology, namely, Marxism–Leninism, with the help of which Russia made significant and, indeed, stupendous achievements in the realms of science, technology, and economics, but, because the emotional world of many Russians, including many members of the Soviet elites, continued to clash

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[136] Fascism/Nazism is an instrument through which capitalist elites control and manipulate disillusioned and agitated popular masses (especially petty-bourgeois) during major capitalist crises in order to prevent them from contesting the established capitalist system. For instance, the German economist and banker Hjalmar Schacht, the German politician Rudolf Hess (a prominent member of the Thule Society), and leading German industrialists, such as Fritz Thyssen and Alfred Krupp, supported Adolf Hitler's rise to power in order to promote the interests of the German capitalist elite in combination with the interests of the German "deep state" through the Nazi regime and to destroy liberal socialist movements. See: Gluckstein, *The Nazis, Capitalism and the Working Class*; Langer, *Mind of Hitler*; Young, *Heidegger, Philosophy, Nazism*.

[137] On 27 September 2018, Vladimir Pozner gave a lecture at Yale University, explaining the impact of U.S. foreign policy toward Russia after the dissolution of the Soviet Union.

with the rationale of Marxism–Leninism and to be mainly non-modern, Marxism–Leninism was radically distorted in Russia, Russia's attempt to implement Marxism–Leninism was led to a dramatic failure, and Russia reproduced many of the defects of the traditional tsarist social model, instead of rationally pursuing a creative, progressive synthesis between positive traditional Russian qualities (such as Russian-Byzantine Christian morality and esotericism, resilience, a noteworthy capacity for accomplishments and perseverance, inner sociality, etc.) and positive modern European qualities (such as rational individuation, rational planning, rational organization, rational historical action, etc.); in fact, the philosophy of rational dynamicity yields such a synthesis.

It is important to mention that the Russian-Soviet philosopher and scientist Alexander Bogdanov, one of the acknowledged founders of the science of planning and organizational theory, argued that World War I underlined the cultural deficiency of the working class, in the sense that, "inadequately organized and hidebound by tradition, industrial workers had succumbed to the primitive nationalism of the petty-bourgeoisie and the peasantry."[138] In addition, according to Bogdanov, the socialist intelligentsia was not better equipped to effect a socialist transformation of society, because "the cultural development of the socialist planners themselves was a precondition of socialism, but most social scientists, as members of the ruling class, were imbued with the individualism of private enterprise."[139] Therefore, Bogdanov argued that socialism is meaningless without a "universal organizational science," which would "combine and coordinate all the individual disciplines."[140] Bogdanov's structuralist approach to socialism in general and to Marx's thought in particular is a very important contribution to the intellectual development and reinforcement of structuralism, because it gives rise to a science of planning, and, given that the philosophy of rational dynamicity, which I expound and propose in this book, is founded on structuralism, I have utilized elements of Bogdanov's research work in order to articulate the method that I call the "dialectic of rational dynamicity" (see section 1.3.3).

As a philosophical criterion, rational dynamicity stems from consciousness, but it is actually activated and implemented when it is actually possible to be applied to objective reality. However, it is important to mention that, as I shall explain in the following section of this chapter, rational dynamicity implies that the reality of the world can be restructured by the intentionality of consciousness. Objective reality is multidimensional and complex, but it becomes significant to consciousness only when objective reality is rationally dynamized in accordance with the intentionality of consciousness. Therefore, the method of rational dynamicity is in agreement both with the nature of the world and the nature of consciousness. Furthermore, the method of rational dynamicity is in agreement with (and provides a creative way of interpreting) Genesis 1:26–30: in that biblical excerpt, we read that the divine creative Cause not only created the human being in Its image, but also assigned authority and responsibility to the human being for the rational organization and management of the world, thus proclaiming the reality of the world, the reality of human consciousness, the structural continuity between the reality of the world and the reality of consciousness, and the submissiveness of the reality of the world to the intentionality of human consciousness.

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138 See: Biggart, "The Rehabilitation of Bogdanov," pp. 11–12.
139 Ibid.
140 Ibid.

1.2.4. The Modes of Being

The essence of being is a system consisting of qualities that can be attributed to it (that is, they can be identified, and they can be assigned to it). The act of being is a state in which an existent can be in one of the following ways: (i) absolutely positively, and then it is said to be a beingly being; (ii) absolutely negatively, and then it is said to be a beingly non-being; (iii) intermediately, and then it is said to belong to an intermediate ontological category between beingly beings and beingly non-beings. Thus, in his dialogue *Sophist*, Plato identifies what he calls the "greatest kinds" ("mēgista gēne")—namely: motion, rest, sameness, difference, and the relationship between them—which Aristotle would call "categories" of being, and which are actual models of the modes of being. From the perspective of Aristotle's *Categories*, a being exists with regard to its substance, with regard to its form, with regard to the relationship between substance and form, with regard to its time, with regard to its space, with regard to its activity, and with regard to its passivity.

When Aristotle says that a being exists with regard to its substance, he refers to the "material" of which a being is composed, namely, to the "material cause" of a being. The material cause of a being allows a being to be what it is, and it underpins the differentiation of a being from every other being that is composed of the same material. This "substantial" mode of being is a qualitative attribute of a being, and it is complemented by form (i.e., by the "formal" mode of being), which is a quantitative attribute that is due to species. In fact, in his *Metaphysics*, Aristotle replaced the Platonic term "idea" with the concept of species. Form, namely, the external shape of a being, is a mode of being that is assumed by substance, and, due to its form, a being is even more sharply differentiated from every other being. Due to the important role that substance and form play in Aristotle's philosophy, the latter has been characterized as a hylomorphism.

Besides substance (i.e., the material cause of a being) and form (i.e., the formal cause of a being), Aristotle studies the efficient cause of a being and the final cause of a being. The efficient cause of a being refers to that event which has produced the given being, and which underpins and controls the existence of the given being. The final cause of a being refers to that event which is the end-purpose of the existence of the given being, and which can be accomplished due to the presence of the given being. No being is totally self-contained, since its existence is due to an external cause. According to Aristotle, the only exception to the aforementioned rule is the "prime mover" (i.e., the first uncaused cause), which is the cause of itself, and whose character is logically necessary. Furthermore, the purpose of a being (i.e., the event at which a being is aimed) represents the converse of a being's dependence on an efficient cause, and it vindicates a being's existence. The purpose, the *telos*, of a being is innate in being, and, if a being does not fulfill its purpose, then it is meaningless and irrational. All the aforementioned causes of being can be understood through the following example, which is due to Aristotle[141]: Let us consider the production of an artifact, such as a bronze statue. The bronze is the material cause of the bronze statue. However, the bronze is not only the material out of which the statue is made, but it is also the thing that undergoes the change and results in a statue. The shape of the statue is the formal cause, which actually makes the bronze statue a bronze statue and not, for instance, a bronze vase. The art of

[141] Aristotle, *Physics*, 195a6–8; and *Metaphysics*, 1013b6–9.

bronze-casting the statue and the artisan who manifests specific knowledge of that art constitute the efficient cause of the bronze statue. The use of the bronze statue is its final cause (*telos*).

The cohesive bond between substance and form is the structure of a being. The deepest reality of a being is its substance, the external aspect of that reality is the form of the given being, namely, an element that animates the given being, and these two elements (modes of being) concur with each other in the context of the structural mode of being. As mentioned in section 1.2.3, “structure is an internal reality that is governed by each own order, which it creates and recreates by itself.” Thus, the structural mode of being underpins the rational, free, and unique order of the internal and the external elements of a being, because the coexistence of the internal and the external elements of each being is expressed through the structure of the given being, and the structure of each being is the cohesive bond between the internal and the external elements of the given being, and it guarantees the coexistence of these elements. In addition, structure allows a being to adapt to changeable situations without changing itself, since it remains (structurally) immutable. In other words, structure is the animate program of being, and it is actualized in terms of an existential journey that is determined by the corresponding being itself. Consequently, the structural mode of being allows a being to subsist and to remain in existence by being identified with itself, despite the various circumstantial changes that a being may undergo in the context of its activities and/or due to tactical adaptations, which a being may make in order to facilitate its action.

Because of its intermediary and relational role, structure underpins and secures what Aristotle has called “entelechy,” namely, the continuity of the presence of a being as that which the given being is and not as something else. Entelechy signifies the *a priori* existence of a specific existential model within a being, and, according to this model, the corresponding being determines its existential journey. Intimately related to structure are two other modes of being, which have also been identified and studied by Aristotle, specifically, “being potentially” and “being actually.” Being potentially is an existential state in which the existential program of a being has not been completed, and it may only be in an early stage of its structural formation, but it has a clear and definite orientation, and, thus, there is a program that governs every aspect of the behavior of the being under consideration. The potential being of a being contains aspects and consequences of every forthcoming particular mode of being of the given being, and, even though the manifestations of these forthcoming modes of being may be conceived in infinitely many different ways, all forthcoming modes of being are determined by the innate existential program of the corresponding being (this is the meaning of “structural stability”).

Being actually is an existential state in which the existential program of a being has been totally completed. The completion of the existential program of a being determines both the given being itself and the impact of the given being’s behavior on other beings and on the existential states of other beings. According to Aristotle, being actually (actuality) is to being potentially (potentiality) as “someone walking is to someone sleeping, as someone seeing is to a sighted person with his eyes closed, as that which has been shaped out of some matter is to the matter from which it has been shaped.”[142] Being actually is an ultimate mode of being, but it is not an exclusive mode of being, in the sense that there exist intermediate existential states that correspond to several degrees of being.

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[142] Aristotle, *Metaphysics*, 1048b1–3.

An important problem is to determine the critical points (or critical values) of the function of a being's existence (in this case, I use the terms "function" and "critical point of a function" as they are used in mathematics, and I study their mathematical significance in Chapter 2). By a critical point of the function of a being's existence, I mean a dynamized degree of being *before* which the given being is not an actuality *yet*, and *after* which the given being is not a mere potentiality *anymore*. If we want to formulate the corresponding mathematical model, then we should define a being's existence as a function of time and find its critical points (namely, the points at which the corresponding function is not differentiable or its derivative is equal to zero). In other words, at a critical point of the function of its existence, a being is simultaneously a quasi-actuality and a quasi-potentiality, and, therefore, it is present enough and malleable enough to be restructured according to the intentionality of one's consciousness. In fact, this argument underpins the dialectic of rational dynamicity, which I shall study in section 1.3.3.

In view of the foregoing, actuality, potentiality, and every other fundamental mode of being can be relativized. In fact, at the level of the absolute being, the modes of being can be considered in an absolute way, but, at the level of any other being, the modes of being should be considered in a relative way. Therefore, at the level of any relative being, the fundamental modes of being can be replaced by other existential qualities that derive from the fundamental modes of being themselves, and I would call them "existential derivatives" (since they derive from the fundamental modes of being). The aforementioned "existential derivatives," namely, the relativized varieties of the fundamental modes of being of classical ontology, underpin the creativity of the active presence of consciousness in the world, and they confirm and semantically enrich Thomas Jefferson's argument that "the earth belongs in usufruct to the living."[143] As I shall argue in section 1.3, the world belongs in usufruct to consciousness, and there is a structural continuity between the reality of the world and the reality of consciousness.

1.3. The Dialectic of Rational Dynamicity

The concept of structure can be associated with and ascribed to both the reality of the world and the reality of consciousness whenever the reality of the world and the reality of consciousness are considered with regard to their functions and energies. As I shall argue in this section, the concept of dynamization underpins the conception of the character of the structural synthesis between the reality of the world and the reality of consciousness, it indicates that consciousness is fundamental to reality, and it highlights the ability of consciousness to interpret the reality of the world and assign meaning to it.

1.3.1. Dynamized Time

The philosophical and the scientific theories of cosmology that belong to the "school" of philosophical realism maintain that the concept of time corresponds to an objective reality.

[143] Jefferson, "The Earth Belongs in Usufruct to the Living."

The philosophical and the scientific theories of cosmology that belong to the "school" of idealism maintain that the concept of time is a conscious construction that serves as a means for understanding the world and the relations between consciousness and the world. Thus, the comprehension of the concept of time depends on the comprehension of the relation of between consciousness and the world.

In general, time is perceived as an order on the set of the states through which a being passes successively (for a rigorous study of the concept of an ordering relation, see Chapter 2). Thus, the notion of time seems to be intimately related to the notion of being, and, for this reason, Aristotle has argued that time is one of the modes of being.[144] From this perspective, Aristotle's perception of time belongs to the "school" of philosophical realism. Before Aristotle, Plato had set the foundations for the development of a realist philosophy of time by arguing that time is a moving image of eternity, specifically, a fluid-like phenomenon that corresponds to the stable, unchangeable, and infinite reality that exists continuously and uninterruptedly.[145]

From the perspective of Platonism, the relation between "eternity" and "time" is similar to the relation between the "idea" (as a transcendent reality) and the "phenomenon": eternity, just like the idea, is immovable, free from passion, and hardly conceivable in itself, but it is dynamically reflected in the fluid-like realm of phenomenality. Moreover, from the perspective of Platonism, by harmonizing itself with the mobility of the world (as a reflection and sensory manifestation of eternity), consciousness conceives both itself and the reality of the world, irrespective of whether one may think that the reality of the world contains consciousness or it is contained in consciousness.[146]

According to Platonism, eternity is conceptually equivalent to infinity and the absolute, which exists unchanging over time, and all particular events take place within the absolute. Furthermore, from Plato's perspective, eternity is the underlying fabric of time. As Aristotle has pointed out, because eternity has the property of infinity, it cannot be experienced directly, but only indirectly, through the intellect (namely, the rational faculty of consciousness), as a concept, except for those cases of metaphysical intuition which are not reducible to discursive reasoning, and they are called experiences of "grace" by religious mystics and theologians. However, as Conor Cunningham stresses in his scholarly work in philosophical theology, many doctors of the Christian church have taught that, far from contradicting nature, grace signifies the ultimate reason of nature and its fulfillment.[147] As Thomas Aquinas has written, "grace does not destroy nature, but perfects it."[148]

In short, "grace" can be construed as the very possibility of nature, as we read in the *Hermetica* (Egyptian-Greek wisdom texts from the second and the third centuries A.D. that are mostly presented as dialogues in which a teacher, generally identified as Hermes Trismegistus ("thrice-greatest Hermes"), enlightens a disciple).[149] According to Hermeticism,

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[144] Aristotle, *Metaphysics*, 1b25–2a4.

[145] Plato, *Timaeus*; *Parmenides*; and *Republic*. Moreover, see: Brague, *Du Temps chez Platon et Aristote*; Leyden, "Time, Number, and Eternity in Plato and Aristotle."

[146] Ibid.

[147] Cunningham, *Genealogy of Nihilism*.

[148] Thomas Aquinas, *Summa Theologiae*, Q. 1.

[149] Around 1460, a Greek manuscript (known as the *Hermetica*) was delivered to Cosimo de' Medici, containing mysterious tracts attributed to the mythical exalted mystic Hermes Trismegistus, who, according to the symbolic language of those tracts, had gained his knowledge and wisdom directly from the divine "Nous"

namely, the cult of Hermes Trismegistus, there is a reciprocal relationship between the physical world (the physical "microcosm") and the spiritual world (the spiritual "macrocosm"): the world is a "dynamic world," specifically, a lawful, intelligent, and active system (or "order") directed by a transcendent, wise efficient and final Cause in accordance with the Hermetic maxim "as above, so below." In particular, according to Sir Isaac Newton's translation of the "Emerald Tablet" (one of the most important pieces of the *Hermetica* reputed to contain the secret of the "prima materia" and its transmutation), "That which is below is like that which is above that which is above is like that which is below to do the miracles of one only thing." On this account, the human being has agency, making one's fate within the framework of divine traditions (systems of ultimate values) and according to one's rational dynamicity at every moment.

Furthermore, "grace" can be experienced and understood as initiation into a path of existential integration and as the pursuit of and the impetus toward existential integration, combined with a way of life that discloses the meaning of grace.[150] The pursuit of and the impetus toward perfection are inextricably linked to the concept of infinity and, more specifically, to the infinite mobility of the human mind, which, when operating according to its nature, moves unceasingly toward ever higher levels of understanding. The presence of grace is intimately related to and indicated by a way of life that is characterized by the orientation of consciousness toward infinity, according to which a transcendent synthesis between perception and intuition is achieved. On the other hand, the lack of grace is intimately related to and indicated by a way of life that is characterized either by the entrapment of consciousness in finitude or by an ineffectual effort to escape to infinity, which results in irrationality. Hence, in the context of rational dynamicity, what one calls "god" is construed and experienced as a model and a force of ontological perfection that humanity is called to achieve and manifest through the dialectic of rational dynamicity (see section 1.3.3).

Eternity constitutes the continuous, underlying fabric of the movements of a being, whereas time is the means by which these movements are measured. Furthermore, eternity is the absolute Infinity, the infinity *par excellence*, whereas time is a dimension of a world of several relative infinities, which I shall mathematically analyze in Chapter 2, according to Georg Cantor's transfinite arithmetic, and it is infinitely divisible into infinitesimals, which underpin the development of infinitesimal calculus by Isaac Newton and Gottfried Wilhelm von Leibniz, as I shall explain in Chapter 2. Thus, time is not so much a reality as a measuring instrument used by consciousness whenever the latter tries to determine its own reality. As Immanuel Kant has argued, time is an *a priori* (specifically, pre-experiential and pre-perceptive) "schema" (figure) that underpins the presence of the world within consciousness, and it structures and makes possible the cognition of objects as appearances.[151]

In Kant's philosophy, a transcendental "schema" (plural "schemata") is a procedural rule, specifically, a mediating function between the active consciousness (the activity of the understanding) and the passive sensibility (the receptivity of the senses), by which a category is associated with a sensory perception. In his essay "On a Discovery Whereby Any New Critique of Pure Reason Is to Be Made Superfluous by an Older One," Kant explains that he relies on a method through which objects are constructed according to rules of the

(mind), Pimander, and who later himself became deified. The *Hermetica* teaches a tolerant philosophical religion of the illumined mind and a spiritualist variety of globalism.

[150] See: Cunningham, "*Natura Pura*, the Invention of the Anti-Christ."

[151] See: Guyer, ed., *The Cambridge Companion to Kant*.

understanding, and he argues that it is also requisite for a philosopher to make abstract constructs (concepts) sensible, specifically, to display the objects that correspond to concepts in intuition, because otherwise concepts remain without sense and, hence, insignificant. In the same essay, Kant argues that mathematics can achieve this goal by means of the construction of figures, which, even though brought about *a priori*, are appearances present to the senses.[152]

In the twentieth century, the Swiss epistemologist and psychologist Jean Piaget took up Kant's terminology while refashioning the function of schemata to his theory of cognitive development.[153] In fact, Piaget has shown that the development of the concept of time within consciousness takes place in parallel with the development of consciousness itself, in the following sense: initially, time does not exist "in itself" as a monotonous, uniform metric time, but it exists as a collection of partial, "impure" conceptions of time, which are mutually irrelevant and underpin different functions of consciousness (for instance, according to Piaget, for children up to the age of operative intelligence, time exists embedded in contexts of action, specifically, in pragmatically oriented processes of movement and action, and it is bound to objects). During this initial stage, consciousness refers to these partial, "impure" conceptions of time only when consciousness wishes to integrate some of its experiences into a particular timeline. During subsequent stages of the development of consciousness, as consciousness expands itself and accumulates more experiences, the aforementioned unrelated, partial conceptions of time gradually become united.

Piaget's perception of time is, in a sense, an extension of Bergson's perception of time, which is based on the comparison between time and duration. According to Descartes, the comparison between time and duration leads to the conclusion that time is abstract, while duration is concrete. However, Bergson maintains that duration is the only indisputable reality, and it constitutes the depth of both the reality of the world and the reality of consciousness, while time is a practical substitute for duration. Furthermore, according to Bergson, whereas eternity is infinite, duration is finite, it is essentially memory, and it can be conceived by means of a logical concept, just as time is conceived through alterations of consciousness. Analyzing Bergson's philosophy, the Japanese philosopher Daisaku Ikeda argues as follows:

> According to Bergson's theory of time, the division into past, present, and future is the product of human consciousness . . . Bergson considered the true nature of consciousness to be in flux, and he spoke of "flowing time." Time perceived from the physical, objective viewpoint is time past. In contrast, "flowing time" is the flow of consciousness or of life itself. In essence, there is no distinction between past, present, and future, since they are created by the flow of consciousness. What is inseparable becomes separated in our minds.[154]

From Bergson's perspective, duration is a continuous flow that cannot be conceived logically or empirically, but it can be conceived through intuition, which is a function of consciousness through which, instead of going around its object from the outside in a cognitively ineffective way, consciousness enters directly into its object in order to consider it

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[152] See: Allison and Heath, eds., Immanuel Kant: Theoretical Philosophy After 1781, pp. 271–336.

[153] Kitchener, *Piaget's Theory of Knowledge*; Piaget, *Biology and Knowledge*; Piaget and Inhelder, *Memory and Intelligence*.

[154] Ikeda, *Life an Enigma*, p. 73.

from the inside. For Bergson, intuitive thinking consists in thinking in terms of duration, and, according to Bergson's conception of duration, consciousness knows itself as duration, consciousness knows that it is the consciousness of the duration of the existence to which it refers, and consciousness knows that, by being duration, it is part of reality. Thus, Bergson conceives duration as "the multiplicity of conscious states"[155] and as a "qualitative multiplicity" that can be defined as "a unity that is multiple and a multiplicity that is one,"[156] so that, as John Mullarkey (the editor of the journal *Film-Philosophy*) has pointed out, Bergson's notion of duration is "a group of mutually interpenetrating elements."[157] According to Bergson's sensuous and psychological type of intuition, the subject and the object of intuitive conception interact and mingle with each other in the context of the reality of duration.

Summarizing Bergson's philosophy of time, we conclude that, according to Bergson, time is always a concept that is formed *a posteriori* as a consequence of the interplay between the divisible concept of time and the consciousness of the indivisible duration. Due to the aforementioned interplay, time is an inauthentic concept, in the sense that it does not correspond to anything, but it is created by consciousness, because the latter seeks to overcome its difficulties by analyzing them—in Cartesian fashion—in order to make them measurable. Moreover, as I have already mentioned, in Bergson's philosophy, eternity, as reality, is conceived in a static way, but duration is conceived in a dynamic way—specifically, as a continuous directed flow—and it imparts its direction to the concept of time.

As the ancient Greek philosophers Proclus (fifth century A.D.)[158] and Aenesidemus (first century B.C.)[159] have explained, we conceive the direction of duration and of time in terms of the relationship between an obsolete state and a forthcoming state. Let us assume that a horizontal straight line l represents the flow of time. Moreover, let us consider an arbitrary point p moving on l. The motion of p is meaningless, unless we define two relevant directions on l: the direction of the "predecessor" and the direction of the "successor" (for a mathematically rigorous explanation of these terms, see Chapter 2). Only then can we determine the magnitude and the direction of the motion of p. However, the point p can be interpreted either as a segment of transient time, which comes in order to leave and to become part of the past, or as the consciousness of existence that moves from an obsolete state to a forthcoming state. In each case, following the reasoning of Proclus and Aenesidemus, we realize that time is divided into smaller segments that correspond to the concepts of before and of after due to a moving point that represents the concept of now, which is defined as a continuously changing state of becoming that moves in the direction of the perceived motion of time or of consciousness itself.

Furthermore, in view of Bergson's and Piaget's philosophies of time, the motion of time is not homogeneous, since time can flow at different rates in different reference frames; for instance, in some cases, the rate of time's flow is intense, while, in other cases, it is imperceptible. These variations can be interpreted in two ways: first, there are different kinds of time, such as cosmological time, biological time, psychological time, and historical time; second, the dynamic nature of time allows its restructuring by consciousness.

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155 Bergson, *Time and Free Will*, p. 75.
156 Bergson, *Creative Evolution*, p. 258.
157 Mullarkey, *Bergson and Philosophy*, p. 19.
158 Proclus, *Elements of Physics*.
159 Aenesidemus, *Pyrrhonian Discourses*.

The relation between time and light and, particularly, the dynamic nature of physical time have been explained by Albert Einstein's theory of relativity.[160] According to the special theory of relativity, time can flow at different rates in different frames of reference, since time depends on the velocity of one frame of reference relative to another. Time dilation is the slowing down of a clock as determined by an observer who is in relative motion with respect to that clock; so that the faster one moves through space the slower one moves through time. Given that the speed of light is the same in all reference frames, the moving clocks run slow. In particular, time intervals have different values when measured in different frames of reference. Hence, by the term "time dilation," we refer to the lengthening of the time interval between two events for an observer in a frame of reference that is moving with respect to the rest frame of the events, in which the events occur at the same location. The relation between a time t_s measured by a stationary observer (i.e., the time measured by an observer inside the given frame of reference), and the time t_m measured by an observer moving with velocity v (i.e., the time measured by an observer outside the given frame of reference) is given by the following formula, known as the equation for "time dilation":

$$t_m = \gamma t_s \text{ where } \gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}},$$

v denotes the speed of the moving observer, who sees the clock moving (or, equivalently, the speed of the clock relative to the observer who is outside the given frame of reference), and c is the speed of light in a vacuum. In fact, at low speeds, there is only a small change in time dilation, namely, the flow of natural time does not change very much, but, at speeds over about seventy-five percent of the speed of light, the effect of time dilation is very dramatic (for instance, imagine the following case: an astronaut left on a space flight today, and, on that space flight, he flew around outer space at near the speed of light for about three years according to his clock; then, during his space flight, your clock on Earth was moving much faster relative to his clock, and, in fact, more than sixty years would have passed on Earth).

In the 1940s, the renowned American theoretical physicist John Archibald Wheeler (largely responsible for reviving interest in general relativity in the United States after World War II), studying positrons as electrons travelling backward in time, proposed his one-electron universe postulate, according to which there was, in fact, only one electron, bouncing back and forth in time.[161] Wheeler's idea that positrons were electrons travelling backward in time was studied further by another renowned American theoretical physicist, Richard Feynman, who, in fact, incorporated the idea of the reversibility of time into his famous "Feynman diagrams," which are pictorial representations of the mathematical expressions that describe the behavior and the interactions of subatomic particles.

On the basis of the foregoing philosophical and physical ideas, time depends not only on the concepts of before and of after, which refer to the flow of time, but also on the concept of

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[160] See: Sears, Zemansky, and Young, *College Physics*, pp. 927–28.

[161] A positron (alternatively known as an antielectron) is the antiparticle or antimatter counterpart of the electron, and it has an electric charge of $+1e$, a spin of $\frac{1}{2}$ (the same as an electron), and the same mass as an electron. According to the Breit–Wheeler process, a positron–electron pair is created from the collision of two photons, and this process is the simplest mechanism by which pure light can be potentially transformed into matter. The inverse process, according to which an electron and a positron collide and annihilate to generate a pair of gamma photons, is known as electron–positron annihilation, or the Dirac process.

now, and, therefore, time can (and, in fact, should) be parametrized in terms of two parameters, one of which represents the flow of time, while the other represents the concept of now. Consequently, in order to define any temporal point, we need a two-dimensional coordinate system, consisting of the horizontal axis, namely, the x-axis, and the vertical axis, namely, the y-axis (in this case, by the term "axis," we mean a straight line with respect to which a body or structure is symmetrical), and the intersection of the two axes is the origin O of the coordinate system (regarding the meaning of a coordinate system, see also Chapter 2). In this case, the x-axis represents the flow of time, and it runs left and right, where the terms "left" and "right" are defined with respect to the origin O of the coordinate system; the left part of the x-axis represents the concept of before, and the right part of the x-axis represents the concept of after. Moreover, in this coordinate system, the y-axis, which is perpendicular to the x-axis (their intersection being the origin of the coordinate system), represents the concept of now, which is determined by the manner in which consciousness experiences the flow of time. The y-axis runs down and up, where the terms "down" and "up" are defined with respect to the origin O of the coordinate system, the downward part of the y-axis represents the "past," namely, the manner in which consciousness organizes the temporal points that correspond to the concept of before and to obsolete conscious states, and the upward part of the y-axis represents the "future," namely, the manner in which consciousness organizes the temporal points that correspond to the concept of after and to forthcoming conscious states. The parameter of now is a conscious construction, and, therefore, it is the aspect of time that is determined by consciousness. This is the reason why this "now" (like a rational equivalent of Wheeler's positron) can assume values over the entire y-axis, running down (past) and up (future). Consequently, an arbitrary temporal point t is parametrically defined as an ordered pair

$$(x, y) = (flow\ of\ time, now).$$

This is the way in which I define the dynamization of time, namely, as the parametrization of time in terms of the flow of time and the concept of now (regarding the mathematical concepts involved in this analysis, see Chapter 2).

In the beginning of the twentieth century, the German philosopher Edmund Husserl originally and emphatically argued that consciousness can extend to capture past moments of experience and temporal objects therein by "retaining" and "protending" elapsed and yet to come phases of experience, so that past words that do not actually exist in the present (when one is in a particular stage of a process of receiving information) can, indeed, remain related to the present experience.[162] For instance, when we listen to a piece of music, we have direct access to a certain note, but the piece of music is not composed of a single, isolated note, since it is a coherent unity of different notes, and we perceive it as such. Thus, according to Husserl's phenomenology, the direct access to a certain note is the "primal impression," but our experience is not exhausted in that note, since we simultaneously retain the notes that are no longer heard (and this is the process that Husserl has called "retention"), and we anticipate the subsequent notes (and this is the process that Husserl has called "protention"). From Husserl's perspective, the dilation of the present that corresponds to protention is underpinned

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[162] Husserl, *On the Phenomenology of the Consciousness of Internal Time*. Moreover, see: Zahavi, *Husserl's Phenomenology*.

by retention, and, therefore, Husserl perceives the dilated present in terms of retention. This peculiar way in which consciousness refers to time is due to what Husserl has called the "intentionality" (or the "referentiality") of consciousness, namely, due to the fact that consciousness is the consciousness of its contents, which thus become experiences for it. However, this is not the only way in which consciousness intervenes in the flow of reality.

It goes without saying that consciousness is not passive, because not only does consciousness treat the presence of experiences within itself in a critical way, it also causes their presence, as it is implied by the term "intentionality," which Husserl has ascribed to the activity of consciousness. However, from Husserl's perspective, intentionality reduces to referentiality, and, therefore, the creativity of consciousness is constrained to referentiality, whereas, from Bergson's perspective, intentionality is not only the ability to refer to something, but also the ability to cause something. Hence, if we espouse Bergson's argument that intentionality consists of both the ability to refer and the ability to cause, then we realize that the term "intentionality" expresses the dynamism of consciousness, and the dynamism of consciousness manifests itself in the manner in which consciousness intervenes in temporality. In order to understand and appreciate the dynamism of consciousness and the meaning of the "dynamized time," we have to realize that the parameter of now (namely, the movement of a temporal point t with respect to the y-axis) allows consciousness to restructure time by evaluating and characterizing the various segments of time and, thus, by seeking some of them, recalling others, and transcending or repelling others. This dynamic process can be called "dynamic recursion" (as I shall explain in Chapter 2, "recursion" is a programming term that means calling a function from itself). This dynamic and creative conception of "now" as well as its dynamic interplay with the "past" and the "future" were originally studied by Proclus and Aenesidemus.

According to dynamic recursion, consciousness does not merely retain the present, preventing it from becoming part of the past, but also intervenes in the flow of time in several other ways. For instance, consciousness makes temporal jumps by recursively dynamizing segments of the past and/or of the future and, thus, transforming them into "now-experiences." Whenever consciousness recursively dynamizes a segment of the past and, thus, transforms it into a now-experience, the given now-experience can be characterized as a type of present that is "recursive up to the past." Whenever consciousness recursively dynamizes a segment of the future and, thus, transforms it into a now-experience, the given now-experience can be characterized as a type of present that is "recursive up to the future." Whenever consciousness recursively dynamizes both a segment of the past and a segment of the future and, thus, transforms both of them into a now-experience, the given now-experience (which is a synthesis of the two aforementioned dynamized segments) can be characterized as a type of present that is "recursive." In other words, the term "recursive present" refers to the synthesis between the present that is recursive up to the past (thesis) and the present that is recursive up to the future (antithesis).

The aforementioned conception of the "recursive present" gives rise to a dynamical model of the consciousness of time, according to which consciousness is capable of the following: (i) dynamizing and, thus, retaining the present in order to prevent it from becoming part of the past; (ii) dynamizing and, thus, retaining segments of the past, even of the distant past (and not only—as Husserl suggests—segments of the recent and still formless or incompletely formed past); and (iii) dynamizing and, thus, protending segments of the future, even of the distant future, by transforming them into now-experiences, meaning that consciousness can conceive a future possibility and transform it into a present actionable situation, thus *creating* the favorable conditions required for achieving a goal. Consequently, the most important aspect of time is a now-experience that can run the entire continuum of the flow of time, namely, the aforementioned x-axis (the mathematical terminology that I have just used will be clarified in Chapter 2). According to the terminology that I have used until now, I define the "recursive present," denoted by RP, as a concept that denotes accumulated now-experience, denoted by NE, divided by the flow of time, denoted by FT, on either an instantaneous or an average basis, symbolically:

$$Instantaneous\ RP = Change\ in\ accumulated\ NE = \frac{NE}{FT} \text{ at time } x_0;$$

and

$$Average\ RP = Rate\ of\ Change\ in\ accumulated\ NE = \frac{change\ in\ NE}{change\ in\ FT} = \frac{\Delta y}{\Delta x} for\ the\ time\ interval\ x_1 to\ x_2.$$

Given the aforementioned dynamical system of categories, "now-experiences" may correspond to temporal points (namely, to points of the x-axis) that are far away from the actual, immediate present. Furthermore, from the previous perspective, a "now-experience" can be interpreted as a local extremum of time caused by the intervention of consciousness in the flow of time (the mathematical significance of an extremum will be clarified and studied in Chapter 2). In fact, now-experiences, as extreme points of time, represent the results of the intervention of consciousness in the flow of time, since the fact that dynamized time is actionable is manifested in the ability of consciousness to discern and utilize possibilities of action in dynamized time and to find different ways of dynamizing time. These now-experiences, representing local extrema of time, encompass dynamized time, and, in general, the essence of dynamization. The aforementioned interpretation of "now-experiences" is a (re)formulation and modification of the following mystical approaches to time in purely philosophical and scientific terms: (i) that mystical Christian approach to time according to which the dynamization of time is construed as the experience of the encounter between "ens creatum" and "ens increatum"[163]; and (ii) the concept of "discrete time," proposed by the French philosopher, theologian, and Iranologist Henry Corbin in his analysis of the structure of time in the Shia and Sufi Islamic traditions.[164]

In general, as Jean Piaget,[165] among others, has pointed out, time is an "intellectual construction" that facilitates the activity of consciousness. However, far from becoming a

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[163] It is worth mentioning that the word "encounter" is one of Pope Francis's favorites, and, for instance, it was used thirty-four times in his apostolic exhortation *Evangelii Gaudium* (Rome, 24 November 2013).

[164] According to Corbin, a mystic following the path of Islamic gnosis should make time somewhat personal; one can personalize time by discovering its unique features (name, figure, character, etc.). By doing so, a mystic achieves the transformation of time into space. That was the original meaning of the ancient concept of "Aeon"; namely, a personalized "time entity." Acquainting oneself with this "time entity," a mystic avoids the doom of the "horizontal" time and finds the way into the imaginary one, "alam-al-mithal," the inner realm of the "malakut" ("beyond birth and death"); this is the very place where the "hidden Imam" lives (Corbin, *La Topographie Spirituelle de l'Islam Iranien*).

[165] Kitchener, *Piaget's Theory of Knowledge*; Piaget, *Biology and Knowledge*; Piaget and Inhelder, *Memory and Intelligence*.

prisoner of its own constructs, such as time, consciousness forms systems of dynamized time by means of which consciousness restructures temporality, and, ultimately, it affirms its freedom and imposes its intentionality on the world. By dynamizing time, consciousness, ultimately, rationalizes and manages the world, given that, through the dynamization of time, consciousness understands the world in a more intelligent and a more creative way. Whereas time is an "intellectual construction" that derives from the reference of consciousness to the world, the dynamization of time is an "intellectual construction" that derives from the intentionality of consciousness. In particular, dynamization can be construed as the dynamic expression of the intentionality of consciousness whenever the intentionality of consciousness consists in a continuously updated strategic plan of action, formed by consciousness for the sake of consciousness. From this perspective, the aforementioned way of defining and studying the dynamization of time is akin to Heidegger's notion of "Ereignis": in terms of Heidegger's philosophy, the dynamization of time can be understood as an event, specifically, as something "coming into view," or as "enowning" (in German, "Ereignis"), and, more precisely, it refers to the transition of "Dasein" ("being there" or "presence") from an inauthentic mode of being to the authentic mode of being; this is the time of authentic being in contrast to the time of inauthentic being, and, in the time of inauthentic being, one always hesitates whether to be or not to be (yet). Thus, from Heidegger's perspective, dynamization signifies the moment of decision (in German, "Entscheidung") that implies whether it is possible or not for "gods" to return.

In conclusion:

- *eternity* is the characteristic mode of being of what we construe by the term "absolute being" (as I have already explained, according to Plotinus, eternity is not the whole time, but the everlasting moment of being always equal to itself, and, from this perspective, dynamization is the moment of rapture and instant elevation to the utmost levels of being);
- *duration* is the characteristic mode of being of every being that continuously tries to preserve and affirm its own substance and to discard any alien substance;
- *time* is the characteristic mode of being of the world as the latter is perceived and organized by consciousness; and
- *dynamized time* is the characteristic mode of being of consciousness, because, as I have already explained, consciousness perceives the reality of the world and thinks of the reality of the world, while simultaneously having intentionality and will, according to which consciousness acts in order to restructure the reality of the world in terms of dynamized time.

Through the aforementioned analysis of time, we can understand both the mechanical way in which the states of being succeed each other and the dynamic way in which consciousness refers to time. Dynamization signifies both a dynamic attitude of consciousness toward the world and a way in which consciousness manages to intensify its presence in the world. The presence of consciousness in the world restructures time, and, therefore, it calls for the study of dynamized time. Moreover, as I shall argue in the following section, intimately related to the dynamization of time is the dynamization of space.

1.3.2. Dynamized Space and the Problem of the Extension of the Quantum Formalism

At the highest level of abstraction, specifically, in the context of pure mathematics, which I shall study in Chapter 2, the term "space" is construed as a structured set, namely, as a collection of elements, called the points of the given space, equipped with a set of rules that determine the relations and, in general, the behavior of the given space's elements. Because a structured set is, more generally, a "system," we can define "space" as a geometric system. Therefore, if we think of space as a geometric system, then the three dimensions of the physical space of our everyday experience and the dimension of time can be thought of together as four dimensions of the same geometric system, namely, of the same *abstract space*. In fact, an abstract space can have as many dimensions as are the independent variables of the model that we study in the corresponding space. As I have already mentioned, from Aristotle's perspective, space is a category of being. From a physical perspective, one can argue that, whereas time can be thought of as the set of all points through which reality passes *successively*, space can be thought of as the set of all points over which reality is extended *simultaneously*. In fact, Leibniz has highlighted the distinction between the notion of succession and the notion of co-existence, and, in his fifth letter to the English metaphysician and theologian Samuel Clarke, he wrote the following: "place is that, which is the same in different moments to different existent things, when their relations of co-existence with certain other existents, which are supposed to continue fixed from one of those moments to the other, agree entirely together," and "space is that, which results from places taken together."[166]

In his *Principles of Philosophy*, Descartes maintains that, just as abstract time is distinct from duration, since the latter is concrete, so abstract space is distinct from extension, since the latter is concrete. In particular, Cartesianism semantically equates the defining property, namely, the "essence," of material substance with three-dimensional spatial extension: "the extension in length, breadth, and depth which constitutes the space occupied by a body, is exactly the same as that which constitutes the body."[167] According to Descartes's *Principles of Philosophy*, the surface on which a body ends constitutes a set of boundary points: if this set is considered with regard to its external side vis-à-vis the given body, then it can be thought of as the extension of the given body; but, if the same set is considered with regard to its internal side vis-à-vis the given body, then it can be thought of as the place of the given body. According to the aforementioned argument, which is largely in agreement with Aristotle's physics, the boundaries of "place" and "extension" coincide with each other, but "place" and "extension" differ from each other with regard to the perspective from which their boundaries are considered.

According to Cartesianism, extension and place are data that underpin the fact that the internal and the external boundaries of every physical body coincide with each other, and, just as the internal and the external boundaries of every physical body are infinitely divisible, so extension and place are also infinitely divisible. Furthermore, in the context of Cartesianism, space, construed as the abstract setting of the particular extended bodies and of the particular places, is also infinitely divisible. Bergson maintains that the degree to which indivisible real

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[166] Leibniz, "Fifth Letter to Samuel Clarke," par. 47.
[167] Descartes, *Principles of Philosophy*, II, 10.

duration is substituted by divisible conceptual time is analogous to the experience of the divisibility of space. In general, space is an abstract generalization of extension, and, therefore, it has been studied from several perspectives.

The first scientifically rigorous perception of space was formulated by the ancient Greek geometers. Around 300 B.C., Euclid published the definitive treatment of Greek geometry and number theory in his thirteen-volume *Elements*, building on the experience and the achievements of previous Greek mathematicians: on the Pythagoreans for Books I–IV, VII, and IX, on Archytas for Book VIII, on Eudoxus for Books V, VI, and XII, and on Theaetetus for Books X and XIII. The axiomatic method used by Euclid is the prototype for the entire field of "pure mathematics," which is "pure" in the sense that we need only pure thought, no physical experiments, in order to verify that the statements are correct, that is, we need only to check the reasoning in the demonstrations. All mathematical theorems are conditional statements, namely, statements of the form

If (hypothesis) *then* (conclusion),

that is, one condition (hypothesis) implies another (conclusion). In particular, in a given mathematical system, the only statements that are called "theorems" are those statements for which a proof has been supplied. By a "proof," we mean a list of statements that is endowed with a justification for each statement, and it ends up with the conclusion desired. The following are the six types of justifications allowed for statements in proofs: (i) "by hypothesis . . ."; (ii) "by axiom . . ."; (iii) "by theorem . . ."; (iv) "by definition . . ."; (v) "by step . . ."; (vi) "by rule . . . of logic"; and a justification may involve several of the aforementioned types (see Chapter 2).

In particular, Euclid articulated:

i. *A set of definitions, such as the following:*
   - A point is that which has no part or magnitude (i.e., it does not have a concrete size).
   - A line is length without breadth.
   - The ends of a line are points. A straight line is a line that lies evenly with the points on itself.
   - A surface is that which has length and breadth only.
   - The edges of a surface are lines.
   - A plane surface is a surface that lies evenly with the straight lines on itself.

ii. *A set of fundamental rules (axioms):*
   - Things that are equal to the same thing are equal to each other.
   - If equals are added to equals, then the wholes are equal.
   - If equals are subtracted from equals, then the remainders are equal.
   - Things that coincide with each other are equal to each other.
   - The whole is greater than the part.
   - Things that are double of the same things are equal to each other.
   - Things that are halves of the same things are equal to each other.

iii. *A set of fundamental propositions (postulates):*

- Postulate 1: A straight line may be drawn from one point to any other point. Given two distinct points, there is a unique straight line that passes through them.
- Postulate 2: A terminated straight line can be produced indefinitely.
- Postulate 3: A circle can be drawn with any center and any radius.
- Postulate 4: All right angles are equal to each other.
- Postulate 5 (known as the Parallel Postulate): If a line segment intersects two straight lines forming two interior angles on the same side that sum to less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles.

According to Euclidean geometry, space is three-dimensional and isotropic (i.e., it has the same value when measured in different directions). This scientific conception of space clashes with several mythical and folk perceptions of space, according to which space is connected with a form of temporality, and it is unisotropic (for instance, the "upward" and the "forward" directions are evaluated as superior to the "downward" and the "backward" directions). The Euclidean perception of space, combined with the concept of gravity, found its fullest expression in Isaac Newton's calculus and mechanics (which are systematically studied in Chapter 2). The modern conceptions of space and time are largely dependent on Newton's arguments regarding their indisputable reality, their divisibility, and their correspondence to empirical observations.

In contradistinction to Newton's philosophical realism, Kant articulates an idealist argument, according to which both space and time are *a priori* (pre-perceptive) schemata of consciousness, through which the intellect articulates synthetic explanations of the world, of which the senses form fragmented perceptions. In general, Kant emphasizes that judgments can be distinguished into two categories: analytic judgments and synthetic judgments. In an "analytic judgment," the predicate merely elucidates what is already contained in the subject; for instance, the judgment "body is an extended thing." Therefore, such judgments are tautological, namely, they are by definition true. On the other hand, "synthetic judgments" add something to the predicate; for instance, the judgment "every material body has specific gravity." However, some synthetic judgments derive from experience, namely, they are *a posteriori*, and, therefore, they are lacking in necessity and in universality (for instance, the judgment "the cat is black"), whereas other synthetic judgments are (or rather deemed) necessary and universal (at least in a certain context), namely, they are *a priori*, which have their source in reason, namely, in the understanding itself. Logicians, mathematicians, and natural scientists wish to find synthetic *a priori* judgments in the foundations of physics and mathematics, and, as I shall explain later, this is a very arduous task (see Chapter 3).

In the fifth century A.D., the Greek philosopher Proclus criticized Euclid's parallel postulate (i.e., "if a line segment intersects two straight lines forming two interior angles on the same side that sum to less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles") by arguing that it should be struck out of the axioms of geometry altogether, because, actually, it is a theorem involving many difficulties. Proclus offered the example of a hyperbola that approaches its asymptotes as closely as one likes without ever meeting them (see Chapter 2), thus indicating that the opposite of Euclid's conclusion is at least conceivable. Consequently, according to Proclus, Euclid's parallel postulate should be treated as a theorem, which should be proved from the other axioms. In fact, the first known such attempt was made, without success, by the second-century A.D. Greek mathematician, astronomer, and geographer Claudius Ptolemy. For about seventeen centuries, some of the best mathematicians unsuccessfully tried to prove Euclid's parallel postulate, which is equivalent to the statement that, given a line l and a point P not on l, there exists a unique line through P that does not intersect l.

In 1824, the German mathematician and physicist Johann Carl Friedrich Gauss wrote to the German mathematician Franz Adolph Taurinus, who had attempted to inquire into the theory of the parallels, that "the assumption that the angle sum [of a triangle] is less than 180^{o} leads to a curious geometry, quite different from ours [the Euclidean], but thoroughly consistent, which I have developed to my entire satisfaction," and he added that, in this new geometry, he could "solve every problem . . . with the exception of the determination of a constant, which cannot be designated *a priori*," and that "the greater one takes this constant, the nearer one comes to Euclidean geometry, and when it is chosen infinitely large the two coincide."[168] However, Gauss was afraid to publish his research work in non-Euclidean geometry, because, as he wrote to another important German mathematician and physicist, Friedrich Wilhelm Bessel, on 27 January 1829, he feared "the howl from the Boeotians [an allusion to prejudiced, obtuse persons]" if he were to make public the results of his research work.[169] The first mathematician to publish an account on non-Euclidean geometry was the Russian mathematician Nikolai Ivanovich Lobachevski (1792–1856), who initially called this geometry "imaginary" and, later, "pangeometry." In the 1830s, Lobachevski openly challenged Kant's argument that space is an *a priori* schema of consciousness, and he mentioned that, in order to establish the validity of his non-Euclidean geometry, he needed the aid of experiments, such as astronomical observations, as in the case of other natural laws.[170]

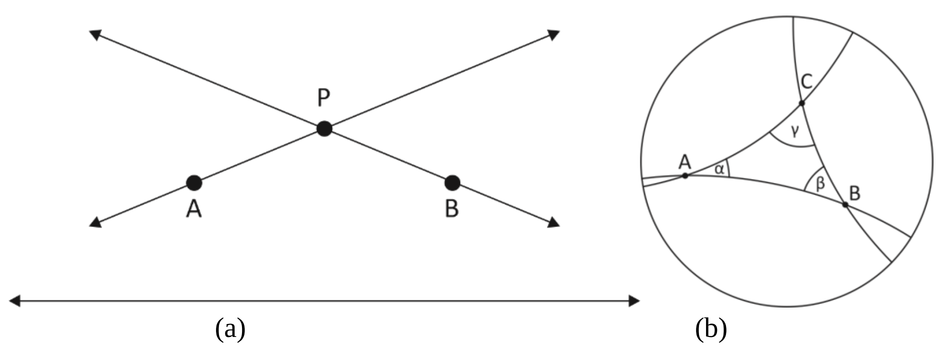


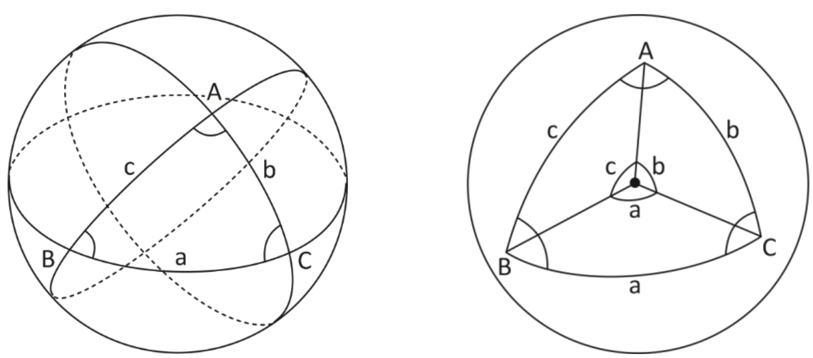
Figure 1.1. Hyperbolic Axiom and Hyperbolic Triangle.

In Gaussian–Lobachevskian geometry, known as hyperbolic geometry, Euclid's parallel postulate is replaced by the so-called "hyperbolic axiom": for any given line L and point Pnot

[168] Quoted in: Wolfe, *Introduction to Non-Euclidean Geometry*, pp. 46–47; Katz, ed., *Using History to Teach Mathematics*, p. 80.
[169] Quoted in: Katz, ed., *Using History to Teach Mathematics*, p. 80.
[170] See: Bell, *The Search for Truth*.

on L, in the plane containing both line L and point P, there exist at least two distinct lines through P that do not intersect L, as shown in Figure 1.1(a). In Euclidean geometry, the sum of the three interior angles of a triangle is always equal to π radians (i.e., 180°, a straight line), but, in hyperbolic geometry, the sum of the three interior angles of a triangle is always strictly less than π radians, as shown in Figure 1.1(b); the difference is referred to as the "defect."

The renowned German mathematician Bernhard Riemann (1826–66), who was a student of Gauss, had the most profound insight in non-Euclidean geometry (see also Chapter 2). In the 1850s, Riemann invented the concept of an abstract geometric surface that need not be embeddable in Euclidean three-dimensional space, and, on this surface, the "lines" can be interpreted as geodesics, and the intrinsic curvature of the surface can be precisely defined, as shown in Figure 1.2(a): a "geodesic" is the shortest path between two points on a curved surface (i.e., the non-Euclidean equivalent of a Euclidean straight line); like, for instance, on the surface of the Earth (e.g., airplanes, wishing to minimize the time that they spend on the air, do not follow Euclidean straight lines, but they follow shortest curves known as geodesics). In other words, Riemannian geometry is geometry on the ellipsoid or on the sphere, and, thus, it exists on surfaces that have constant positive curvature; Gaussian–Lobachevskian geometry exists on surfaces that have constant negative curvature; and Euclidean geometry exists on surfaces that have constant zero curvature. This is the way in which modern geometers construe the reality of non-Euclidean planes. Therefore, whereas hyperbolic triangles are "thin" triangles (i.e., their angle sum is strictly less than 180°, as shown in Figure 1.1(b)), Riemannian triangles (i.e., triangles on the ellipsoid or on the sphere) are "fat" triangles (i.e., their angle sum is strictly greater than 180°, as shown in Figure 1.2(b)).



(a) (b)

Figure 1.2. Riemannian Geometry on the Sphere (where "Lines" are Geodesics) and a Spherical Triangle.

For instance, Riemannian geometry was used by Albert Einstein in order to formulate the general theory of relativity: According to Newtonian mechanics, which is formulated in the context of Euclidean geometry (assuming zero curvature), the natural trajectory of a physical body that is not acted upon by any external force is a straight line. According to the general theory of relativity, gravity manifests itself as space-time curvature, and, therefore, what Newton has called natural straight-line trajectories should be generalized into curved paths known as geodesics (great circle arcs). Moreover, a significant relationship between the special theory of relativity and hyperbolic geometry was pointed out and analyzed by the

German physicist Arnold Sommerfeld in 1909 and by the Serbian-Croatian mathematician Vladimir Veričak in 1912. In particular, during a series of lectures in Munich, in 1909–10, Sommerfeld showed the manner in which hyperbolic geometry "facilitated the derivation of the formula for the addition of velocities in special relativity and made it seem natural."[171]

If Euclidean geometry is consistent, then non-Euclidean geometries, specifically Gaussian–Lobachevskian geometry and Riemannian geometry, are also consistent, since we can construct models (projections) for the latter within Euclidean geometry. Conversely, if non-Euclidean geometries, specifically Gaussian–Lobachevskian geometry and Riemannian geometry, are consistent, then Euclidean geometry is also consistent, because the lines in non-Euclidean geometries (specifically, the "horocycles" on the "horosphere" in hyperbolic space and the "geodesics" in Riemannian space) form a model of the lines on the Euclidean plane. Thus, the aforementioned geometries are equally consistent.[172] Geometry on the sphere is known as "embedded geometry," since the spherical surface is thought of as embedded in (i.e., as part of) the three-dimensional space, whereas geometry on the plane, which is a two-dimensional continuum, is known as "intrinsic geometry," since the plane representation of the world uses only the two dimensions that are intrinsic to the surface of the sphere (for instance, aviation is based on geodesics, and, hence, it uses embedded geometry, whereas two-dimensional maps of the world use intrinsic geometry).

It goes without saying that engineers, architects, and real-estate developers use Euclidean geometry, because they are concerned with ordinary measurements that are not too large. Nevertheless, the representational accuracy of Euclidean geometry is less certain when one is concerned with the measurement of larger distances. According to Albert Einstein, space and time are inseparable, and the geometry of space-time is affected by matter, so that light rays are curved by the gravitational attraction of masses. Therefore, physicists have ceased to think of space as an empty Newtonian box whose contours are unaffected by the masses of heavy bodies. Einstein has made the following statement regarding the non-Euclidean interpretation of geometry: "To this [non-Euclidean] interpretation of geometry I attach great importance, for should I not have been acquainted with it, I never would have been able to develop the theory of relativity."[173]

When the great French mathematician and philosopher Henri Poincaré (1854–1912) was asked which geometry is true, he answered as follows:

> If geometry were an experimental science, it would not be an exact science. It would be subjected to continual revision . . . The geometric axioms are therefore neither synthetic *a priori* intuitions [as Kant has contended] nor experimental facts [as Newton has assumed]. They are conventions. Our choice among all possible conventions is guided by experimental facts; but it remains free, and is only limited by the necessity of avoiding every contradiction, and thus it is that postulates may remain rigorously true even when the experimental laws which have determined their adoption are only approximate . . . One geometry cannot be more true than another: it can only be more convenient.[174]

[171] See: Gray, *Plato's Ghost*, p. 322.
[172] See: Kulczycki, *Non-Euclidean Geometry*.
[173] Quoted in: Pickover, *The Math Book*, p. 224.
[174] Poincaré, *Science and Hypothesis*, p. 50.

For instance, Euclidean geometry is the most convenient geometry for ordinary engineering, but it is not the most convenient geometry for the theory of relativity or for aviation. Moreover, computations show that the geometry of perspective spaces is non-Euclidean (for instance, in perspective spaces, collinearity, instead of parallelism, is preserved, and angles are not invariant under translation and rotation). In particular, the German-American mathematician Rudolf Karl Luneburg has argued that the most convenient geometry in order to study the "visual space," namely, the space that we perceive through vision, is hyperbolic geometry.[175]

Guided by Max Planck's research work in quantum physics, by Bernhard Riemann's research work in non-Euclidean geometry, and by Constantin Carathéodory's research work in mathematical analysis and the axiomatization of thermodynamics, Albert Einstein concluded that space and time are functions of each other, so that, by referring to space, we actually refer to a temporal correspondence of space, and vice versa. Einstein's theory of relativity implies that, in contrast to Newton's perception of a three-dimensional space, we should perceive a four-dimensional space whose fourth dimension is time, and, thus, time is part of the substance of space; and, conversely, time is underpinned by the three classical dimensions of space. The experiments on which Einstein was based in order to articulate his general theory of relativity, according to which mass and energy are, in essence, mutually transformable forms of the same reality, presupposed the existence of a four-dimensional continuum (space-time) whose curvature is determined by gravity.

Henri-Louis Bergson was deeply interested in the study of Einstein's theory of relativity, but he attempted to transcend the concept of space-time through the distinction between conceptual time and real duration. Following Bergson's thought, the French psychiatrist Eugène Minkowski (1885–1972) incorporated the four-dimensional space-time model of physics into psychoanalysis, arguing that the entire physical reality is directly related to the reality of consciousness, and that consciousness determines both the relations that govern itself and the relations that govern existence with respect to consciousness.[176] According to Eugène Minkowski, schizophrenia is a "spectrum disorder": it is characterized by the arbitrary domination of the schizoid dimension of personality (consisting of interpersonal withdrawal, solitude, and a tendency to indulge private cognitions) over syntonia (an extroverted, world-oriented sociable attitude).

Eugène Minkowski's aforementioned arguments are intimately related to the development of two alternative cosmological models: a continuous one (associated, under certain conditions, by Minkowski with "syntonia") and a discrete one (associated, under certain conditions, by Minkowski with "schizoïdia"). Every physical theory contains the following two theoretical ingredients: first, the "state" of the system, namely, a complete description of the object for which one wants to make predictions (for instance, in the context of the classical theory, which is not quantized, the state would consist of the positions and the velocities of the particles, whereas, in order to describe the position in a quantum theory, one should take the wave-functions); and, second, a "dynamical law," which is often called an "evolution equation" (namely, an equation that tells one how the state of the corresponding system changes over time). With regard to time, the distinction between the continuous cosmological model and the discrete one can be explained as follows: the evolution of a

[175] Luneburg, *Mathematical Analysis of Binocular Vision*.
[176] Minkowski, *Lived Time*.

system can be described either as a continuous trajectory in the space of system states (called the system's "phase space") or as a discrete sequence of successive states. With regard to space, the distinction between the continuous cosmological model and the discrete one can be explained as follows: the underlying space (which has $n \geq 3$ dimensions, depending on the corresponding model) can be thought of either as a continuum, where positions are defined by n real-valued coordinates, or as a tiling of discrete cells (i.e., completely covered by identical plane shapes that do not overlap with each other), or as a lattice (i.e., a partially ordered set in which every two elements have a unique least upper bound and a unique greatest lower bound), where positions are defined by n integers.

In classical mechanics, we have mechanical waves (such as water waves, sound waves, and seismic waves) and light waves, which are described in terms of a continuous cosmological model[177]: During wave motion, a particle with equilibrium position x is displaced some distance y in the direction perpendicular to the x-axis, and the value of y depends on x (namely, on the specific particle) and on the time t when we observe it, so that y is a function of x and t, symbolically: $y = f(x,t)$. Let us consider a string kept at a constant tension F_T in such a way that one end is fixed and the free end oscillates between $y = +A$ and $y = -A$ due to a mechanical device or a constant frequency. Given that the sine function of an angle θ oscillates between $+1$ and -1 and repeats every 2π radians, and that the y-position of the medium, or the wave-function, oscillates between $+A$ and $-A$ and repeats every wave-length λ, we obtain the classical wave equation for the computation of the motion of point x at time t, as the wave disturbance travels from $x = 0$ to some point x to the right of the origin in an amount of time given by x/c, where c is the wave speed:

$$y(x,t) = Asin\theta(t - \frac{x}{c}) = Asin2\pi f(t - \frac{x}{c}),$$

where c is the wave speed, $c = \frac{\lambda}{T}$, where λ denotes the wavelength, and T denotes the time period ($time\ period = \frac{1}{frequency}$).

On the other hand, in the context of quantum physics, the Danish quantum physicist Niels Bohr has articulated a model of the atom, according to which electrons in atoms move in circular orbits around a certain nucleus, and they can only orbit stably in certain fixed circular orbits at a discrete set of distances from the nucleus. These orbits, called energy shells or energy levels, are associated with definite energies. Nevertheless, as the theoretical physicist David Tong has argued, in contrast to Bohr's discrete cosmological model, integers are not inputs of the theory but outputs, and, therefore, ultimately, quantum-mechanical models are underpinned by an underlying continuous process. For instance—according to David Tong—in the hydrogen atom, the processes described by the theory mold discreteness from the corresponding system's underlying continuity.[178] David Tong argues that the building blocks of modern physical theories are not particles but fields, namely, continuous, fluid-like substances that are spread throughout the entire universe, and they ripple in ways that have very interesting geometries.[179] Far from maintaining a sharp distinction between continuous cosmological models and discrete ones, modern physics tends to highlight the underlying

[177] See: Sears, Zemansky, and Young, *College Physics*, pp. 477–93.
[178]Tong, "The Unquantum Quantum."
[179] Ibid.

structural continuity of the world. This situation is similar to the following debate in computer science: all computers are analog computers, because digital computers consist of analog parts, and, even though one may argue that those analog parts depend on discrete quantum phenomena, another may counter-argue that those discrete quantum phenomena depend on continuous fields.[180] The aforementioned scientific research results seem to corroborate Bergson's insistence on a continuous worldview, such as that of classical wave mechanics, as opposed to the French epistemologist Gaston Bachelard's insistence on a discrete worldview, inspired by earlier formulations of quantum mechanics.[181] However, the debate does not end here.

The perception of the relation of dynamized time to both the couple $(time, duration)$ and the couple $(space, extension)$, which play fundamental roles in the theory of relativity as well as in the continuous and the discrete cosmological models, can be studied on the basis of the fact that the observed, fluid-like objective continuity can be broken ("discretized") as a consequence of the experienced continuity of conscious states whenever consciousness manifests its intentionality in the external world and exerts its intentional influence on the reality of the world. When this is the case, consciousness restructures and reorganizes the world in a way that, far from being arbitrary, is determined by the capabilities of consciousness and their relation to the capabilities of the world. Thus, the fact that consciousness may interrupt the flow of particular states of the world does not signify an interruption of the operation of the world itself, but it signifies that consciousness controls and manages the operation of the world.

In view of the foregoing, the cohesive bond between temporal presence and spatial presence is manifested in history. History is a series of acts through which consciousness controls and manages regions of the world. In other words, history is a series of interventions of consciousness in the space-time continuum, which underpin the distinction between the notion of before and the notion of after. Moreover, the relationship between time and space as well as the dynamization of both time and space underpin the notion of dynamized space, which complements the notion of dynamized time.

On the basis of the foregoing ideas, space is not a three-dimensional concept, or magnitude, but a six-dimensional one, because space depends not only on the concepts of "somewhere" and "nowhere," which refer to the place of bodies, but also on the concept of "here," which is determined by consciousness. Therefore, a spatial point should be defined in a six-dimensional coordinate system, consisting of the three classical "big" dimensions of space and of three "small" spatial dimensions: besides each of the three classical "big" dimensions of space (length, width, height/depth), there is a corresponding dimension of "here" (i.e., "here-length," "here-width," "here-height/depth"), which is determined by the manner in which consciousness experiences and organizes the corresponding dimension of extension (i.e., length, width, height/depth). Because the aforementioned "here-dimensions" are conscious constructions, they can be thought of as spatial objects that are determined by consciousness. Therefore, a spatial point p is an ordered pair $(x_1, x_2, x_3, x_4, x_5, x_6)$, defined by length, width, height/depth, here-length, here-width, and here-height/depth; and total real space can (and, in fact, should) be parametrized in terms of the aforementioned six spatial dimensions (regarding the mathematical concepts involved in this analysis, see Chapter 2).

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[180] Ibid.
[181] Bachelard, *The Poetics of Space*.

This parametrized space is a manifestation of the fact that consciousness can restructure and reorganize the world whenever consciousness intentionally refers to the world, and it is inseparable from the parametrized time studied in section 1.3.1. Therefore, we obtain an eight-dimensional extension of the four-dimensional space-time. This conclusion is in agreement with the applications of "octonions" (an eight-dimensional analog of complex numbers) in string theory, special relativity, and quantum logic.[182]

Through the dynamization of space-time and through its operation in the context of dynamized space-time, consciousness expresses its dynamic reference to the world. Dynamization underpins and vindicates the intentionality of the consciousness of a being that is governed not only by a biological program but also by an evaluative one. By the term "evaluative program," I mean a program in terms of which consciousness structures/restructures itself and the world. In the context of an evaluative program, consciousness:

- determines the content of a scholarly discipline by tackling the philosophical problems (particularly, the ontological, epistemological, and ethical problems) that the corresponding object evokes;
- is committed to logic and reasonable explanations;
- is committed to history, which, as I have already argued, refers both to a methodology (i.e., historiography) and to a way of understanding reality (in terms of the interventions of consciousness in the world); and
- is committed to a system of values and to a moral criterion.

Thus, both the spatial mode of being and the temporal mode of being are enriched with the possibility of experiencing and utilizing a dynamized mode of being. Consequently, space and time are not universal conditions, or the conditions in which consciousness operates, but consciousness creates its spatio-temporal existential conditions by dynamizing space-time. In particular, consciousness integrates itself into the world in order to dynamize the world and, thus, in order to create dynamized space and dynamized time, structuring/restructuring the world according to the intentionality of consciousness. In this way, consciousness utilizes the world as a source of energy that allows a conscious being to develop into an enhanced, superior version of itself according to its own structure in the context of an organism.

Contrary to what some reductionists maintain, the fact that the integration of beings into the world underpins their organization, which, in turn, underpins the completion and the integration of their presence, does not imply that the presence of beings is passive or that the study of beings as agents is of secondary significance vis-à-vis the study of beings as organisms of the world. Being as agency is best understood as dynamic intentionality. The reality of a being as an agent is manifested through discarding those possibilities which do not comply with the given being's nature or with its program of development as well as though the identification and the embracing of those elements which are akin to the given being's nature and reinforce it. In other words, being as agency is manifested through self-affirmation. In the context of their integration into the world, beings are structurally interrelated and interdependent, and they affirm themselves within each other, but the nature of each being is not altered by this interaction.

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[182] See: Baez, "The Octonions."

The constitutive elements of being, namely, the categories of being, interact with the constitutive elements of the reality of the world, and, therefore, apart from being *modes of being*, they become *modes of enhanced being*. The intentionality of the structure of a being, namely, the character of a being's intentionality, is manifested through the dynamization of the existential conditions of that being by that being, and, therefore, consciousness restructures both the world and itself in a way that ontologically upgrades the corresponding being into an enhanced, superior version of itself, so that its organic integration does not bring about its substantial alteration. The aforementioned perspective paves a new way of thinking about the problem of the extension of the quantum formalism, which pertains to the contradiction between the formalism of quantum mechanics and the formalism of classical physics (including the general theory of relativity).

It goes without saying that the general theory of relativity and quantum mechanics are the two scientific theoretical systems that most explain the physical world: the former explains the macro-structure of the physical reality, while the latter explains the micro-structure of the physical reality. Many physicists maintain that these two theories must be integrated into a unified physical theory, while others, such as Niels Bohr and Freeman Dyson,[183] maintain that such a theoretical unification is neither needed nor plausible. In particular, according to Freeman Dyson, the classical worldview, to which the general theory of relativity belongs, underpins our knowledge of the past and of indisputable facts (such as the fact that the Earth condensed out of a cloud of dust, the fact that heat from the Earth's core creates convection currents that cause tectonic plates to move, the fact that uranium isotopes are radioactive, and—because the nuclei of radioactive elements are unstable—they are transformed into other elements in the context of a process known as radioactive decay, generally resulting in the emission of alpha or beta particles from the nucleus, etc.), whereas quantum mechanics is concerned with future possibilities and the calculation of probabilities (for instance, the probability of an atom of uranium decaying at time t_0 in the future, etc.). Everything that we can say about the physical world with certainty and everything that we can definitely say about the past of the physical world are based on the classical worldview, which is founded on two major theoretical pillars (depending on the scale of our analysis): Newtonian mechanics and the general theory of relativity. In fact, the general theory of relativity is a geometric theory of gravitation and of space-time, explaining the behavior of the universe on the large scale. On the other hand, the quantum world is not directly observable, and it can be used only for calculating probabilities (see also Chapter 2).

The distinction between being actually and being potentially, the concepts of dynamized time and of dynamized space, as well as the distinction between the modes of being and the modes of enhanced being, as I have already expounded them, provide new insights to the understanding of the fundamental difference between the general theory of relativity and quantum mechanics: the objects with which the general theory of relativity is concerned are actual beings, and, therefore, Einstein's theory of gravity and of space-time deals with actuality (being actually) in the physical world, whereas the objects with which quantum mechanics is concerned are potential beings, and, therefore, quantum mechanics is concerned with potentiality (being potentially) in the physical world. Therefore, the reality of the physical world is not *one*, since it consists of two different realms, namely, the realm of actuality and the realm of potentiality; but the reality of the physical world is *unified,* in the

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[183] Dyson, *Dreams of Earth and Sky*.

sense that there is a structural continuity between the reality described by quantum mechanics and the reality described by the general theory of relativity, since both of them are parts of the intrinsic program of development of the physical world. David Tong's argument that the building blocks of physics are fields can be better understood and appreciated in the context of the aforementioned argument about the *structural* continuity between the worldview of quantum mechanics and the worldview of the general theory of relativity.

Moreover, in accordance with the foregoing inquiries into dynamized time and dynamized space, the worldview of quantum mechanics and the worldview of the general theory of relativity do not represent the *world* (the reality of the world) more than they represent *views* (the reality of consciousness). In fact, consciousness is fundamental to the way in which "existence" is perceived in the context of modern positive science. In the context of modern positive science, a (successful) scientific theory (such as the general theory of relativity, quantum mechanics, etc.) is a mathematical framework, that is, an abstract system, from which we can derive predictions that agree with observation. Therefore, physical objects (such as time, black holes, quarks, etc.), which are said to "exist" in the physical world, are names that physicists give to mathematical structures that are parts of successful hypothetico-deductive systems, and, in physics, a hypothetico-deductive system is said to be successful if the predictions, that is, the generalizations, that derive from it agree with observations and logic (see also Chapter 2). This is the meaning of the term "existence" in the context of the natural sciences. However, as I shall explain in section 3.6, the definition of empirical significance and especially the articulation of criteria of empirical significance in the context of modern science are highly controversial and complex issues.

1.3.3. Consciousness, the World, and the Dialectic of Rational Dynamicity

My foregoing analyses and arguments point us in the direction of the awareness that reality is composed of both the presence of the world within consciousness (namely, of intellectual representations of the world) and of the reality of the world. In fact, even cosmology—namely, the branch of philosophy that is preoccupied with the study of the nature and the meaning of the world—is ultimately based on consciousness, because the latter determines which part of reality is external to and independent of consciousness. Furthermore, philosophical research, in general, is based on the elucidation of the relations between consciousness and extra-conscious reality. Hence, those natural scientists who are insensitive to ontological questions and do not consider ontology to be an integral part of their research work do not really know what they do. The renowned French mathematician and philosopher René Thom has recognized the importance of ontology for the natural sciences, in general, and he has conceded that "modern science has made the mistake of foregoing all ontology by reducing the criteria of truth to pragmatic success," and that, even though "pragmatic success is a source of pregnance and so of signification," pragmatism can produce only "an immediate, purely local meaning," and it "is hardly more than the conceptualized form of a certain return to animal nature."[184]

As I mentioned in section 1.2.2, there are two fundamental "schools" of ontology: philosophical realism and idealism. According to philosophical realism, the fact that

[184] Thom, *Semio Physics*, pp. 218–20.

experience provides images—even unrelated to each other—of a reality that seems to lie outside the dominion of consciousness implies that the reality of the world is the cause of the particular images of the world that are present within consciousness. From the realist perspective, the principle of causality points us in the direction of the claim that the autonomous existence of reality is naturally and logically necessary. In addition, as I explained in section 1.2.2, the philosophical "school" of realism is subdivided into several particular views that differ from each other, namely: the monist variety of realism, which is further subdivided into the materialist type of monism and the spiritualist type of monism; and the dualist variety of realism. It is worth pointing out that, in the context of Neoplatonism, Plato's dualism was transformed into a spiritualist theory, since the "One" is the beingly being *par excellence*, whereas matter does not really exist, but, during subsequent stages of Neoplatonism, specifically, in the context of Proclus's philosophical work, Neoplatonism assigned being to matter.

The second fundamental "school" of ontology is idealism. According to idealism, the nature of consciousness is not totally different from or opposite to the nature of extra-conscious reality. The representatives of idealism, as it was formed in the context of modern philosophy, highlight the logical principle of identity (in contradistinction to the logical principle of causality, which is highlighted by the representatives of philosophical realism), and their way of thinking can be summarized as follows: if the nature of reality were totally different from the nature of consciousness, then the human being would be unable to know reality. Thus, ultimately, idealism construes and studies the world not as something reflected in consciousness, but as an extension and a projection of consciousness outside itself and as part of consciousness. The "school" of idealism presupposes a radical form and a very high degree of individuation, and, for this reason, the fundamental arguments of modern idealism were inconceivable in the ancient and the medieval societies, which were characterized by a high degree of collectivism, and, as I shall explain shortly, the philosophical "school" of idealism was articulated in the eighteenth century under particular social-anthropological and cultural conditions.

In the Middle Ages, after the dismantling of the Western Roman Empire, knowledge was largely subjugated to belief, and philosophy was largely subjugated to religious belief systems, "faith," which required humanity to exist for the sake of a separate, divine world. The confinement of human consciousness within the realm of belief brought about the confinement of humanity within the realm of emotion, which, in turn, underpinned the development of mystical theology as an internal, apophatic, experience of the deity. However, after the ninth century A.D., and especially during the twelfth and the thirteenth centuries, scholastic theologians developed a new spiritual research program: they attempted to utilize reason and to integrate reason into theology. In the context of scholasticism, philosophy—as the "handmaiden of theology" ("ancilla theologiae"), according to Petrus Damianus's and Thomas Aquinas's thinking—undertook to achieve a great intellectual compromise between revealed truth and ontology in compliance with the teachings of the church fathers.[185] The scholastics' acquaintance with Aristotle's philosophy (a magnificent example of dualist realism), which was reinforced by the Arabs' translations of Aristotle's works, opened new philosophical horizons, and it made the Western intellectual elite capable of reconsidering the existing stock of knowledge and of recognizing the autonomy of sciences, such as

[185] See: Gracia and Noone, eds., *A Companion to Philosophy in the Middle Ages*.

mathematics, astronomy, and medicine. In particular, Thomas Aquinas managed to inquire into theological issues by articulating philosophical arguments and to create a synthesis between theology and philosophy, while recognizing and explaining the methodological differences between theology and philosophy. Thomas Aquinas argues that God knows all things in One, namely, in Himself, and, therefore, He does not need any methodologies, syllogisms, analyses, or syntheses, whereas the human being knows only under specific conditions and through specific mental processes. In *Scripta super libros Sententiarum*, III, 31, Thomas Aquinas argues that, "in the present life, it is true what the Philosopher [Aristotle] says, namely that, without images ['phantasmata'], the soul could neither develop science nor revise the things that it already knows; since images are for the intellect what sensibilia [i.e., sensory-sensuous data] are for the senses." Thus, positive, cataphatic, theology arose, and the inquiry into divine reason underpinned the legitimation of human reason.

One of the most important consequences of the legitimation of human reason was that logic gave rise to the problem of certainty, which, in turn, gave rise to heated debates about human thinking and its validity. In the fourteenth century, philosophy was preoccupied with the problem of universals: do concepts (that is, "genera" and "species") exist in nature ("subsistentia"), or are they mere abstractions ("nuda intellect")? Thus, as I explained in section 1.2.2, medieval Western philosophy was divided into two mutually competing philosophical "camps": realism and nominalism. However, in the late medieval period, the philosophy that prevailed was a form of Aristotelianism adapted to the spirit of Christianity, and the development of this philosophy was mainly due to the work of Thomas Aquinas. However, Platonism and Neoplatonism survived in a dynamic way, and they inspired subsequent philosophers, such as George Berkeley and Georg Wilhelm Friedrich Hegel, respectively, whose philosophies exerted an important influence on the development of modern philosophy.

The Renaissance led European civilization to modernity, combining elements of the medieval civilization and new findings. The term "Renaissance" was coined by the French historian Jules Michelet (1798–1874), who used it in his seminal book *Histoire de France* (*History of France*) in order to describe the historical period that roughly covers the time from 1400 to 1600 as "the discovery of the world, the discovery of man."[186] The Renaissance was guided by the idea of reviving classical Antiquity, but it attempted to do so in a creative and unique way that was underpinned by the post-medieval human being's self-confidence and humanistic spirit. Hence, the Renaissance is associated with the following events:

- the founding of Italian republics;
- the development of political science by the Italian diplomat, political philosopher, and writer Niccolò Machiavelli;
- the placing of emphasis on the principle of harmony (methodically studied by the Spanish mathematician, music theorist, and composer Bartolomé Ramos de Pareja, as well as by the Italian music theorists and composers Franchinus Gaffurius, Giovanni Spataro, and Pietro Aaron);
- the development of the idea that the universe is infinite (proposed by the Italian Dominican friar, philosopher, and mathematician Giordano Bruno);

[186] Michelet, *Histoire de France*, vol. 7.

- the invention of the mechanical movable type printing press (by the German goldsmith and printer Johannes Gutenberg);
- the manufacturing of high-quality gunpowder and firearms;
- the construction and the systematic use of the nautical (magnetic) compass;
- the achievement of important advances in machinery, mining, and chemistry (as exemplified in Georg Bauer's treatise *De Re Metallica*, published in 1556);
- the rigorous formulation of Heliocentrism (i.e., the astronomical model in which the Earth and other planets revolve around the Sun at the center of the solar system; this astronomical model was originally proposed by the ancient Greek astronomer and mathematician Aristarchus of Samos, and it was reformulated in a scientifically more rigorous way by the Polish mathematician and astronomer Nicolaus Copernicus);
- the Lutheran Reformation (despite the fact that Matin Luther's teaching about the three "Solae" has caused agitation and controversies, Martin Luther's attempt was focused on liberating Western Christians from legalism and from a feeling of guilt that was deliberately cultivated by particular authoritarian clerical elites, and, therefore, he emphasized that the primary and most important factor that determines whether one can achieve eternal life is one's psychical openness toward Christ, the belief that God comes to serve humanity[187]);
- and the mastering of perspective space and perspective drawing (Renaissance artists replaced the extra-temporal and extra-spatial symbolism of medieval painting with the subject's own logical way of seeing the world).

In the context of the Renaissance, humanism, as a cultural movement, was based on classical ancient literature in order to teach trust in education, art, and science, to defend rational thinking and its link to action, to propose the use of a scholarly advanced language in contradistinction to "common speech," and to highlight personal expression. Thus, Dante Alighieri's poetry was largely displaced by Francesco Petrarca's poetry, whose lyricism expresses the sensitivity of the emerging individual.

Moreover, it is important to mention that the Renaissance was characterized by a new individualist spirit, which highlights the human individual as a moral, independent, and autonomous being, but it was expressed in different and sometimes contradictory ways, including science, art, mysticism, modern rationalism, and modern forms of communitarianism. Thus, the history of the Renaissance is identified with artists, engineers, and inventors, such as Filippo Brunelleschi, Donatello, Sandro Botticelli, and Leonardo da Vinci, as well as with mystical philosophers, occultists, and scientists, such as Giordano Bruno, Henry C. Agrippa, Jakob Böhme, Robert Boyle, John Dee, Paracelsus, and Sir Walter Raleigh.

The natural sciences and their relationship with philosophy, the recognition of the significance of individual consciousness, the development of towns and bourgeois culture, the creation of sovereign nation-states as a result of the Treaty of Westphalia, which was signed

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[187] The fundamental principles held by Luther to be central to the doctrine of salvation are the following: "Sola scriptura" ("by Scripture alone"); "Sola fide" ("by faith alone"); and "Sola gratia" ("by grace alone"). Furthermore, in the context of the controversy between the church of Rome and Luther, Lutheranism contributed to the revival of patristic theology, because the Lutheran movement had to delve into the early church fathers in order to try to refute the arguments of the Roman Catholic church against Luther's theology. For more details, one may study the works of the eminent theologian Martin Chemnitz (1522–86).

in 1648, marking the end of European religious wars, the establishment of standards of international political and economic behavior, and the theory of natural law as a counterbalance to “monarchy by divine right” gave rise to a new historical reality that was characterized by rationalism and empiricism, and it was spiritually founded on a reflective human being, known as the “modern subject.” This is the historical period of the seventeenth century, which proclaimed consciousness to be a category distinct from the world, and it became strongly preoccupied with the theory of knowledge, namely, with the inquiry into the relations between consciousness and its objects.

As I have already mentioned, some types of dualism emphasize the significance of matter, while others emphasize the significance of spirit, namely, of those aspects of a being that are not exhausted in physics and biology. The type of dualism that prevailed in modern philosophy is Descartes’s dualism, which is based on the distinction between “extension” and “cognition.” Descartes assigned primary importance to cognition, in contrast to his philosophical opponent, the French philosopher, scientific chronicler, experimentalist, and Roman Catholic priest Pierre Gassendi, who proposed a neo-Epicurean cosmology according to which the material constitution of the world should not be considered as one of the primary aspects of the world at all. Moreover, as I have already mentioned, Descartes also opposed scholasticism. Descartes is considered to be the father of modern philosophy, because he founded his spiritualist variety of dualism on his perception of the self-reliance of reason.

In fact, Descartes’s perception of the self-reliance of reason is the common attribute of all theories that belong to the philosophical “school” of Cartesianism: First, the French Cartesian philosopher and Oratorian priest Nicolas Malebranche articulated a synthesis between Augustinian theology and Cartesianism by arguing that spirit is the substance of the absolute, and cognition is the imposition of the spirit through the perception of extended matter by consciousness. In other words, according to Malebranche, there is only one supreme Reason encompassing the ideas of all possible things, and the material world is *terra incognita* (i.e., we do not know whether it exists or not). Second, in Spinoza’s monist system, cognition (which is considered to be the most important attribute of consciousness) and extension are interconnected due to a mutual transition from one extended thing to another, since, according to Spinoza’s *Ethics*, God, who is considered to be equivalent to nature, is an “extended thing” (“res extensa”), and bodies are “modes of extension” (“modi extensionis”). Spinoza’s ontological sequence of extended things indicates a strong Neoplatonic influence on his philosophy, but, in the case of Spinoza’s philosophy, Neoplatonism is adapted to Descartes’s rationalism, and it gives rise to a deterministic and pantheistic model of the universe. Third, Leibniz gives primacy to motion over extension. In Leibniz’s philosophy, the concept of inertia plays a major role, and it is interpreted as an expression of force rather than as the absence of motion. Leibniz’s monadology, namely, his theory of the existence of real, unique, indivisible, fundamental things (“monads”) that constitute the world, is a type of spiritualist atomism: whereas “atoms” were meant to be the smallest unit of extension out of which all larger extended things are built, Leibniz’s monads are not extended, and they are “pregnant” with the future and “laden” with the past (so that, according to Leibniz, space is an illusion). With regard to the substance of the monads, Leibniz disagrees with Descartes by arguing that there exist only spiritual, and not material, monads. Nevertheless, Leibniz espouses Descartes’s rationalism by arguing that the order of the monads is determined by an *a priori* and definite harmony, and, therefore, Leibniz’s monads are governed by reason.

As I have already mentioned, materialism is a kind of monist realism that espouses Descartes's rationalism, while discarding Descartes's dualism. Descartes maintains that the entire reality is subject to a mechanistic organization, and that, by exception, the human being is composed of two separate substances: matter/body and soul/mind. In his *Treatise of Man*, Descartes argues that the pineal gland (or "conarion," or "epiphysis celebri")[188] is the principal seat of the soul and the place in which all thoughts are formed, and, in his *Passions of the Soul*, Descartes argues that, according to the mechanism of the human body, whenever the pineal gland is moved in any way by the soul/mind, or by any other cause, it drives spirits toward the pores of the brain, which, in turn, direct them to the muscles through the nervous system, and, in this way, the pineal gland makes the spirits move the limbs.[189] Descartes described the aforementioned animal spirits as "a very fine wind, or rather a very lively and pure flame,"[190] and as "a certain very fine air or wind,"[191] and he thought that the pineal gland is full of animal spirits brought to it by the surrounding arteries. However, the ancient Greek anatomist Galen had already discovered that the pineal gland is surrounded by veins rather than arteries, and the Italian anatomist Niccolò Massa had already discovered that the ventricles are filled with liquid rather than Descartes's airy spirits.

Descartes explains the distinction and the interaction between the soul/mind and the body as follows: "as regards the body in particular, we have only the notion of extension, which entails the notions of shape and motion"; "as regards the soul on its own, we have only the notion of thought, which includes the perceptions of the intellect and the inclinations of the will"; and, "as regards the soul and the body together, we have only the notion of their union, on which depends our notion of the soul's power to move the body, and the body's power to act on the soul and cause its sensations and passions."[192] Moreover, in his *Third Mediation*, Descartes argues that the soul/mind, which is more real than the body, causes the latter's motions according to the aforementioned mechanism. According to Descartes, the soul/mind is "the true substantial form of man," and, as he wrote to Denis Mesland, on 9 February 1645, the soul/mind is "substantially united" with the human body, thus implying that the reality of the human being ultimately reduces to the reality of the human soul/mind.[193] The biological fallacies and the philosophical gaps of Descartes's treatment of the mind–body problem led post-Cartesian mechanical philosophy to discard Descartes's concept of the soul/mind and to apply Descartes's concept of an animal-machine to the study of the human brain, and, thus, post-Cartesian mechanical philosophy maintains that the brain secretes cognition just as, for instance, the liver secretes bile. As I shall explain shortly, this argument is both logically and biologically mistaken, but, in the eighteenth and the nineteenth centuries, it was endorsed by naive materialists, such as the French physician and philosopher Julien Offray de La Mettrie, the German-Swiss physician and philosopher Karl Vogt, and the German physician, biologist, and philosopher Ernst Haeckel.

In contrast to the aforementioned naive varieties of materialism, Karl Marx formulated a materialist theory that is a reversed form of the Hegelian dialectic. In particular, Marx endorsed Hegel's concepts of antithesis and synthesis, and he applied them to the study of

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[188] The pineal gland is located in the epithalamus (a part of the forebrain), near the center of the brain, between the two brain hemispheres, and it produces melatonin, a serotonin-derived hormone that modulates sleep patterns.
[189] Descartes, *Oeuvres de Descartes*, 3:19–20, 11:354.
[190] Ibid, 11:129.
[191] Ibid, 11:331.
[192] Ibid, 3:665.
[193] Ibid, 3:508, 4:166.

economics, in the context of his theory of surplus value, as well as to cosmology, attempting, like Hegel, to articulate a philosophy of nature. According to Marx, surplus value is the new, additional value (antithesis) created by the labor force in excess of the workers' own labor cost (thesis), and it is appropriated by the capitalists as profit when the production is sold (synthesis). Marx's dialectic is the converse of Hegel's dialectic, in the sense that Marx interchanged the hypothesis and the conclusion in Hegel's dialectic, in accordance with Marx's thesis that people's social being determines consciousness rather than the converse. However, ultimately, Marx did not manage to articulate a complete philosophy of nature, since he did not conceive of nature as separate from society, and he argued that the explanation of space, nature, history, consciousness, and, in general, of every aspect of existence is reducible to the following three material laws: law of opposites (i.e., every existent is a combination of opposites), law of negation (i.e., every entity tends to negate itself in order to reproduce itself in higher quantity), and law of transformation (i.e., a continuous quantitative development by a particular class of entities may give rise to a qualitative change, and, therefore, to the production of a completely new form or entity).

Even though neither Marx nor Hegel managed to articulate a complete philosophy of nature, it is worth pointing out that Marx was more careful than Hegel and Comte in applying the dialectical method in the context of dialectical materialism, because, in contrast to Hegel and Comte, Marx avoided determining the precise moment of the beginning of the final stage of the development of the desired type of society. According to Marx, the beginning of the final stage of the development of the desired type of society (namely, communism) corresponds to a future moment that is approaching gradually. Thus, Marx's dialectical model is more flexible than Hegel's and Conte's dialectical models. Intimately related to the intellectual flexibility of Marx's dialectical model is his argument that human emancipation is based on critical self-consciousness, whereas, in Hegel's model, the request for human emancipation and the principle of critical self-consciousness are rather meaningless, since, in the context of Hegelianism, humanity is subjugated to the autonomous logic of historical becoming.

Marx argues: "Only when man has recognized and organized his *forces propres* as *social* forces, and consequently no longer separates power from himself in the shape of political power, only then will human emancipation have been accomplished."[194] According to Marx, theory and practice will lead to human emancipation only when consciousness and reality are brought into unison, namely, only when humanity's conception of consciousness, conceived as "theory," will be historical, and, as Marx contends, "theory becomes a material force once it has gripped the masses."[195] Furthermore, as I have already mentioned, Antonio Gramsci has highlighted the importance of culture and discourse as catalysts for creating those subjective conditions (namely, conscious contents and states) which, together with objective conditions, are necessary in order to bring about desired historical changes and to provide a new political and economic order. Gramsci's interpretation of Marx's theory of communism signifies reason in revolt, and it elaborates on the subjective requisites of social transformation, which were also recognized by Marx himself, since, in 1843, Marx wrote to Arnold Ruge:

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[194] Marx, "On the Jewish Question," p. 68.
[195] Marx, *Critique of Hegel's Philosophy of Right*, p. 137.

> The internal obstacles seem to be almost greater than external difficulties . . . The reform of consciousness consists entirely in making the world aware of its own consciousness, in arousing it from its dream of itself, in explaining its own actions to it . . . Hence, our program must be: the reform of consciousness.[196]

As regards the naive materialists' approach to the mind–body problem, it should be mentioned that modern biology implies that the functioning of a biological organism can be compared to a bio-chemical factory, but it does not legitimate the argument that consciousness is *caused* by the brain. It goes without saying that modern biologists have discovered many correlations between neural activity and conscious experiences, but a "cause-and-effect relationship" between neural activity and conscious experiences cannot be deduced solely on the basis of an observed association, or "correlation," between them. According to an old statistical adage, "correlation does not imply causation." Hence, for instance, in the case of anxiety disorders, symptomatic pharmacotherapy (e.g., through the administration of: neuroleptic or antipsychotic drugs, such as derivatives of phenothiazine; narcotic–hypnotic drugs, such as phenobarbital; and tranquilizers and antidepressants, such as benzodiazepines) proves that there exist important neural correlates of consciousness (i.e., a biological substrate of conscious experiences), but the explanation of the phenomenon of consciousness cannot be exhausted in or entirely reduced to the explanation of its biological substrate, for which reason, for instance, the treatment of anxiety disorders cannot be constrained to symptomatic pharmacotherapy, but it calls for other types of treatment, too, such as psychoanalytic therapy. Indeed, if consciousness were only a consequence of bio-chemical processes in the brain, or the central nervous system, then the academic discipline of psychology should be totally abolished and totally replaced by bio-chemistry.

Furthermore, apart from the "down-top" pathways that feed forward data from the sense organs up to the brain, there are also "top-down" response pathways, which are intimately related to the intentionality of consciousness, and, in several cases, their effects (positive or negative) are more important than the effects of the "down-top" pathways. Far from passively responding to or reflecting sensory-sensuous data, consciousness creates abstract models of them, which are often structurally intertwined with the world, and, in fact, the natural sciences are based on such models. Thus, scientific theories are not occurrences seen and the associations recorded, but they are explanations of them. The Austrian physicist and philosopher Ludwig Boltzmann (1844–1906) has argued that a theory is not a reproduction of external reality, but is a picture, mentally formed, of a bounded realm or domain of activity, namely, it is an intellectual depiction of the organization of a domain and of the connections between its parts.[197]

In the beginning of the twentieth century, Gestalt psychology highlighted the active role of consciousness in perception. Gestalt psychology was founded by Max Wertheimer (1880–1943), an Austro-Hungarian psychologist.[198] Wertheimer noted that we perceive motion where there is nothing more than a rapid sequence of individual sensory events. This argument is based on observations that he made with his stroboscope at the Frankfurt train station and on additional observations that he made in his laboratory when he experimented with lights flashing in rapid succession (like the Christmas lights that appear to course around

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[196] Marx and Ruge, *Deutsch–Französische Jahrbücher*, Letters to Ruge, September 1843.
[197] Boltzmann, "Theories as Representations."
[198] See: Asch, "Max Wertheimer's Contribution to Modern Psychology."

the tree, or the fancy neon signs in Las Vegas that seem to move). Wertheimer called this effect "apparent motion," and it is actually the basic principle of motion pictures.

According to Wertheimer, apparent motion proves that people don't respond to isolated segments of sensation but to the whole (*Gestalt*) of the situation. Gestalt psychologists have shown, through various experiments, that consciousness does not respond to isolated segments of sensation but to the whole (*Gestalt*) of the situation, and they have argued that, in perception, there are many organizing principles called "Gestalt laws."[199] Examples of such laws are the following: the law of closure: if something is missing in an otherwise complete figure, we shall tend to add it (e.g., a triangle with a small part of its edge missing, will still be seen as a triangle, and also we shall "close" the gap); the law of similarity: we shall tend to group similar items together, to see them as forming a whole (*Gestalt*), within a larger form; the law of proximity: things that are close together are seen as belonging together.[200] Thus, consciousness perceives and thinks in nonlinear ways, and it actively influences perception. In addition, Gestalt psychology has shown that, in perception, the method of trial and error coexists with psychological intuition (which plays a protagonist role in Bergson's philosophy).

In 2001, Harold Koenig (professor of Psychiatry and Behavioral Sciences at Duke University), Michael McCullough (professor of Psychology at the University of Miami), and David Larson (adjunct professor of Psychiatry and Behavioral Sciences at Duke University) published the *Handbook of Religion and Health*, in which they inquire into the interplay between religion—perceived in a more encompassing manner including "spirituality"—and mental health, and, in particular, they argue as follows:

> . . . religious beliefs may prevent sufferers from complying with medical treatments by encouraging them to rely on faith rather than on traditional medical care; they may therefore refuse potentially life-saving blood transfusions, prenatal care, childhood vaccinations, or other standard treatments or prevention measures.[201]

They likewise maintain that the mental patients who "present with bizarre and distorted religious ideas" or who use "religious beliefs and practices" in "pathological ways" suffer negatively on account of their religious beliefs or practices.[202] Nevertheless, as regards the positive effects of spirituality on general health, they concede that "it is clear that much of the general public and a growing number of health professionals believe that religion and good health are somehow related."[203] Even though "some religious attitudes are associated with worse health outcomes,"[204] and allowing that "religious and health professionals may debate the benefits or risk to health that religion conveys," it still remains that:

> . . . people with serious health problems, people fighting against life-threatening or life-disabling diseases, tell us the most about how religion relates to health. Even if no relationship existed, religion would be relevant to health care if patients perceived that it improved their

[199] See: Köhler, *Gestalt Psychology*.
[200] Ibid.
[201] Koenig, McCullough, and Larson, *Handbook of Religion and Health*, p. 77.
[202] Ibid.
[203] Ibid, p. 59.
[204] Ibid, p. 94.

> coping with health problems and therefore wished health care providers to address spiritual issues as part of their medical or psychiatric care.[205]

It is important to recall that Koenig, McCullough, and Larson subsume the concept of spirituality under the term "religion," and, therefore, they use the term "religion" in a more encompassing way, but they have clarified the difference between "religion" and "spirituality" as follows: religion is a system of beliefs, practices, rituals, and symbols that aims to facilitate a person's attempt to access the sacred and to develop and further one's relationship with and moral responsibility toward the others in the context of a community; spirituality is one's personal quest for, or encounter with, the sacred and ultimate existential questions, and it "may (or may not) lead to or arise from the development of religious rituals and the formation of community."[206] Harald Walach (professor of Research Methodology in Complementary Medicine at Viadrina European University Frankfurt and former researcher in Clinical Psychology at the University of Northampton), Niko Kohls (Human Science Center, Ludwig-University-Munich), Nikolaus von Stillfried (Institute for Environmental Medicine and Hospital Epidemiology, Freiburg), Thilo Hinterberger (Institute for Environmental Medicine and Hospital Epidemiology, Freiburg), and Stefan Schmidt (Institute for Environmental Medicine and Hospital Epidemiology, Freiburg), in their scientific research paper entitled "Spirituality," argue that, apart from "transcendence," which is a "common denominator of different concepts and definitions of spirituality," spirituality can be interpreted as "alignment of the individual with the whole," and "the Whole would be a transcendent reality as well."[207] Moreover, in the aforementioned research paper, Walach et al. have clarified the meaning of spirituality as follows:

> Spirituality is the experiential realization of a transcendent reality. This is variably called meaning or purpose, sometimes it is called a relationship with a transcendent goal or reality reaching beyond the ego . . . spirituality has at least two core aspects: It refers to a relationship with a reality that reaches beyond the ego. The second aspect is about its experiential manifestation, i.e., a holistic type of knowing that includes cognition, affect, and motivation.[208]

Using the term "religion" in a way that encompasses "spirituality," Koenig, McCullough, and Larson argue that "religion provides a powerful source of comfort and hope for many persons with chronic mental illness,"[209] and they add that "the primary influence of Judeo-Christian beliefs and practices on schizophrenia and other psychotic disorders is in providing comfort, hope, and a supportive community to individuals who must cope with their emotionally devastating, largely biological illnesses."[210] Hence, these are characteristic cases of what I earlier called "top-down" response pathways.

From the perspective of structuralist philosophy, and according to the dialectic of rational dynamicity, which I expound and support in this book, religion as spirituality (in contradistinction to religion as ritualism, superstition, and spiritual despotism) and the idea of

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[205] Ibid, 78.
[206] Ibid, p. 18.
[207] Walach, Kohls, von Stillfried, Hinterberger, and Schmidt, "Spirituality," p. 279.
[208] Ibid, p. 297.
[209] Koenig, McCullough, and Larson, *Handbook of Religion and Health*, p. 165.
[210] Ibid, p. 163.

the absolute (or the deity) itself elevate and orient consciousness to the vision and the ideal of the ontological perfection of humanity, and they help one to envisage the human being as a god-in-the-making instead of being confined to a bounded historical horizon. This empowering perception of spirituality was highlighted by the young Karl Marx in his poems and theatrical plays.[211] Moreover, the German philosopher Ernst Bloch (1885–1977) has pointedly observed that the human being (in spite of the gloomier littérateurs and in spite of particular psychoanalysts' tendency to emphasize humanity's inner litter) is a hoping animal.[212] At the most fundamental level, humanity expresses its urge to hope by being unsatisfied and by wishing to envisage an alternative (specifically, better) state of the world. At the highest level, humanity's urge to hope is expressed through a strategic existential vision, or a philosophico-theological (as opposed to a political) utopia: an ideal type of perfection that human beings seek or try to realize and guides human action like an intellectual sun. According to Bloch, "utopia" is the bond and the interplay between that which does not exist yet and that which already exists. The guiding idea of Bloch's philosophy is that the ever present now, conceived as the one and only creative event, is pregnant with that which is about to be. Thus, the now as "noch-nicht-Sein" ("not-yet-Being") is a "that" ("ein Das") on the way toward its "what" and, hence, the location of newness.

Not only has modern natural science *not* proved that physical phenomena, such as the brain, *cause* consciousness, but also it lacks a physical explanation of the choices made by quantum systems. In quantum mechanics, we cannot predict the outcome of measurements with certainty, we can only calculate probabilities, and, even though quantum physicists have gradually managed to reduce the significance of observers in quantum mechanics, quantum mechanics cannot explain the aforementioned probabilities, namely, it cannot explain what drives a quantum system toward one way or another at each junction (quantum transition). This strange quantum image of the world implies that, as I have already argued, there is an underlying structure, namely, the ontological program of the world, which is structurally united with consciousness.

In view of the foregoing, regarding the mind–body problem, one can reasonably argue that modern biology and, in general, modern natural science, far from confirming naïve materialist arguments, corroborate Aristotle's hylomorphism and Thomas Aquinas's Aristotelianism. Aristotle argues as follows:

> It is not necessary to ask whether soul and body are one, just as it is not necessary to ask whether the wax and its shape are one, nor generally whether the matter of each thing and that of which it is the matter are one. For even if one and being are spoken of in several ways, what is properly so spoken of is the actuality.[213]

Similarly, Thomas Aquinas argues that the mind and the body are united with each other, like the integral union of the seal and the wax, and, in the spirit of Christianity, he adds that, with regard to the body, the human being is mortal, while, with regard to the mind, the human being is immortal, in spite of the unity between the mind and the body.[214] According to

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[211] See: Johnston, "Karl Marx's Verse of 1836–1837 as a Philosophical Foreshadowing of His Early Philosophy."
[212] Bloch, *The Principle of Hope*.
[213] Aristotle, *De Anima*, 412b6–9.
[214] See: Kretzmann and Stump, eds., *The Cambridge Companion to Aquinas*.

Aquinas, the human soul/mind is the form of the human being, which is a hylomorphic (matter–form) composite. Aquinas specifies that form is the intrinsic constitutive element of the species, matter is the "stuff" of which creation is made, and substantial form is a type of form that is specifically ascribed to the human soul/mind, and informs *materia prima* (prime matter), so that any other form that may be ascribed to a being or thing is posterior to substantial form, and informs an already constituted substance, namely, it is an accidental form.

A very good approximation of hylomorphism in relation to the explanation of the relationship between consciousness and the brain has been formulated by the American philosopher James Porter Moreland as follows[215]: A CD (compact disc) does not actually contain music, but it contains only pits (recessed areas on a CD where data are stored). Moreover, a CD does not "create" music. But, if the configurations on a CD are placed into the adequate retrieval system, then music can be played. If the CD is damaged, then the CD player cannot properly read the configurations, and, therefore, it cannot play the music. By analogy, consciousness can read pathways in the brain and, thus, access and process stored information, and, if these pathways are changed or damaged, then the underlying information (received and stored by the brain) will not be available or could be read in an altered way. Similarly, the English philosopher, theologian, and Anglican priest Keith Ward has explained the interplay between consciousness and the brain by arguing that consciousness reads or interprets the configuration of neurons, which store information that the brain receives from the environment.[216]

Let us now turn our attention to the second major "school" of ontology, namely, idealism. As I have already mentioned, idealism is a creation of the eighteenth century. The realist philosopher Descartes proved to be an involuntary founder of idealism, because he started his philosophical inquiries with a "methodological doubt" regarding the ability of consciousness to predicate the correctness or the falsehood of its elements of knowledge, he discarded such elements of knowledge on the grounds that they are uncertain, and, therefore, he concluded that consciousness is an ontologically sufficient foundation of truth. The aforementioned Cartesian reasoning has been summarized in the statement: "Cogito ergo sum" ("I think, therefore I am").[217]

The English empiricist philosopher John Locke (1632–1704) was the second (after Descartes) involuntary founder of idealism, because he argued that ideas derive from sensation, which supplies the mind with sensible qualities, and from reflection, which supplies the mind with ideas of its own functions (perceiving, thinking, believing, doubting, reasoning, knowing, willing).[218] From Locke's perspective, the mind, in its first state, is a *tabula rasa* ("white paper"), all our knowledge is founded on and, hence, derives from experience, and the primary capacity of consciousness is the intellect's ability to receive the impressions made on it, either through sensation (by external objects) or through reflection (when it reflects on its own functions). The ideas thus received do not constitute knowledge *per se*, but they are elements of knowledge ("simple ideas"), which consciousness can repeat, compare, and combine in different ways, and, therefore, it can make at pleasure new "complex ideas." According to Locke, no understanding can invent or frame a new element of

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[215] Moreland, "In Defense of a Thomistic-like Dualism," pp. 107–08.
[216] Ward, *More than Matter?*, p. 121.
[217] Descartes, *Oeuvres de Descartes*, 7:25, 7:140, and 8a:7.
[218] See: Chappell, ed., *The Cambridge Companion to Locke.*

knowledge ("simple idea"). As a result, Locke's theory of knowledge replaced certainty with the uncertainty of sensation, and, furthermore, it replaced "substantial truth" with "conventional truth." However, Locke failed to bear in mind that, even if ideas are not innate with regard to their content or substance, ideas may be innate with regard to their structure.

Modern idealism has different forms, namely: first, solipsism, which maintains that the only reality consists of one's own intellections; second, the more moderate thesis that the sensory-sensuous world is a degraded sensory-sensuous appearance of an experienced conscious state that is the only reality; and, third, immaterialism, which was put forward by the Irish philosopher George Berkeley (1685–1753), who became Bishop of Cloyne in 1734, and he opposed philosophical realism (especially Newton's natural philosophy) by arguing—under the influence of Neoplatonism—that matter is not real and by replacing dualist realism (which is based on two principles: the reality of the world and the reality of consciousness) with the thesis that there exist two principles: one that cognizes, namely, the spirit, and another that creates, namely, the absolute.[219]

According to Berkeley, the world exists only because it fills human consciousness. The starting point of Berkeley's philosophy is Locke's empiricism, specifically, the argument that perception is a prerequisite for existence and a demonstration of the perceived things and of the consciousness that perceives them. However, by trying to reinforce the consistency of immaterialism, Berkeley ultimately reached a conclusion that is similar to philosophical realism, namely, he concluded that the existence of things consists in *being* perceived, and this conclusion is conceptually very close to the realist argument that things exist *because* they are perceived.

Apart from Berkeley, another major modern scholar who was strongly influenced by Locke's philosophy was the Scottish philosopher, historian, and economist David Hume (1711–76). Hume's variety of empiricism consists in a theory of phenomena that advocates the reality of impressions and rejects every stable substance.[220]By "impressions," Hume means our more lively perceptions when we see, or hear, or feel, or love, or hate, or desire, etc., namely, all our sensations, passions, and emotions as they originally appear in consciousness, and, by "ideas," or "thoughts," he means representations, or copies, of such impressions, namely, the faded perceptions of which we are conscious when we recall an impression or reflect on it. Hume maintains that we cannot assert the reality either of matter, because we have only representations of matter, or of the soul, because we experience only actions.

However, Hume failed to take account of the following three fundamental mistakes of skepticism:

i. Skepticism maintains that we can know very few of the elements of each object of consciousness, and that, because we ignore most of them, we substantially ignore the corresponding object of consciousness itself. This skeptical argument can be refuted by counter-arguing that limited knowledge is not equivalent to invalid knowledge, and that the knowledge of a few significant and actionable attributes of an object of consciousness is equivalent to valid knowledge.

[219] See: Winkler, ed., *The Cambridge Companion to Berkeley*.
[220] See: Norton and Taylor, eds., *The Cambridge Companion to Hume*.

ii. Skepticism maintains that our cognition tends to operate according to circular reasoning (i.e., one begins with what one tries to end with), and that, therefore, we should discredit every logical certainty, and we should endlessly question knowledge. This skeptical argument can be refuted by counter-arguing that the logical fallacy of circular reasoning is less serious and less detrimental than the skeptics' attempt to show that it is *reasonable* to negate the validity of reason, which is an obvious contradiction.
iii. Skepticism maintains that the senses and reason can provide us with false impressions. Indeed, our senses are imperfect (and, thus, fallible), and our intellect may confuse dreams with reality. However, this skeptical argument can be refuted by counter-arguing that the fallibility of sensible knowledge and dreams can be examined and controlled by reason, and that, at least within certain conceptual communities, there exist self-evident truths and epistemological principles, such as the principle of contradiction.

Immanuel Kant has written about Hume: "I freely admit that the remembrance of David Hume was the very thing that many years ago first interrupted my dogmatic slumber."[221] Under Hume's influence, Kant rejected metaphysics as the knowledge of the supersensuous, but, in contrast to Hume, he accepted metaphysics as the knowledge of knowledge, and, therefore, Kant's philosophy, which he called "critical," fluctuates between realism and idealism. In particular, Kant recognizes the existence of a real world and of a "noumenon," namely, a thing-in-itself, and he argues that a "phenomenon" is a faded, dissolved declaration of the corresponding noumenon, the manner in which the corresponding noumenon appears to an observer.[222] In his *Critique of Pure Reason* (first edition), A34 and A249, Kant defines appearances as the undetermined objects of empirical intuitions, and he defines noumena as follows:

> Appearances, to the extent that as objects they are thought in accordance with the unity of the categories, are called *phenomena*. If, however, I suppose there to be things that are merely objects of the understanding and that, nevertheless, can be given to an intuition, although not to sensible intuition (as *coram intuiti intellectuali*), then such things would be called *noumena* (*intelligibilia*).[223]

Furthermore, he argues that consciousness cannot know the substance of the real world, and that the only thing that consciousness can achieve is to organize mutually unconnected segments of the real world that exist within consciousness into systems (structured sets) with the assistance of twelve mental categories with which cognition is *a priori* equipped as well as with the assistance of two pre-perceptive schemata, namely, those of space and time.[224] Kant distinguishes twelve mental categories (general concepts of the understanding), divided into four sets of three as follows: (i) quantity: unity, plurality, totality; (ii) quality: reality, negation, limitation; (iii) relation: inherence and subsistence (substance and accident),

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[221] Kant, *Prolegomena*, p. 10.
[222] The term "noumenon" (plural: "noumena") derives from Greek, and Kant used it in order to refer to something that can be the object only of a purely logical, non-sensuous intuition.
[223] Kant, *Critique of Pure Reason*, p. 347.
[224] See: Guyer, ed., *The Cambridge Companion to Kant*.

causality and dependence (cause and effect), community (reciprocity); (iv) modality: possibility, existence, necessity.[225] Kant's refusal to accept the knowledge of the noumena is epistemological (signifying only *formal* idealism), but not ontological, and, in his *Prolegomena to Any Future Metaphysics that Could Come Forth as Science*, Kant maintains that transcendent reality is indisputable. In particular, in his *Prolegomena*, which is a summary of his *Critique*, Kant explains the following: *first, the difference between his critical philosophy and idealism:* idealism, Kant argues, is founded on the thesis that all cognition through the senses and experience is illusion, and that valid knowledge consists only of the ideas of pure understanding and reason, whereas Kant's *Critique* consistently maintains that bodies exist in space, and consciousness has immediate, non-inferential, knowledge of them; *second, his formal idealism:* Kant's *Critique* is characterized by formal idealism, in the sense that it maintains that the *form* of objects is due to consciousness, but not their *matter*.

In contrast to Kant's formal idealism, which is inextricably linked to Kant's transcendentalism (realism) in the context of his attempt to provide a general way of understanding the overall evolutionary course of the natural world and humanity, romantic idealism, marking a significant philosophical departure from the European Enlightenment, proposes a way of understanding history as the self-disclosure of the spirit in the temporal manifestation of particular egos, whether human individuals or national units. The two first pioneers of romantic idealism were the German philosophers Johann Gottlieb Fichte (1762–1814) and Friedrich Wilhelm Joseph von Schelling (1775–1854).[226] A few years after the publication of Kant's *Critique of Pure Reason* (first edition: 1781, second edition: 1787), Fichte published the *Foundations of the Science of Knowledge* (1794). Fichte recognizes only an "ego" that alone creates the object of its representations, and that becomes self-conscious by opposing everything alien to it. Thus, the elements that are alien to the ego become the means through which the ego affirms itself. By highlighting the importance of the ego, Fichte discards the concept of the noumenon. Following a reasoning that is similar to that of Fichte, Schelling argues that both the "ego" (namely, consciousness) and the things that are alien to it are functions of a unique reality that Schelling calls the "absolute."

The most famous representative of German romantic idealism is Hegel, who systematized the philosophies of Fichte and Schelling, and he focused on the active spiritual reality, which he called the "idea." In fact, Hegel replaced Kant's conception of the noumenon with his conception of the idea, to which, as he contends, everything is reducible. As I explained in section 1.2.2, Hegel's idealism is totally historicized, and, in his philosophy, he described cognition as something that exists in-itself and for-itself, and that, like a distinct subject, reflects on itself. According to Hegel, the "real" is the "rational," in the sense that the core of the rational is the absolute idea, which is embodied by the nation-state, philosophy, art, and religion, and the absolute ("objective") spirit tends to the absolute idea. In particular, Hegel identifies three "sectors" of society, namely, the family, civil society, and the state, and he argues that, in the context of social and political philosophy, family (symbolizing "unity," which absorbs the particularity of each individual) represents the "thesis," civil society (symbolizing the "singularity" of its atomistic subjects and their "particularity" as members of families) represents the "antithesis" (in the context of which, the "becoming citizen"

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[225] Ibid.

[226] See: Brusslan and Norman, *Brill's Companion to German Romantic Philosophy*; Thorslev, "German Romantic Idealism."

gradually recognizes civil society as one's broader family), and the state (symbolizing the optimum form of universality) represents the "synthesis"; and, according to Hegel, the immediacy of family life finds greater fulfillment as part of civil society, and the nation-state is the largest extension of the family, and it unites all its particular families and all the particular relations that are established in civil society into an organic whole (hence, Hegel's communitarian, state-centered idealism).[227] In Hegel's philosophical system, history is the "laborious journey" of the absolute spirit toward the absolute idea in a dialectical way, which generalizes Fichte's and Schelling's teachings about the development of the ego (each "thesis" gives rise to its "antithesis," and both of them are negated and preserved in a subsequent, superior stage, called "synthesis").

In the context of Hegelianism, Fichte's and Schelling's concept of the ego is counterbalanced by Hegel's concept of the absolute ("objective") spirit. In this way, Hegelianism leads to the conclusion that, even though the world constitutes a historical creation of humanity, the world has obtained its autonomy vis-à-vis humanity. In the nineteenth century, Hegel's arguments exerted a significant influence on the philosophical "school" of spiritualism, represented by Antonio Rosmini-Serbati and Vincenzo Gioberti in Italy as well as by Charles-Bernard Renouvier, Octave Hamelin, Léon Brunschvicg, and René Le Senne in France. On the other hand, Bergson managed to transcend the antithesis between realism and idealism, but his work gave rise to a new antithesis, namely, that between intuition and cognition, which, in turn, can be overcome in the context of structuralism. Structuralism, which has assimilated Hegelianism in a creative way, corroborates Bachelard's argument that there is a dynamic continuity between cognizing consciousness and the object of cognition.[228]

Furthermore, as I have already pointed out, realism and idealism are issues of great concern in the foundations of physics (i.e., the area of physics that deals with those natural laws which do not derive from any underlying laws, and it consists of General Relativity, which deals with the behavior of space and time, and of the Standard Model of Particle Physics, which deals with the smallest constituents of matter and the manners in which they interact) and in biology (especially, in neuroscience). The natural sciences are based on mathematics. In order to understand the structure of mathematics, we have to realize that mathematics is an abstract field, and, for this reason, it is very powerful, since it can manifest itself in many different problems. In fact, mathematics is based on a peculiar synthesis between imagination, perception, and scientific rigor, or, equivalently, between intuition, experience, and logic. This awareness is the starting point and the original underpinning of my thesis that mathematics and philosophy are homomorphic. It is due to this homomorphism that mathematics has furnished philosophy with a model of a certain kind of knowledge as well as with several intellectually challenging and significant philosophical problems, and philosophy, in turn, underpins the development of mathematics not only in terms of ontology and epistemology but also in terms of moral values and aesthetic.

Mathematics is done by consciousness. Mathematics provides a model of knowledge of a particular kind, and, in fact, philosophers have highlighted the particular nature of mathematical knowledge and have argued that all knowledge could possibly aspire to the particular nature of mathematical knowledge. Unlike other kinds of knowledge, mathematical

[227] For a synoptic outline and explanation of Hegel's political theory, see: Brown, *International Relations Theory*.
[228] Bachelard, *The Dialectic of Duration*; and Bachelard, *The Poetics of Space*.

knowledge is characterized by rigor, because mathematics is constituted as a logical system, in the sense that mathematical concepts are subject to the relations R_0 and the judgments S_0 of formal logic $T^0(R_0, S_0)$, which I shall study in chapters 2 and 3, and which hold for every kind of concepts. In other words, inherent in the relations and the judgments of mathematical theories is the system $T^0(R_0, S_0)$ of formal logic. The addition of new relations R_1 and new judgments S_1 to $T^0(R_0, S_0)$ determines, under certain conditions, a set M of mathematical concepts, namely, objects that belong to the system $T^0(M, R_0, S_0, R_1, S_1)$, which is formed by R_0, S_0, R_1, S_1 and their corollaries.

As I shall explain in chapters 2 and 3, the basic concepts of the relations R_0 of formal logic, namely, "is a part of" or "belongs to" (∈), "if, then" ($a \rightarrow b$), "a, not a" ($a, \neg a$), "or" (∨), "and" (∧ or &), and the relations that are expressed through the judgments S_0 of formal logic determine the axiomatic system $T^0(R_0, S_0)$ of formal logic. The set of all the new concepts, the new definitions, and the new true propositions that are produced from $T^0(R_0, S_0)$ constitute what is called the theory of formal logic, or the structure of formal logic, or simply formal logic, and it is denoted by $\tilde{T}^0(R_0, S_0)$. In mathematics, by the term "axiomatization," we mean the creation of theories that comply with $T^0(R_0, S_0)$.

Suppose that a mathematician is given an object A whose properties can be described in terms of formal logic, but the substance of this object has not yet been studied. Then this mathematician can study A with the help of$T^0(R_0, S_0)$, that is, through logical abstraction. In this case, $\tilde{T}^0(R_0, S_0)$ is the well-defined initial information. Using this well-defined initial information, namely, $\tilde{T}^0(R_0, S_0)$, one can study the object A and produce a scientific result $\tilde{A}$. By analogy, a mathematical theory (or structure) can be constituted as follows: Let ω_0 denote the empirical, sensory-sensuous world, and suppose that, in ω_0, we have to study "space" as an object $\omega_0^1 = \omega_0^1(\alpha, \beta)$, where α denotes the property of being a "measurable magnitude," and β denotes the property of having a "shape." Then, by using the structure $\tilde{T}^0(R_0, S_0)$, we can study $\omega_0^1(\alpha, \beta)$ and, ultimately, produce the axiomatic system $\omega_1^1 = T^1(M, R, S)$ of Euclidean geometry, which includes the set M of all the objects for which R and S hold. In fact, within the structure $\tilde{T}^1(M, R, S)$, we can look for theorems and algorithms, and, through a system of algorithms, we can deduce new results.

With the help of the logic of the formal system $\tilde{T}^1(M, R, S)$, mathematical consciousness created Euclidean geometry $T^1(M, R, S)$ from the object $\omega_0^1(\alpha, \beta) \subset \omega_0$, and, gradually, over the course of the history of mathematics, mathematical consciousness created new mathematical structures from other objects $\omega_0^i(\alpha, \beta) \subset \omega_0, i = 2,3, \ldots$ For instance, the mathematical theory that is deduced from the object ω_0^2 and whose basic concepts are those of a function and of a real number, is known as real analysis, and it can be denoted by $\omega_0^{2.1}$. If we create the theory of complex numbers, then we can also create the corresponding structure $\omega_0^{2.2}$, which is known as complex analysis, etc. All these theories will be studied in Chapter 2.

The incorporation of logical relations R_0 and logical judgments S_0 into mathematics underpins the creation of mathematical concepts. The mathematical model of the property of being a "measurable magnitude" gives rise to the concepts of a subset and of the arithmetic operations, which correspond to the relations R_0 of formal logic. Given the fact that there is a homomorphism (structural similarity) between R_0 and the basic concepts of number theory, as I shall explain in chapters 2 and 3, the basic concepts of logic become mathematical concepts, thus giving rise to mathematical logic.

In view of the foregoing, the model of knowledge that is provided by mathematics has the following characteristics: (i) certainty (in the sense that, if something is true and known in mathematics, then it is undoubted), (ii) incorrigibility (in the sense that the development of mathematical knowledge is internally consistent), (iii) eternity (in the sense that mathematical knowledge is not subject to time), and (iv) necessity (in the sense that mathematical truths are not contingently true but necessarily true). Being aware of these attributes of mathematical knowledge, Plato had the phrase "Let no one ignorant of geometry enter" engraved at the door of his Academy. In the context of Plato's philosophy, geometry is concerned with the understanding of the reason ("logos") of the world. Thus, *Plato*, in his *Republic*, 527c, argues that "geometry is the knowledge of the eternally existent," and that, therefore, geometry "would tend to draw the soul to truth, and create the spirit of philosophy, and would be productive of a philosophical attitude of mind."

By the term "mathematical model," we mean the description of an object or a phenomenon by means of mathematics. Let C denote the set of all basic conceptual objects, R the set of all basic conceptual relations, and A the set of the axioms of a structure. Then the corresponding structure is denoted by $\mathcal{S}(C, R, A)$. A segment of a structure is a set of concepts, definitions, and judgments of the given structure that it satisfies the axioms of the given structure as well as some additional conditions, and it is denoted by $\bar{\mathcal{S}}(\bar{C}, \bar{R}, \bar{A})$. Suppose that a phenomenon of the sensory-sensuous world has been described by a structure $\mathcal{S}(C, R, A)$ or by a segment of this structure. Both the phenomenon and its mathematical model can be regarded as two homomorphic models, since the original phenomenon is initially modeled by our perception of it, or, more precisely, by the initial reference of our consciousness to it, and its mathematical model is $\mathcal{S}(C, R, A)$ or a segment of $\mathcal{S}(C, R, A)$. The creation of homomorphisms between mathematics and other sciences or human activities, namely, the creation of mathematical models, is called mathematical modelling. Thus, mathematical modelling consists of two stages: (i) the formulation of the mathematical model of the object that one studies, that is, the transformation of the given problem into a mathematical one, and (ii) the solution of the corresponding mathematical problem, namely, the processing of the information that is contained in the given problem by means of mathematics and mathematical informatics. Consequently, we realize that mathematical modelling is a method of studying every particular science, including mathematics itself, and this fact leads to philosophy's aspiration to universality and philosophy's intention to evaluate the object of its inquiry and the inquiry into its object according to a general criterion. From this perspective, one can argue that mathematics is a way of doing philosophy through mathematical concepts, mathematical methods, and mathematical structures. In addition, the American mathematician Jordan Ellenberg has pointed out that, unlike computing machines, a mathematician's work is not just to compute formulas, but to develop and implement a way of thinking that involves the evaluation of research questions, the formulation of the right research questions, and the investigation of the assumptions that underlie research work.[229]

[229] Ellenberg, *How Not to Be Wrong*. In his aforementioned book, Ellenberg writes that, during World War II, a group of U.S. military officers visited the Statistical Research Group (a classified scientific research program in the context of which mathematicians and statisticians were working on problems related to the conduct of war), and they stated that they had noticed that the U.S. military aircrafts that were returning from flying missions over Germany were riddled with bullet holes, and that the damage was not uniformly distributed across the aircraft, since there were more bullet holes in the fuselage and less in the engines, and, therefore, those U.S. military officers asked the members of the Statistical Research Group (SRG) to optimize the armor of the U.S. military aircrafts on the basis of those findings. Then one of the most distinguished members of the

The intellectual capacity of a human being derives from one's genetic and, generally, biological state as well as from one's contact with reality. In other words, the intellectual capacity of a human being derives from one's physical abilities, innate traits, and culture. For instance, the origin of mathematics lies, arguably, in counting, which is not even an exclusively human trait (since other animals can count as well[230]), and evidence of human counting goes back to prehistoric times, when people were using "tally marks" marked on bones, namely, a primitive unary numeral system. However, with the advancement of civilization, several mathematical innovations were achieved. For instance, the ancient Egyptians developed the first equation (related to architecture and agriculture), the ancient Greeks made great strides in geometry and arithmetic, negative numbers were invented in ancient China, zero as a number was first used in ancient India, Persian and Arab mathematicians, during the golden age of Islam, made significant contributions to the further development of algebra, a great flourishing of mathematics and the natural sciences took place in the Renaissance, etc.[231]

If, in a certain domain, a human being can form new intellectual images, reinterpret and reorganize old intellectual images, and create meaningful models by identifying and understanding harmonious phenomena and by synthesizing new harmonies from them, then one can be creative in this domain. In these creative processes, both one's innate abilities and one's overall culture are crucially significant. These factors determine the relationship between a creative conscious being and the world. In the context of the rational dynamization of reality, one can conceive a new phenomenon and express it through a model, and/or can combine existing images of one's culture into beautiful, inspiring forms. The rational dynamization of reality by consciousness is a complex intuitive process, which is inextricably linked to one's culture, and it underpins creativity in arts and sciences.

Let us denote the underlying structure of external ("objective") reality by S_{ER}. In the present book, I maintain that S_{ER} cannot be opposite to the structure of that kind of existence which is inextricably linked to S_{ER} and manifests itself as consciousness. As the French philosopher and Jesuit priest Pierre Teilhard de Chardin (1881–1955) has pointedly argued, there is a continuity between the energy (and, hence, the evolutionary process) of the world, the energy of life, and the energy of consciousness, and this continuity is condensed into and manifested as an energy field that is subjective with regard to experience, while it is objectified with regard to its effects in the realm of creative activity.[232] As I have already argued, the structure of the universe (at all levels, namely, those of astronomy, life, subatomic particles, and consciousness) does not consist in a "unique" universal structure, but it consists in a "unified" system of structures, and, for this reason, the structure of the universe can be expressed in limitless ways, all of which designate and reflect the energy structure of the universe. Consequently, the energy structure of the universe is malleable, and, just as the mass of each celestial body deforms the fabric of physical space-time (as I explained in

SRG, Abraham Wald, pointed out that those U.S. military officers had formulated their problem in a wrong way, and that they had to put the armor where there were no bullet holes. In particular, Wald argued that they had to armor the engines, not the fuselage, because the fundamental assumption should not be that the Germans could not hit the U.S. military aircrafts on the engines (because they could), but the fundamental assumption should be that the U.S. military aircrafts that got hit on the engines were unable to return from their flying missions (for which reason they did not).

[230] See: Angier, "Many Animals Can Count."

[231] See: Ball, *A Short Account of the History of Mathematics*.

[232] Chardin, *The Phenomenon of Man*; Chardin, *Human Energy*; and Chardin, *Activation of Energy*.

section 1.2.3), so the energy structure of the universe is susceptible to change in accordance with the intentionality of that type of consciousness to which the universe refers and into which the universe can be condensed, and that type consciousness has been called the "Omega Point" by Teilhard de Chardin. The aforementioned type of consciousness can actualize its worldview and make it meaningful. In fact, according to the dialectic of rational dynamicity, the history of civilization can be interpreted as humanity's "laborious journey" toward the aforementioned type of consciousness, namely, toward the "Omega Point," and, more specifically, the history of civilization can be interpreted as humanity's attempt to attain the mode of being that corresponds to the "Omega Point."

Isaac Newton's "classical scholia" (explanatory notes intended for use in a future edition of his seminal *Principia*) and his library (which included many books on mathematics, the natural sciences, philosophy, theology, mythology, and occultism) indicate that he achieved major scientific breakthroughs by studying certain phenomena deeply, analytically, and synthetically throughout his life, by living within his intellectual images, ideas, and intellectual representations. This is the way in which a creative conscious being lives in general, continuously trying to expand one's consciousness in order to ascend to and become the aforementioned "Omega Point." Every creative conscious being is continuously oriented toward one's intellectual images, ideas, and intellectual representations, and inquires into them systematically on the way to the "Omega Point." After the accomplishment of arduous tasks and the overcoming of various obstacles, and with the contribution of several conscious and unconscious factors, one enters into a state of dynamized reality, and then one suddenly sees the goal that one has achieved. Regarding the unconscious processes of creativity, the distinguished Hungarian-American mathematician George Polya (1887–1985) has argued as follows:

> The fact is that a problem, after prolonged absence, may return into consciousness essentially clarified, much nearer to its solution than it was when it dropped out of consciousness. Who clarified it, who brought it nearer to the solution? Obviously, oneself, working at it *subconsciously*.[233]

The moment of intuitive enlightenment is a creative discontinuity in one's life, a unique, sudden event through which something new comes into view, but it comes to fruition only because the subject has the necessary culture in order to be able to see the corresponding phenomenon, and is internally prepared to see it. Without the requisite culture, the same subject might be unable to see the given phenomenon.

The 2004 American film *What the Bleep Do We Know?* (co-directed and co-authored by William Arntz, Betsy Chasse, and Mark Vicente) mentions the invisible-ships phenomenon: When the great Italian explorer and navigator Christopher Columbus (1451–1506) first approached the shore of Hispaniola, the natives sitting on the shore were unable to see his ships approaching, not due to their distance from the shore, but because of the fact that, until then, the natives in the Caribbean had no objects in their lives that even remotely resembled Columbus's galleons, and, therefore, they did not have any categories in which they could place these objects. In other words, the natives in the Caribbean were unable to see Columbus's galleons as they were approaching to them because they lacked a mental model

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233 Polya, *How to Solve It*, p. 198.

in terms of which they could register and process this stimulus. The first person to notice Columbus's galleons was a local shaman, because shamans were accustomed to seeing strange things. In general, if we do not have the requisite concepts in order to understand something, then our consciousness may be unable to process it, and it may even fail to notice it at all.

One can argue as follows: we have mathematical models of space, because we move in space; we have mathematical models of time, because we move in time; we have counting systems, because we see objects; we study lengths, areas, the paths that objects with mass in motion follow, velocities, and slopes, because we throw objects (e.g., stones, bullets, etc.), we are involved in building activities, we travel, and we cultivate the land. From the aforementioned perspective, one could argue that our consciousness is largely determined by everyday experience. However, if we restrict our analysis of mathematics to the aforementioned perspective, then we are urged to think that mathematical concepts are "local," namely, that mathematics ceases to be effective (cognitively relevant) in a new realm, for instance, in the realm of the "very large," the "universe," or in the realm of the "very small," the world of elementary particles. But, remarkably, this is not the case. Mathematics has been extremely successful in conceptually conquering the entire field of physical experience. Thus, for instance, Einstein could not have articulated his theory of relativity unless Riemann had previously articulated his theory of geometry and mathematical analysis.

Nevertheless, quantum theory is a very peculiar case, because of the following reason: the traditional methods of abstract reasoning cannot lead to the articulation of quantum laws (generalizations). Both physicists and mathematicians are bewildered by facts such as the following: in classical mechanics (dealing with "big," massive bodies), one can know both the position and the velocity of an object, whereas, in quantum mechanics, one can know only either the position or the velocity of a particle; in classical mechanics, objects move from one position to another by following the shortest (i.e., the "optimal") path between two positions (e.g., classical straight lines in a Euclidean space, geodesics in a Riemannian space, or horocycles in a hyperbolic space), but, in quantum mechanics, a particle (e.g., an electron moving from one atom to another) is free to follow any possible path between any two positions, and the only thing that quantum physicists can do is to assign a certain probability to each of these paths (possible scenarios), so that, when quantum physicists are faced with the question of whether a particle is in a position A or in a position B, they realize that there is a probability that it is in position A, there is a probability that it is in position B, and it can even be partially in position A and partially in position B at the same time. In other words, the realm of quantum mechanics is not merely the realm of *probability*, but also the realm of *potentiality*.

Moreover, one of the most intellectually challenging and thought-provoking phenomena of quantum physics is quantum tunneling, which has absolutely no analogy in classical physics. From the perspective of classical mechanics, if the energy of a barrier is greater than the energy of the incoming particles, then there is no possibility that any of the particles will reach the other side of the barrier, but, from the perspective of quantum mechanics, the rules of the world are different, and, thus, we have quantum tunneling. Let us suppose that a particle bounces off a barrier, because the energy of the barrier is greater than the energy of the particle. This situation is represented by the wave-function reflecting at the boundary. Inside the barrier, the wave-function behaves as follows: as the distance into the barrier increases, the amplitude of the wave-function decreases exponentially, but the wave-function does not actually reach an amplitude of zero. Now, let us consider a different scenario where the barrier is shorter in length. As in the previous case, the amplitude of the wave-function will decay inside the barrier. But, because the wave-function does not reach an amplitude of zero, the wave-function can exit the barrier on the other side. Once the wave-function exits the barrier, its amplitude does not decay any more. Therefore, a portion of the wave-function passes through each of the two sides of the boundary, and a portion of the wave-function reflects at each of the two sides of the boundary. Consequently, there is a non-zero probability that the particle will pass through the barrier to the other side, and there is a non-zero probability that the particle will bounce off the barrier. Furthermore, let us consider a third scenario where the barrier's length is even shorter. In this case, the wave-function does not have as much distance to decay inside the barrier, and, therefore, we have a larger amplitude for the portion of the wave-function that exits the barrier. In other words, with this smaller barrier, the particle has a greater probability of passing through and a lower probability of bouncing off the barrier, which is represented by a smaller amplitude for the reflected wave. In general, irrespective of the barrier's size, and even if the probability of each individual particle passing through a barrier is inversely proportional to the barrier's size, if there is a very large number of particles ("large" in relation to the barrier's size), then there is a significant probability that at least some of these particles will pass through the barrier.

The bewilderment that overwhelms physicists when they deal with quantum-mechanical problems is due to the fact that classical physics deals with the realm of actuality, and even classical probability theory reflects the underlying intuition of the realm of actuality. Quantum mechanics implies that we should not think in terms of a single path between states, but in terms of the system of all possible paths between states, which is called the "sum over histories" (in this case, "history" means all possible scenarios at the same time). This is the reason why, as I have already mentioned, quantum theory, which deals with elementary particles, clashes with general relativity, which deals with large structures in the universe. In a sense, quantum mechanics is pure physics, whereas general relativity reduces to geometry, and, therefore, in order to properly understand and resolve the clash between the worldview of quantum mechanics and the worldview of general relativity, we have to come up with something even more fundamental than geometry, namely, we have to study "structure" at an even higher level of abstraction. Geometry is an abstract study of *actual* material objects, but, if we zoom in on the material world sufficiently enough in order to enter the realm of quantum mechanics, then actual matter, or material actuality, is replaced by potential matter, or material potentiality, and, of course, we realize that consciousness is fundamental to reality. The material-physical world is characterized by different levels of ontological development (e.g., starting from the level of quantum physics, advancing to the level of classical physics and the general theory of relativity, and then advancing further to the level of biology, culminating in the phenomenon of intelligent life); and, hence, the difference between microscopic physical laws and macroscopic physical laws.

As Proclus argued in his *Elements of Theology* (Proposition 37), of all those beings or things that are maintained through their intrinsic structural program, which determines their ontological development and perfection, "those that are first are more imperfect than those that are second, and those that are second are more imperfect than those that are successive, whereas those that are last are the most perfect," because, if changes come into being according to a structural program, and if change is directed to the actualization and manifestation of an intrinsic program of ontological development and perfection, then change refers to and is guided by that which is most perfect. Thus, as Proclus argues in his *Elements of Theology* (Proposition 32), every change that comes into being according to a program of ontological development and perfection that is intrinsic to being "is brought to completion through the likeness" of those beings or things that change to that which they become. Proclus's aforementioned reasoning can help one to philosophically understand why quantum mechanics cannot be a complete model of the physical reality.

In line with Proclus's principle of "likeness," the dialectic of rational dynamicity implies that our encounter with pure potentiality and, hence, with "chance" can be interpreted as our encounter with the specific structure of a sequence of causes and consequences that are interrelated due to a particular class of homomorphisms that underpin the synthetic organization of homomorphic groups into specific systems (these concepts will be rigorously studied in Chapter 2). Organization and structure are possible only when there exist homomorphisms, namely, relationships of structural likeness (or even isomorphisms, namely, bijective homomorphisms, which imply sameness). The existence of homomorphisms between different groups depends on whether the corresponding groups are suitably structured and on whether the behavior of these groups has an attractor, namely, a state in which most of the given groups' particular tendencies and orientations settle (I explained the concept of an attractor in section 1.2.3). In addition, according to the dialectic of rational dynamicity, the aforementioned sequence of causes and consequences is governed by the Aristotelian principle of the reduction to the first cause ("prime mover"), and it underpins a universal sequence of homomorphisms, giving rise to a worldview that is similar to Pierre Teilhard de Chardin's model of the world.

Conclusively, neither philosophical realism nor idealism can stand as a general theory of reality, but, as I argued earlier in this section, particular aspects of realism and particular aspects of idealism tend to approach truth. Philosophical realism is corroborated by the indisputable awareness that the world is different from consciousness, for which reason consciousness has to try hard in order to grasp the reality of the world. Idealism is corroborated by the indisputable awareness that, from a certain perspective, the structure of the world is not fundamentally different from the structure of consciousness, for which reason consciousness can partially and increasingly grasp the reality of the world. Consequently, reality consists of both the world and consciousness, and, thus, consciousness refers to both itself and the world. This is the reason why, if we want to be philosophically (and scientifically) rigorous, then we should not discourse on the relationship between "reality" and "consciousness," but we should discourse on the relationship between the "reality of the world" and the "reality of consciousness." The difference between the reality of a being A and the reality of a being B is determined by each of these beings' degree of ontological integration and completion. Thus, the manner in which the philosophy of rational dynamicity interprets history is based on Robert W. Cox's argument that "three categories of forces (expressed as potentials) interact in a structure: material capabilities, ideas and institutions"; and, in particular: (i) "material capabilities are productive or destructive potentials"; (ii) ideas are either "intersubjective meanings" or "collective images of social order held by different groups of people"; and (iii) "institutionalization is a means of stabilizing and perpetuating a particular order."[234]

[234] Cox, "Social Forces, States and World Orders," p. 218–19.

In view of the foregoing, the *dialectic of rational dynamicity*, as a method for the operation of consciousness and as a model of the operation of reality in general, consists of the following five stages (i.e., it is a five-fold dialectic):

Stage I: Vision and Orientation: Consciousness forms a clear intellectual image of an existential state that it wants to achieve, or, in Bloch's terms, a "utopia," and it is clearly oriented toward that intellectual image. Thus, in this stage, consciousness determines the teleology of its action.

Stage II: Strategy: In general, "strategy" refers to "the orientation of the organization in the long term, within its environment."[235] Consciousness makes the strategic decision to act upon the reality of the world and upon itself in accordance with its teleology, that is, in order to bring about intended changes.

Stage III: Planning: Consciousness articulates a plan, namely, a method of deliberate, self-conscious activity, involving the consideration of outcomes before choosing among alternatives. The primary functions of planning are the following: (i) optimization (i.e., improving efficiency of outcomes); (ii) balancing the agent's teleology (which is aimed at restructuring reality) and the goal of maintaining the continuity of existence (i.e., counterbalancing systemic failures); (iii) widening the range of decision-making (i.e., enhancing the consciousness of choice); and (iv) organizing and enriching codes and networks of communication.

Stage IV: Control: Consciousness continuously tries to maintain control over its action (and its consequences) in two ways: first, by intensifying its action (namely, its intervention in the reality of the world and in itself) whenever its action is unreasonably sub-optimal (namely, whenever it can improve its existential conditions even more, according to its strategic plan); second, by counterbalancing its original action (specifically, by reversing its original action and by following alternative paths of action) whenever the "negative externalities" of its original action, namely, the costs of its original action for the world (or the "environment"), in general, and/or for itself, in particular, tend to exceed a critical value that represents the maximum existential risks that consciousness is determined to undertake in order to continue acting in the same way.[236] Additionally, it should be mentioned that the term "dialectic," in general, implies a transition from one state to another without the total elimination of the previous state, in the sense that the previous state leaves its traces in the new one, and, therefore, according to the dialectic of rational dynamicity, an agent of change does not bring about a totally new state, which would be uncontrolled by the agent of change. In general, change cannot go beyond certain limits without running the risk of systemic collapse, and, for this reason, the dialectic of rational dynamicity highlights the importance of preventing uncontrolled systemic turbulence and of continuously maintaining control over the consequences of our actions. Furthermore, the aforementioned reasoning is exemplified in economics by

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[235] Schwaninger, "Governance for Intelligent Organizations," p. 36.

[236] For instance, every serious research paper that proposes the emission of extra aerosols (i.e., suspensions of liquid, solid, or mixed particles with highly variable chemical composition and size distribution) to the atmosphere in order to reflect a larger portion of the Sun's energy back to space is accompanied by the remark that, because there are many things that are not fully understood or fully controlled by scientists, we should try this type of intervention in the atmosphere on such a scale that allows us to reverse it in case anything goes wrong.

the investment strategy that is called hedging, and it consists in securing oneself against a loss on an investment "by investing on the other side," that is, hedging is insuring or protecting against adverse changes in the market (often using financial derivatives, through which a loss on one investment is mitigated or offset by a gain in a comparable derivative).[237]

Stage V: Development: Consciousness seeks to ensure and enhance its capabilities and to create favorable conditions for the continuation of its action in the future. However, consciousness realizes that the achievement of its ultimate goals is a work in progress. Thus, consciousness seeks to restructure the world according to the intentionality of consciousness without, however, jeopardizing the possibility of future interventions in the reality of the world.

Finally, it is worth mentioning that, as a method of historical action, the dialectic of rational dynamicity is inextricably linked to and essentially in consonance with the twelve basic characteristics of the personality of a "normal person" that I mentioned in section 1.1. Thus, in line with Plato's philosophy, the philosophy of rational dynamicity emphasizes the significant yet elusive interplay between intellectual development and psychological health.

1.3.4. Matter, Life, and Consciousness

Before inquiring into conscious life and into the functioning of philosophizing and scientific consciousness, we must have a clear understanding of life and of the major philosophical and biological perceptions of life, which is one of the most important manifestations of existence. The term "life" refers to a set of phenomena (such as reproduction, development, and homeostasis or maintenance) that characterize organisms. The term "organism" refers to any entity that embodies the properties of life, and it is contrasted to those objects which, lacking an organic constitution, are characterized by inertia and apparent stability.

It goes without saying that life contains organic matter. However, life restructures organic matter in an organic way, thus differentiating it from inorganic matter. In other words, life is entwined with inanimate matter and consciousness, and it underpins the structural continuity between them. For this reason, there are both differences and similarities between organic matter and inorganic matter.

Even though there is a structural continuity between inorganic matter and organic matter, life—by transforming inorganic matter into organic matter—implies an important differentiation in matter. Thus, the differences between inorganic matter and organic matter can be summarized as follows[238]:

Inorganic matter is governed by inertia, which is the resistance of any inorganic body to any change in its velocity (see section 1.2.3). On the other hand, organically structured living beings sense things, react to external stimulus, and move on their own.

Inorganic matter reacts according to Newton's third law of motion (see section 1.2.3). In other words, the reaction of an inorganic body is quantitatively determined by external

[237] Arditti, *Derivatives*.
[238] See: Raven and Johnson, *Understanding Biology*.

mechanical forces (specifically, by tensile force, compressive force, and shear force) that are applied to it. On the other hand, the reactions of organically structured living beings manifest peculiar qualitative features that are not strictly analogous to the stimuli that cause reaction, and they depend on organic relations that govern each living being according to its structural program.

According to the Standard Model of particle physics, the minimal constituent matter elements of inorganic bodies are uniform, that is, subatomic particles are identical (so that no exchange of two identical particles, such as electrons, can lead to a new microscopic state). Thus, all the atoms of which any inorganic body is composed are identical to each other. By contrast, the minimal constituent matter elements of organic matter (such as the DNA) are subject to differentiations, which underpin the actualization and the manifestation of the structural program of an organic being. In fact, due to their differentiation, the cells of an organic being underpin its organic constitution, which determines the corresponding organic being's unity and cohesion (i.e., the attraction of molecules for other molecules of the same kind). Furthermore, it is important to mention that eukaryotes (that is, organisms whose cells have a nucleus enclosed within a nuclear envelope), such as the human being, have two types of DNA: the DNA of the cells (namely, the agent of the genetic information of the cells) and the mitochondrial DNA (namely, the DNA located in mitochondria, which are double membrane-bound organelles supplying cellular energy and controlling the cell cycle and the cell growth; mitochondrial proteins, that is, proteins transcribed from mitochondrial DNA, vary depending on the tissue and the species).

Inorganic bodies are connected with each other under specific conditions in order to form chemical compounds, which are always characterized by the same quantitative data, described and explained by Antoine Laurent Lavoisier's "law of conservation of matter" ("matter is neither lost nor gained during a chemical reaction"[239]), Joseph Louis Proust's "law of constant composition" ("in a compound, the constituent elements are always present in a definite proportion by weight"[240]), and John Dalton's "law of multiple proportions" ("in the formation of two or more compounds from the same elements, the weights of one element that combine with a fixed weight of a second element are in a ratio of small whole numbers (integers), such as 2 to 1, 3 to 1, 3 to 2, or 4 to 3"[241]). On the other hand, organically structured living beings exchange some of their constituent elements with some of their environment's constituent elements in the context of a dynamic process that is called assimilation (in biology, assimilation is the absorption and digestion of food or nutrients by an organism).

Inorganic bodies exist in definite and fixed quantities according to Lavoisier's "law of conservation of matter," Proust's "law of constant composition," and Dalton's "law of multiple proportions." On the other hand, organically structured living beings ("parents") create new living beings ("offsprings") similar to them in the context of the reproductive process.

With few exceptions (such as radioactive nuclides (nuclear species), which "are unstable structures that decay to form other nuclides by emitting particles and electromagnetic radiation"[242]), inorganic bodies are incapable of self-transformation. On the other hand,

[239] See: Jones, Johnston, Netterville, and Wood, *Chemistry, Man and Society*, p. 21.
[240] Ibid. p. 23.
[241] Ibid.
[242] Sears, Zemansky, and Young, *College Physics*, p. 1031.

organically structured living beings follow life cycles (developmental stages that occur during an organism's lifetime).

Furthermore, consciousness is a state in which a being can understand, process, and modify one's internal and external environment, and, therefore, it can be described as a complex system of concepts. Consciousness is manifested by the creation of multiple feedback loops whereby a conscious being can create models in order to pursue certain goals. Animals can understand their position in space, and many of them can also understand their relationships with other beings, but only humans can understand the future and restructure their spatio-temporal existential conditions according to their intentionality, thus creating history.

The continuity of living organisms is ensured by the succession of generations. On the one hand, each living organism is organically self-contained, but, on the other hand, the succession of generations ensures the continuity of the corresponding species. Intimately related to the study of the continuity of living organisms are the neo-Darwinian concept of a mutation (namely, an abrupt jump in the continuity of living organisms, specifically, an alteration in the nucleotide of the genome of an organism), the classical Darwinian theory of natural selection, and biological structuralism, which I explained in section 1.2.3.

However, the aforementioned scientific approaches to the properties of life cannot sufficiently address the issue of the nature and the substance of life, because, as I have already argued, "scientific explanation" is founded on experience. Therefore, ontology is necessary in order to inquire into the nature and the substance of life. It goes without saying that materialist ontological theories have formulated over-statements and over-simplifications by arguing that the properties of life are reducible to chemical reactions, while spiritualist ontological theories have formulated over-statements and over-simplifications by articulating interpretations that are founded on mere intellectual speculation and ignore empirical data. Nevertheless, the careful study of the history of philosophy with regard to the issue of the nature and the substance of life can provide us with useful information and ideas.

According to ancient philosophy, there is a kind of continuity between life and spirit. Inspired by pre-Socratic philosophy, Epicurus formulated a theory of hylozoism that is founded on the concept of a primal breath animating matter. This hylozoist perspective is similar to the Biblical Jews' and the Kabbalists' teachings about God's "ruach," namely breath and spirit (in fact, Greek philosophy exerted a significant influence on ancient Judaism and the Kabbalah of the Jews[243]). The Stoics' hylozoism is founded on the concept of a divine fire animating matter, and this hylozoist perspective is similar to several Biblical passages, such as Acts 2, where the divine spirit is symbolized by fire. Furthermore, the conception of a principle that animates the body underpins both Plato's philosophy and Aristotle's philosophy. In Plato's philosophy, the soul is placed between spirit (whose energies are the

[243] See: Gruen, *Heritage and Hellenism: The Reinvention of Jewish Tradition*. In particular, the spirituality of the Kabbalah is a synthesis of Pythagoreanism, Neoplatonism, and Biblical mysticism. Moreover, the Renaissance saw the birth of Christian Kabbalah (often transliterated as Cabalah to be distinguished from the Jewish Kabbalah). Christian Kabbalah reinterpreted Kabbalistic texts and symbols from a distinctly Christian perspective, and Ramon Llull (1232–1316), a philosopher, logician, Franciscan tertiary, and writer from the Kingdom of Majorca, was the first Christian scholar to acknowledge and appreciate the Kabbalah as a tool of conversion. Among the first and most important systematic propagators of Kabbalistic studies beyond exclusively Jewish circles were the Italian philosopher Giovanni Pico della Mirandola (1463–94), the Venetian Franciscan friar Francesco Giorgi (1466–1540), the German scholar Johann Reuchlin (1455–1522), and Paolo Riccio (1480–1541), a German Jewish convert to Christianity who became a professor of Philosophy at the University of Pavia, and, subsequently, he was physician to Emperor Maximilian I.

ideas) and matter, and, in Aristotle's philosophy, the ontological differentiation of the principle that animates matter underpins Aristotle's argument that this principle is the organizing form of the body and governs the entire body. In line with Plato's arguments regarding the placing of the soul between spirit and matter, Neoplatonism has articulated its own ontological hierarchies. The aforementioned arguments regarding the existence of an external principle that animates matter are based on a sense of logical necessity, and, therefore, they deal with the ontological component of the issue of life more in terms of logical reduction than in terms of ontology itself.

In the seventeenth century, and in relation to important advances in the scientific discipline of medicine, the philosophical inquiry into the nature and the substance of life was systematized, and it started considering clearly scientific data. Thus, in the context of modern philosophy, the philosophical inquiries into the problem of life can be distinguished into two general categories: mechanism (known also as mechanical philosophy) and dynamism (known also as dynamical philosophy).[244] According to mechanism, which is largely inspired by ancient atomism, the constitution of reality, including life, is a result of random physical-chemical phenomena. However, Descartes's philosophy replaced the previous materialist variety of mechanism with a spiritualist variety of mechanism, according to which, in contrast to animals, the human being is governed by spirit, which makes the human being a cognizing organism.

As I have already explained, Descartes's attempt to explain the life of animals by means of a monist philosophy and the life of the human being by means of a dualist philosophy is characterized by important flaws. In the twentieth century, several distinguished representatives of mechanism, such as Daniel Auger, Jacques Loeb, and John Searle, while endorsing an anti-materialist ("anti-physicalist") perspective, argue that there is a kind of continuity between matter and life, including consciousness as an outgrowth of life. In addition, such careful and thorough proponents of mechanism reject the argument that life is a transcendent principle by maintaining that—in spite of the continuity between organic matter and inorganic matter, and in spite of the fact that both organic matter and inorganic matter are subject to the same natural laws—life consists in the set of the differences between organic matter and inorganic matter. According to John Searle, in particular, consciousness is a higher state of the brain just as ice is a higher state of water, and the brain can be in a conscious state just as liquidity and solidity are states in which water can be.[245] However, as I have already argued, the scientific corroboration of the aforementioned arguments of mechanism does not imply their definitive confirmation, either in the context of philosophy or in the context of science itself. Even though mechanical philosophy can provide epistemologically satisfactory propositions, it cannot properly address the fact that there exists a substantial difference between life and matter. Moreover, mechanical philosophy analyzes the data of life in a way that cannot give rise to a synthetic study of the principle of life and of structural questions.

The argument of classical mechanism according to which any living organism is merely a set of physical-chemical phenomena contradicts the second law of thermodynamics, specifically the minimum energy principle (see section 1.2.3). The attempt to reconcile the aforementioned argument of classical mechanism with the second law of thermodynamics by arguing that the world is in a state of maximal entropy, consisting of beings that are imperfect

[244] See: Glennan and Illari, eds., *The Routledge Handbook of Mechanisms and Mechanical Philosophy*.
[245] Searle, *The Rediscovery of the Mind*.

and weak manifestations of life, is also unsuccessful, because it contradicts both the continuity and the dynamism of life itself.

In contrast to mechanism (mechanical philosophy), dynamism (dynamical philosophy) highlights the differences between matter and life. The major representative of the first historical phase of dynamism was Leibniz. Leibniz founded his variety of dynamism on the Stoics' hylozoism, thus departing from Descartes's mechanism. The major representative of the second historical phase of dynamism was the French physician, physiologist, and encyclopedist Paul Joseph Barthez (1734–1806). Barthez employed the expression "vital principle" as a convenient term for the cause of the phenomena of life, distinguishing it from both the principle of matter and the principle of spirit, and, thus, refusing to commit himself to either spiritualism or materialism. The major representative of the third historical phase of dynamism was the French anatomist and pathologist Marie François Xavier Bichat, the acknowledged father of modern histology (1771–1802). In his famous physiological research works, Bichat, rejecting reductionism, recognized three essential "vital systems," namely, animal life, sensible organic life, and insensible organic life; he located the primary seat of animal life in the brain, of sensible organic life in the heart, and of insensible organic life in the lungs; and he argued that various physical-chemical factors tend to destroy organic life. The aforementioned varieties of dynamism converge to the argument that there is a discontinuity between physical-chemical phenomena and life, but, in the end of the nineteenth century, dynamism started following an alternative intellectual path, according to which physical-chemical phenomena constitute the basis of life, but, apart from them, life has also a final cause (or purpose), which consists in the preservation of the unity of each and every organism through which life is manifested. The aforementioned teleological approach to life has been called "neofinalism" (in French, "néo-finalisme") by the French philosopher Raymond Ruyer (1902–87).[246]

Every rigorous inquiry into the phenomenon of life and every rigorous attempt to understand the significance of a being are necessarily dependent on the study of the structure of a being. As I argued in section 1.3.3, the most important components of the dialectic of rational dynamicity consist in preserving and changing structures. From the perspective of the dialectic of rational dynamicity, "development" signifies a smooth growth and expansion of an organically structured living being according to the given being's structure, whereas "evolution" signifies a sequence of smooth and rather slow transformations according to a procedural logic, and, in its pure form, the notion of evolution is associated with the passive role that British empiricism assigns to consciousness. Thus, "development" signifies a deliberately organized process of amelioration, which can be studied in terms of a model of constrained optimization (see Chapter 2).

Intimately related to the study of life is the study of consciousness. In accordance with the dialectic of rational dynamicity, consciousness proceeds from life, but it is not ontologically posterior to life, because consciousness exists potentially within the tendency of a being to exist, and it is intrinsic to instinct, which is a condensed form of logic. Moreover, consciousness underpins the adaptation of the organically structured living beings to their environment. Finally, as Bergson has correctly pointed out, consciousness is inextricably linked to action. When the human being ascends to the highest levels of consciousness, which correspond to reason and morality, it spiritualizes matter. Three characteristic, easily

[246] Ruyer, *Neofinalism*.

understood ways in which human consciousness spiritualizes matter are art, technology, and political action, which signify the integration of ideas into matter and the restructuring of matter according to the intentionality of consciousness. It is worth pointing out, for instance, that the American economist Robert Solow (who was awarded the Nobel Prize in Economics in 1987) has found that, in the United States, during the period 1909–49, about one-eighth of the increment in labor productivity could be attributed to increased capital per man hour, and the remaining seven-eighths to a factor that is called "Solow residual" and consists of technological progress and other cultural factors that improve efficiency.[247] Moreover, the American economist Edward F. Denison has studied the contribution of different elements to growth in real Gross National Product in the United States during the period 1929–82, and he has shown that advancements in knowledge, education, and other cultural-institutional factors play the most important role in economic growth.[248]

Classical political economy is founded on the hypothesis that resources are limited, and it leads to the conclusion that we should expect a "limit to growth." In 1972, the Club of Rome published a book entitled *The Limits to Growth*, according to which, within a time span of less than one hundred years with no major change in the physical, economic, or social relations that have traditionally governed world development, society will run out of the non-renewable resources on which the industrial economy depends.[249] However, the dialectic of rational dynamicity implies that consciousness can rearrange the resources and create an additional resource base. Even if resources are limited, rational dynamicity enables us to get more from the existing resources by transforming them. Energy transitions from wood to coal, from coal to oil, and from oil to other energy resources provide important examples of the contribution of the dialectic of rational dynamicity to economic growth. Indeed, Paul M. Romer, an American economist and entrepreneur associated with the New York University Stern School of Business and with Stanford University, has argued that "economic growth occurs whenever people take resources and rearrange them in ways that are more valuable," and that "a useful metaphor for production in an economy comes from the kitchen," in the sense that "economic growth springs from better recipes, not just from more cooking," and "new recipes generally produce fewer unpleasant side effects and generate more economic value per unit of raw material."[250]

Finally, the dialectic of rational dynamicity, as I expound and defend it in the present book, extricates human consciousness from the intellectual and the material shackles of capitalism. As I have already mentioned, Marx's social philosophy is one of the components of my philosophy of rational dynamicity, since Marx's analysis of capitalism helps one to understand that capitalism is not only an exploitative system, but also one that is characterized by self-complacent nihilism and by an attitude that constrains human consciousness to the established, systemic mechanism.

Karl Marx has pointedly emphasized the difference between "use-value" and "exchange-value" in order to explain the essence of the capitalist system. The difference between the "use-value" and the "exchange-value" of a commodity corresponds to the difference between the usefulness of a commodity and the exchange equivalent in terms of which a commodity is compared to other objects traded in a market. In particular, Marx argues that use-value is

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[247] Solow, "Technical Change and the Aggregate Production Function."
[248] Denison, *Trends in American Economic Growth: 1929–1982*.
[249] Meadows, D. H., Meadows, D. L., Randers, and Behrens III, *The Limits to Growth*.
[250] Romer, "Compound Rates of Growth."

inextricably linked to “the physical properties of the commodity,”[251] namely, to the human needs that it fulfills, whereas the exchange-value (i.e., the “exchange relation”) of a commodity is characterized precisely by its “abstraction” from its use-value.[252] In capitalism, money takes the form of the aforementioned equivalence, and it conceals the real equivalent behind the exchange, namely, labor. Given that the more labor (physical and/or mental) it takes in order to produce a product, the greater its value, Marx concludes that, “as exchange-values, all commodities are merely definite quantities of congealed labor-time.”[253] In fact, the fundamental difference between the political economy of traditional, pre-capitalist societies and the political economy of capitalist ones is that the political economy of traditional, pre-capitalist societies gives primacy to use-value over exchange-value, and, in particular, it refuses to valorize usury,[254] whereas the political economy of capitalist societies valorizes usury, and it gives primacy to exchange-value over use-value. In this way, in the capitalist system, “money” is transformed into “capital,” and labor is fully commodified.

Furthermore, Marx has clarified the manner in which capital transforms the simple circulation of commodities: In commodity trading, money is a medium of exchange, a store of value, and a unit of account, and economic actors exchange commodities for money, and then they exchange money for some other commodities. In other words, in commodity trading, economic actors sell something in order to buy something else that they need. Hence, according to Marx, the structure of commodity trading can be described by the formula

$$C \to M \to C, \tag{1}$$

namely, $Commodity \to Money \to Commodity$. However, Marx has observed that financial speculation, which consists in buying in order to sell at a higher price, allows money to transform formula (1) into the following formula:

$$M \to C \to M, \tag{2}$$

namely, $Money \to Commodity \to Money$. According to Marx, formula (2) is the general formula for capital. In the context of capitalism, which is governed by formula (2), “the circulation of money as capital is an end in itself, for the valorization of values takes place only within this constantly renewed movement,” and “the movement of capital is therefore limitless.”[255] In this context, as Marx argues in his *1844 Manuscripts*, capitalism essentially negates and aims to nullify the role of labor as self-realization or as a self-affirmative process, and to bring about the transformation of the human individual into an economic object shaped by external, alien forces. In particular, explaining Marx’s thought on the issue of labor, the German-American philosopher and sociologist Herbert Marcuse has argued that, within the historical facticity of capitalism, “labor is not ‘free activity’ or the universal and free realization of man, but his enslavement and loss of reality,” in the sense that “the worker is not man in the totality of his life-expression, but something unessential, the purely physical

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[251] Marx, *Capital*, vol. 1, p. 126.
[252] Ibid, p.127.
[253] Ibid, p. 130.
[254] An eloquent description of pre-capitalist European societies’ repulsion against usury and usurers is William Shakespeare’s play *The Merchant of Venice*.
[255] Marx, *Capital*, vol. 1, p. 253.

subject of 'abstract' activity," and "the objects of labor are not expressions and confirmations of the human reality of the worker, but alien things . . . 'commodities.'"[256] It is worth mentioning that Adam Smith, one of the acknowledged founders of classical political economy, has conceded that, "in the process of division of labor," on which the industrial development of the capitalist world has been based, the worker "whose whole life is spent in performing a few simple operations . . . generally becomes as stupid and ignorant as it is possible for a human creature to become."[257]

Moreover, Marx has observed that, ultimately, the aim of the capitalist becomes "the unceasing movement of profit-making,"[258] and, due to usury (namely, the act of lending money at a significant interest rate) and the growth of financial speculation, formula (2) reduces to

$$M \to M, \tag{3}$$

namely, $Money \to Money$. Formula (3) expresses the culmination of nihilism under capitalism. In particular, the mathematical formula of compound interest is the following: Assume that you borrow an amount P of money (the "principal") at an (annual) interest rate of $r > 0$, and that, at the end of each year, you have to pay back a fixed amount (a "deposit") d. Let A_n be the total amount of money owed after n years. The formula for computing A_n in terms of P (the principal of the loan), r (the interest rate of the loan), and d (the loan deposits) is the following:

$$A_n = A_{n-1}(1+r) - d = P(1+r)^n - d(1+r)^{n-1} - d(1+r)^{n-2} - \cdots - d$$
$$= P(1+r)^n - d\frac{(1+r)^n - 1}{(1+r) - 1} \Leftrightarrow A_n = P(1+r)^n - \frac{d}{r}[(1+r)^n - 1], r \neq 0;$$

so that the initial condition is $A_0 = P$; at the end of the first year, you owe P (the principal) plus an interest equal to rP minus the deposit you have agreed to pay each year, and, therefore, $A_1 = P + rP - d = P(1+r) - d$; by analogy, at the end of the second year, you owe $A_2 = A_1(1+r) - d = P(1+r)^2 - d(1+r) - d$, etc. By allowing the owners of large sums of money to lend (that is, trade) money on interest, we give them power to immunize themselves against loss (in fact, this is the ultimate purpose of charging interest on loans: to immunize the lender of money against loss), while socializing loss and risks, and, thus, to create an exceptionally privileged financial oligarchy. In general, "financial fascism" consists in socializing loss and privatizing profits; and the most extreme form of financial fascism is an economic system dominated by usurers.

According to an old adage, originally attributed to the German statesman Otto von Bismarck, "there are two things you don't want to see being made, sausage and legislation." It is a telling statement in many ways. In general, this adage means that one's established consumption, trading, working, and entertainment practices as well as various established mentalities would be spoiled by intimate familiarity with the underlying principles and the very fabric of the actual state of affairs in political economy. Furthermore, it is worth mentioning that, in 1893, the German philosopher Friedrich Engels, in a letter that he wrote to

[256] Marcuse, *Heideggerian Marxism*, p. 104.
[257] Smith, *An Inquiry into the Nature and Causes of the Wealth of Nations*, Book V, Chapter 1, Article II.
[258] Marx, *Capital*, vol. 1, p. 254.

the German historian and politician Franz Mehring, used the term "false consciousness" in order to refer to the deliberate manipulation of one's awareness of reality and to anyone suffering the burden of an established ideological "monopoly" or "oligopoly."

Nevertheless, even though Marx and Engels managed to articulate a thorough criticism of capitalism, the fact that, as I have already mentioned, Marx's thought was imbued with the prophetism of Hegel's philosophy, the fact that, as I have already mentioned, Marx has not clarified whether his theoretical work should be interpreted as a general method or as a model of particular objective processes that he seeks to interpret and evaluate, and the fact that Marx has written very little about the exact structure of the regime that should be established after a socialist revolution and about the exact structure of his ideal communist society can lead to the use of Marx's thoughts and political visions by slicker and devious persons in order to establish a system of state capitalism (or bureaucratic socialism) as an end-in-itself and in order to implement an expansionist policy under the pretense of cosmopolitanism. For instance, in the 1930s and in the 1940s, the Soviet leader Joseph Stalin created and imposed a peculiar amalgam of Marxism, nationalism, and authoritarian statism, and, in the 1970s, under the leadership of Deng Xiaoping, China officially adopted a model of state capitalism/bureaucratic socialism, so that, in the 1970s, it became clear that the ruling elite of the Chinese Communist Party, like the Communist Party of the former Soviet Union, established a system of state capitalism/bureaucratic socialism not as a transitional phase toward socialism *proper*, but as an end-in-itself, primarily serving and reflecting the selfish calculations and expediencies of a ruling coalition between professedly "communist" politicians, state bureaucrats, and private speculators.

As regards the destiny of Marx's thought in China, it should be mentioned that, during Mao Zedong's "Cultural Revolution," the major ideas and the major visions of socialism and communism were replaced by a hybrid system of state capitalism/bureaucratic socialism and Confucianism, thus, in essence, maintaining and prolonging China's political tradition of autocracy, totalitarianism, and dictatorial rule, and, in the aftermath of Deng Xiaoping's reformations, the established Chinese system of state capitalism/bureaucratic socialism was combined with higher levels of economic speculation and institutionalized corruption and with a plan for the total algorithmization and, hence, dehumanization of social organization. In his best-selling book *Wolf Totem*, Jiang Rong (which is the pseudonym of the Chinese dissident author Lü Jiamin) explains that the regime of the People's Republic of China wants people to become sheep, and, from this perspective, it follows Confucianism, whose central tenet was obedience to the emperor and the established social order, while, simultaneously, in the economic sphere, the prevailing character of the Chinese so-called "communist" businessmen is that of a fierce wolf.[259] In particular, Han Fei Tzu's Legalism (or "school of the method"), of which Tung Chung-shu (179–04 B.C.) is the preeminent representative, grafted Confucian political and ethical notions on an organic conception of society, and, additionally, it adopted an eclectic attitude toward the incorporation of legalist practices in the administration of the empire. As Robert Spalding (retired U.S. Air Force Brigadier General) has explained, in the economic sphere, the CCP (Chinese Communist Party) has embraced the speculative aspects and dynamics of Western capitalism, given that, in China, capitalism is strictly controlled by the CCP's ruling elite, but, in the political sphere, CCP wants to dissociate Western capitalism from bourgeois liberalism, and, being imbued with fundamental

[259] Rong, *Wolf Totem*.

loathing of the United States' Bill of Rights, the CCP, especially during the presidency of Xi Jinping, aims to impose a system whose major constituent components are state capitalism/bureaucratic socialism and totalitarianism.[260]

Regarding imperialism, in particular, it should be stressed that, as the German-American political theorist Hannah Arendt has thoroughly explained, imperialism is an anti-cosmopolitan variety of universalism, since it is an attempt of a particular political actor to universally impose one's own selfish interests and perceptions, whereas genuine cosmopolitanism is based on intrinsically universal values and norms, which transcend every particular actor's own political and economic expediencies.[261] In particular, in the 1950s and in the 1960s, in the United States of America, a group of American intellectuals associated with or influenced by social democracy and/or Trotskyism transformed themselves into "neoconservatives" by isolating the socialist notion of cosmopolitanism and Trotsky's theory of "continuous revolution" from their socialist context and fusing them with Leo Strauss's political conservatism, thus transforming them into underpinnings of an imperialist ideology, which became very influential during the Republican presidential administrations of Ronald Reagan (1981–89) and George Walker Bush (2001–09). In terms of political culture, neoconservatism is a product of the fusion of fundamentalist Evangelical American organizations, the selfish expediencies of the Euroatlantic military-industrial complex, and right-wing Zionist currents (the latter—as Hannah Arendt has pointed out—representing an updated version of neo-imperialism and neo-colonialism[262]). Thus, neoconservatism semantically equates "cosmopolitanism" with "Pax Americana," and it semantically equates "continuous revolution" with a strategy of continuous geostrategic military interventions.[263]

Even though the capitalist West's imperialist dynamic did not characterize the former Soviet Union, the latter was decisively characterized by a bureaucratic-militaristic dynamic. Hence, even though the economic structures of the former Soviet Union did not give rise to an imperialist dynamic, the former Soviet Union was characterized by an expansionist dynamic, which expressed the former Soviet social establishment's need to stabilize its global position irrespective of ideological principles and without pursuing a substantial socialist-communist transformation of the former Soviet society. The declared teleology of the former Soviet Union's policy of stabilizing its global position may differ from Western imperialism, but the fundamental mentalities and the practices of the former Soviet Union's political system were not essentially different from those of Western imperialism (this mentality of selfish pragmatism, bureaucratism, conformity to capitalist norms, and power politics divested of any philosophical and/or ideological principles became the official political doctrine of the post-Soviet Russian establishment during the presidency of Vladimir Putin).

Let us recall an old adage according to which "the best defense is a good offense," implying that offensive expansionist tendencies, plans, and intentions may constitute integral parts of a defensive strategy. In the beginning of the twenty-first century, a current of Russian political scientists exemplified by Sergey Karaganov, invoking *Realpolitik* and pragmatism, simply exchanged Stalin's model of authoritarian statism for a Bismarckian model of authoritarian statism (i.e., one inspired by the Prussian statesman Otto von Bismarck and his policies) and endorsed Henry Kissinger's ethos (thus establishing a reflexive relation between the Kremlin and the White House). To put it succinctly, just as the American neoconservatives developed an imperialist political theory by separating Trotsky's

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260 Spalding, *Stealth War*.

261 Arendt, *Imperialism*.

262 Arendt, *Eichmann in Jerusalem*. Moreover, see: Whitaker, "US Thinktanks Give Lessons in Foreign Policy."

263 McGowan, *American Liberalism*, Book II, 4. Moreover, Genrikh Aleksandrovich Trofimenko has articulated a thorough and authoritative analysis of the U.S. imperialism in his book *The U.S. Military Doctrine*.

internationalism and theory of revolution from Trotsky's socialist vision (espousing the former while dismissing the latter), so too Karaganov and the like developed their own variety of authoritarian statism and expansionism by espousing the authoritarian, statist, and expansionist aspects of the Stalinist legacy while dismissing the declared socialist commitments of Stalinism. In this way, the global capitalist establishment, in general, and the ruling Euroatlantic (NATO-EU) elites, in particular, achieved a subtle yet significant victory over Russia, in the following sense: they managed to assimilate Russia into the established capitalist-euroatlantic "World Order." Hence, Karaganov has been Presidential Advisor to both Boris Yeltsin and Vladimir Putin while being simultaneously a member of the Trilateral Commission (since 1998); and he served on the International Advisory Board of the Council on Foreign Relations from 1995 until 2005.

Whereas the Soviet Union was, at least in principle, an international-political actor trying to establish and promote (and even globalize) a substantially different socio-political model, post-Soviet Russia, officially and explicitly, became part of the dominant Western capitalist system, irrespective of the fact that Putin's Russia may often exhibit higher levels of nationalism, conservativism, and competitiveness than particular Western capitalist elites would wish. However, since the era of the Cold War, what matters most to the Western capitalist elites is the extent to which the Kremlin elites conform to the logic and the ethos of capitalism and to the Western capitalist bourgeoisie's theories of geopolitics and *Realpolitik*. In other words, what matters most to the Western capitalist elites is the manipulation of the ruling Soviet/Russian elites' mentality and intellectual and moral horizons, in order to make the ruling Soviet/Russian elites think and act mainly according to methods that belong to the Western capitalist establishment's political and economic "toolbox," which includes both "liberal-internationalist" varieties of capitalism and "conservative-nationalist" varieties of capitalism. In fact, a military faction of the Soviet Union, indoctrinated with Western theories of geopolitics and *Realpolitik* and seeking to partner with Western military elites for the management of world affairs, managed to impose itself on the KGB and bargained socialism away, thus playing a decisive role in the dissolution of the Soviet Union.

During the Cold War, the language of strategic studies was given to euphemism, jargon, and oxymoron, and the political establishment, both in the Euroatlantic "bloc" and the Soviet "bloc," looked at every attempt to promote free thought and rigorous political discourse with a jaundiced eye.[264] This Cold War legacy was preserved during the first decades of the post-Cold War era in the form of propaganda operations that manipulate the concept of democracy. To put it succinctly, during the end of the twentieth century and during the beginning of the twenty-first century, it became amply clear that—even though democracy originated in ancient Greece encouraging the constant and never-ending reflective re-evaluation of social institutions—Euroatlantic elites abusively and selfishly invoke European theories of democracy in order to justify and disguise their geostrategic and financial ambitions, thus, in essence, identifying "democracy" with "liberal oligarchy," while Eurasian and Asian elites castigate the deviousness of the aforementioned Euroatlantic elites in order to justify their own (Eurasian/Asian) models of authoritarianism by dismissing the European democratic tradition outright as a phenomenon of only local (modern Western) relevance and as a pretense on the part of imperialist Euroatlantic elites. Regarding post-Soviet Russia under Vladimir Putin's government, in particular, it should be mentioned that, however capable and

[264] See: Lifton and Markusen, *The Genocidal Mentality*.

skillful a tactician he may be, Vladimir Putin has failed to realize that his government cannot get around the substantial problems of noopolitics, ideology, and, ultimately, philosophy by means of ordinary power politics and propaganda.

Regarding *Realpolitik*, in general, it should be mentioned that, from the perspective of the philosophy of rational dynamicity, it should be criticized on the following levels: The advocates of *Realpolitik*, such as Hans J. Morgenthau, regard the "will-to-power" as the defining characteristic of politics and as the element with respect to which one can distinguish politics from other spheres of social life (e.g., economics, law, morality, religion, etc.), and they assert the autonomy of politics as a distinct form of social life, which is characterized by the "will-to-power."[265] By abstracting the "political man" from the "real man," by abstracting "political life" from "real life," by identifying political action with power politics, and by confining political science within the realm of such abstractions and one-sided attempts at explaining politics, the advocates of *Realpolitik* lapse into a form of political idealism that also makes it difficult to distinguish between *explanation* and *prescription*. In other words, the pursuit of unitary understanding ("power politics") and the tension between the abstracted (necessity in the form of power politics) and the unabstracted (the realm of freedom and morality, which have been separated from politics by the advocates of *Realpolitik*) undermine the empirical relevance of the theory of *Realpolitik* and the cognitive significance of the theorems that have been formulated by the advocates of *Realpolitik*, such as Morgenthau. For instance, the espousal of the claim that the national interest of the "modern nation-state" could be defined independently of any consideration of a nation-state's dominant culture and independently of the dynamics of a nation-state's elites undermines the empirical significance of any foreign-policy analysis.[266] It is worth pointing out that, according to the eminent American sociologist Charles Wright Mills, societies should be studied in terms of what he has called the "sociological imagination," which has the following three components: (i) History: why society is what it is, how it has been changing for a long time, and how history is being made in it. (ii) Biography: what is the "human nature" in society, and what kinds of people constitute a society. (iii) Social structure: how the various institutional orders in a society function, which ones are dominant, how they are kept together, how they change, etc.[267] Furthermore, as Stanley H. Hoffmann has argued, "it is impossible to subsume under one word variables as different as: power as a condition of policy and power as a criterion of policy; power as a potential and power in use; power as a sum of resources and power as a set of processes."[268]

Analyzing the "modern world system" as an evolving, interlocking world capitalist economy that emerged in its discernible modern form in the sixteenth century, the distinguished American sociologist and economic historian Immanuel Wallerstein has argued that the former Soviet Union and all the professedly socialist states should be characterized as parts of the established capitalist system by virtue of the fact that they traded in a world market and reproduced capitalist relationships.[269] In the post-Soviet era, the Euroatlantic model of active and overt imperialism is primarily based on and primarily guided by the intention and the plans of a capitalist elite to maximize its profits (manipulating politics), and

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265 Morgenthau, *Politics among Nations*.
266 Hoffmann, ed., *Contemporary Theory in International Relations*; Seabury, *Power, Freedom, and Diplomacy*.
267 Mills, *The Sociological Imagination*.
268 Hoffmann, ed., *Contemporary Theory in International Relations*, p. 32.
269 Wallerstein, *Unthinking Social Science*; Wallerstein, *The Modern World-System*.

the Russian-Eurasian model of passive and covert imperialism is primarily based on and primarily guided by the intention and the plans of a political and bureaucratic elite to maintain and maximize its power and authority (manipulating capitalism).

In the twentieth century, the three major conceptual and institutional frameworks within which and due to which the most repellent and the most extreme phenomena of "false consciousness" were manifested were private capitalism, state capitalism (or bureaucratic socialism), and geopolitics. Regarding geopolitics, in particular, the distinguished French economist and political geographer Yves-Marie Goblet (1881–1955) timely diagnosed that it was essentially "political alchemy" and "metaphysics," and he scornfully contrasted "géographie politique" with "Geopolitik spagyrique."[270]

By contrast, the dialectic of rational dynamicity helps one to understand, rationally and creatively criticize, and avoid the structural defects of both private capitalism and state capitalism/bureaucratic socialism. From the perspective of the dialectic of rational dynamicity, the term "market" should mean a free social space. Even though the advocates of capitalism talk of "free trade," the underlying ethos of capitalism does not consist in the sacredness of human freedom, but it consists in the sacredness of the logic of money power (represented by banks and financial oligopolies), which breaks any rule pertaining to noble traditional aspirations until humanity's spiritual life, "culture," is systematically dragged through the mire under deceptive notions, such as "prosperity," "stability," "mutual understanding," "dialogue," and, of course, "security," and it is ultimately sacrificed on the altar of Mammon. The underlying ethos of capitalism renders stillborn any attempt to establish a really free market, because, in essence, a free market is a free social space (such as the ancient "agorā" of Athens), whereas the underlying ethos of capitalism gives rise to "cartelism" (i.e., the control of production and prices through agreements between/among big corporations)[271] and various types of "mafiocracy" (i.e., rule by organized crime).[272] The concept of a free-market-as-a-free-social-space (originally exemplified by the "agorā" of classical Athens) implies that human persons possess certain rights and liberties because of the very fact that they are humans, namely, by nature, and, therefore, they have the right to band together and form and reform their social institutions. However, the aforementioned concept should not be confused with the concept of a free-market-as-a-capitalist-institution, because, in the latter case, corporations—having become persons in law—gradually usurp the rights and the liberties that naturally belong to human persons, and they tend to impose themselves as superior persons vis-à-vis the human persons.

As a conclusion, rational dynamicity is substantially different from both the underlying reasoning of private capitalism and the underlying reasoning of state capitalism (or

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[270] Goblet, *The Twilight of Treaties*.

[271] See, for instance: Tepper and Hearn, *The Myth of Capitalism*.

[272] It is important to mention that, as the French Professor of Legal History and literary critic Jacques de Saint Victor has argued, mafia (specifically, the European and American transnational system of organized crime) was born in the "décombres" (rubble) of the feudal regime, and it was developed further as a consequence of the advent of bourgeois democracy and capitalism from the nineteenth century onward (Victor, *Un Pouvoir Invisible*). In fact, the essence of modern mafia is the result of the merger between a rotten nobility and a criminal bourgeoisie, and various secret/"esoteric" societies (such as the notorious Italian Masonic Lodge "P2") and private exclusive membership clubs operate as front organizations for the mafia, often in collaboration with state bureaucracies. Moreover, in the context of the Cold War, a notorious and powerful alliance was formed between the CIA, the Vatican, and the Mafia, ostensibly, in order to conduct covert operations against communism in general and against the Soviet Union in particular; see: Williams, *Operation Gladio*. Finally, regarding the Russian Mafia, in particular, see: Friedman, *Red Mafiya*.

bureaucratic socialism). Additionally, rational dynamicity is primarily a philosophy, and, for this reason, it transcends action itself, since its purpose is to operate as an ideal type and a guiding principle of action (rather than as a particular set of concrete actions). As a philosophy, rational dynamicity is inextricably linked to a continuous evaluation of the way in which humanity expresses its freedom and historical creativity vis-à-vis cosmic necessity. The rationality and the dynamism that characterize the philosophy of rational dynamicity stem from and presuppose, more than anything else, a deep trust in humanity's creative presence. In view of the foregoing, rational dynamicity can also operate as a method of judging and evaluating civilizations. In my book *Taking the Bull by the Horns: Causes, Consequences and Perspectives in Politology and Political Economy*, originally published in Greek by the Greek scholarly publisher ΚΨΜ (https://kapsimi.gr/), I propose an alternative, integral political and economic theory, which I have called **"critical rational socialism."**

"One must divide one's time between politics and equations. But our equations are much more important to me, because politics is for the present, while our equations are for eternity."
— Theoretical physicist Albert Einstein (1879–1955) speaking to mathematician Ernst Strauss (1922–83); quoted in: Joy Hakim, *The Story of Science*, Washington, D.C.: Smithsonian Books, 2007, Chapter 28.

Chapter 2

FORMAL METHODS OF ANALYSIS: ALGEBRA, CALCULUS, AND ANALYTIC GEOMETRY

2.1. SETS, RELATIONS, AND GROUPS

The history of set theory and, generally, of non-numerical mathematics can be traced back to the era of classical Greece, but the first systematic inquiry into the foundations of set theory was due to the German mathematician Georg Ferdinand Ludwig Philip Cantor (1845–1918). However, before Cantor, George Peacock (1791–1858), Augustus De Morgan (1806–71), and George Boole (1815–64) had already made significant contributions to the formalization of non-numerical mathematics.[273] Inextricably linked to set theory is algebra. Peter J. Cameron has explained the meaning of algebra as follows:

> The word "algebra" is derived from the Arabic *al-Jabr*, meaning "transformation." It refers to a technique derived by Al-Khwarizmi, a Persian mathematician who lived in Baghdad early in the Islamic era (and whose name has given us the word "algorithm" for a procedure to carry out some operation). Al-Khwarizmi was interested in solving various algebraic equations (especially quadratics), and his method involves applying a transformation to the equation to put it into a standard form for which the solution method is known.[274]

However, one of the very influential drivers of the Arabs' and the Persians' algebraic thinking was ancient Greek number theory, which culminated in the work of the third-century A.D. Greek mathematician Diophantus of Alexandria, who published his seminal book *Arithmetica*, which is a collection of one hundred and thirty algebraic problems giving numerical solutions of determinate equations (i.e., equations with unique solutions) and indeterminate equations (i.e., equations with more than one solutions) and using fractions.

Before proceeding any further, I would like to clarify that I use the following symbols of logic and set theory:

$\wedge$ or &: conjunction ("and"),

[273] For a systematic study of the history of set theory, see: Merzbach and Boyer, *A History of Mathematics*; Halmos, *Naive Set Theory*; Stoll, *Set Theory and Logic*.

[274] Cameron, *Introduction to Algebra*, p. 1.

$\vee$: disjunction ("or"),
$\neg$: negation ("not"),
$\rightarrow$ or $\Rightarrow$: material implication ("if . . . then . . .")
$\leftrightarrow$ or $\Leftrightarrow$: biconditional ("if and only if"),
$\forall$: universal quantification ("for every"),
$\exists$: "there exists,"
$\exists!$: "there exists exactly one,"
$\nexists$: "there does not exist,"
$P(x)$: predicate letter (meaning that x (an object) has property P),
$|$: "such that,"
$\vdash$: turnstile ($x \vdash y$ means that x "proves" (i.e., syntactically entails) y; a sentence φ is "deducible" from a set of sentences Σ, expressed $\Sigma \vdash \varphi$, if there exists a finite chain of sentences $\psi_0, \psi_1, \psi_2, \dots, \psi_n$ where ψ_n is φ and each previous sentence in the chain either belongs to Σ, or follows from one of the logical axioms, or can be inferred from previous sentences),
$\vDash$: double turnstile ($x \vDash y$ means that x "models" (i.e., semantically entails) y; a sentence φ is a "consequence" of a set of sentences Σ, expressed $\Sigma \vDash \varphi$, if every model of Σ is a model of φ).

2.1.1. Basic Concepts of Set Theory

The difficulty in defining the concept of a set is that it is a fundamental concept, and, therefore, it cannot be reduced to simpler concepts. Cantor described a "set" as a well-defined gathering together into a whole of definite, distinguishable objects of perception or of our thought that are called elements of the set.[275] By the term "well-defined," Cantor means that, given any object and any set, the given object is either an element of the given set or not an element of the given set, and, by the terms "definite" and "distinguishable," Cantor means that no two elements of a set are the same. Cantor's definition of a set, though rather vague, implies the following properties of sets:

(i) any set A contains "elements" or "members" of A, symbolically:
$x \in A \Leftrightarrow (\mathit{the\ object\ x\ is\ an\ element\ of\ A})$;
(ii) each set is determined by its elements, symbolically:
$A = B \Leftrightarrow (\forall x)[x \in A \Leftrightarrow x \in B]$ for any sets A and B. This property is known as the "property of extension."

The elements of a set may not be related to each other in some way. The "empty set," denoted by $\emptyset$, has no elements, and, by the property of extension, it is unique. A set is "finite" if the number of its members is finite; otherwise, it is an "infinite" set. For instance, the set $\{1,2,3\}$ is fine, whereas the set $\{x|x > 3\}$ is an infinite set. If a set has only one element, then it is called a "singleton."

However, every collection is not a set. Before the first rigorous axiomatization of set theory by the German mathematician and logician Ernst Zermelo (1871–1953), to whose

[275] Cantor, "Beiträge zur Begründung der transfiniten Mengenlehre," p. 481.

work I shall refer extensively in section 3.5, Cantor's set theory was based on his (intuitive) definition of a set and on the General Comprehension Principle.

General Comprehension Principle[276]: For every definite condition P of n variables $x_1, x_2, \dots, x_n$, there exists a set

$$X = \{\vec{x} | P(\vec{x})\}$$

whose elements are the n-tuples $\vec{x}$ of the objects having property P, so that

$$\vec{x} \in X \Leftrightarrow \mathrm{P}(\vec{x}).$$

A condition P of n variables is called "definite" if it is definitely determined whether $P(\vec{x})$ is true or false for any n-tuple $\vec{x}$ of objects $x_1, x_2, x_3, \dots, x_n$. The General Comprehension Principle is restricted to definite conditions in order to avoid mathematically irrelevant obscurities (e.g., subjective judgments, such "New York is a nice city").

The General Comprehension Principle means that, given any condition expressible by a formula $\varphi(x)$, it is possible to form the set of all sets x meeting that condition. Cantor endorsed the General Comprehension Principle mainly because it was in agreement with his intuition about sets. Nevertheless, the British philosopher and mathematician Bertrand Russell (1872–1970) proved that the General Comprehension Principle is not valid by putting forward "Russell's paradox."

Russell's Paradox[277]: Let U be the collection of all sets:

$$U = \{x | x \ is \ a \ set\}.$$

Then U is not a set. We can prove Russell's Paradox by *reductio ad absurdum*. Assume, for the sake of contradiction, that U is a set. However, any ordinary mathematical set (e.g., of numbers, functions, etc.) is not a member of itself and can be naturally regarded as a member of a smaller universe of sets that can be obtained again by the General Comprehension Principle. In particular, let V be an arbitrary set and $V \notin V$. Then, by the definition of U,

$$V \in U. \qquad \text{(i)}$$

Moreover, because U is a set, either $U \in U$ or $U \notin U$. If $U \notin U$, then, by statement (i), $U \in U$. But, if $U \in U$, then, again by (i), $U \notin U$. Therefore, in both of these cases, we reach a contradiction, and, in this way, we prove that U is not a set. The class U is known as "Russell's class," and the aforementioned contradictory situation is known as "Russell's paradox" (i.e., the "universal set" is not a set). According to Russell, the problem in the aforementioned paradox is that we confuse a description of sets of numbers with a description of sets of sets of numbers. In order to overcome such difficulties, Russell and Alfred North Whitehead introduced a hierarchy of objects, which they called "types," namely: numbers,

[276] Cantor, *Gesammelte Abhandlungen*.
[277] Russell, "Mathematical Logic as Based on the Theory of Types."

sets of numbers, sets of sets of numbers, etc. In particular, Russell has described the concept of a type in the following way:

> Every propositional function $\varphi(x)$—so it is contended—has, in addition to its range of truth, a range of significance, i.e., a range within which x must lie if $\varphi(x)$ is to be a proposition at all, whether true or false. This is the first point in the theory of types; the second point is that ranges of significance form *types*, i.e., if x belongs to the range of significance of $\varphi(x)$, then there is a class of objects, the *type* of x, all of which must also belong to the range of significance of $\varphi(x)$, however φ may be varied; and the range of significance of $\varphi(x)$ is always either a type or a sum of several whole types.[278]

Thus, having objects of type 0 (individuals, i.e., any object that is not a range), 1 (classes of individuals), 2 (classes of classes of individuals), etc., relations among them are acceptable under specific conditions. For instance, inclusion, ⊆, is an acceptable relation when it relates objects of type 1 to objects of type 1; belonging, ∈, is an acceptable relation when, on its left, there is an object of type 0 and, on its right, there is an object of type 1.

Almost simultaneously with Russell and Whitehead, Ernst Zermelo proposed a different way to overcome the antinomies of Cantor's set theory, namely, to replace Cantor's intuitions with axioms; thus, the development of modern set theory was initiated (see also section 3.5). In Zermelo's axiomatic system, it is assumed that there exist a "universe of objects" U, some of which are sets, and some "definite conditions and operators" in U, the basic of which are the following:

$$x = y \Leftrightarrow \mathit{the\ object\ x\ is\ identical\ to\ y},$$
$$Set(x) \Leftrightarrow x\ \mathit{is\ a\ set},$$
$$x \in y \Leftrightarrow Set(y)\ \&\ x\ \mathit{belongs\ to\ y}.$$

The objects that are not sets are called "atoms." In fact, in order to overcome Russell's paradox, Zermelo replaced the axiom that, "for every formula $\varphi(x)$, there exists a set $y = \{x|\varphi(x)\}$" with the axiom that, "for every formula $\varphi(x)$ and every set v, there exists a set $y = \{x|x \in v\&\varphi(x)\}$."

If every element of a set B is an element of a set A, then B is said to be a "subset" of A, and we write $B \subseteq A$. Every set is a subset of itself. If A is an arbitrary set, then $\emptyset \subseteq A$, that is, the empty set is a subset of every set. Two sets A and B are "equal" if and only if $A \subseteq B$ and $B \subseteq A$, and then we write $A = B$. If two sets A and B satisfy the condition $B \subseteq A$ and there is at least one element of A that is not an element of B, then B is said to be a "proper subset" of A, and we write $B \subset A$. If $B \subseteq A$ or $B \subset A$, then A is said to be a "superset" of B. When in a particular situation all the sets under consideration are subsets of a fixed set, this fixed set, which is the superset of every set under consideration, is called the "universal set," or the "universe of discourse."

If the elements of a set are sets themselves, then the set is called a "set of sets," or a "family of sets," or a "collection of sets," or a "class of sets." For instance, $\mathcal{C} = \{\{x\}, \{y, z\}\}$ is a class of sets (notice that x is something different from $\{x\}$).

[278] Russell, *The Principles of Mathematics*, p. 523.

Let A be an arbitrary set. Consider a class X of sets such that to each element of A corresponds an element of X. Then X is called an "indexed class of sets," and the set A is called an "index set." For instance, let $X = \{A_n | n = 1,2,3, \dots\}$ and $A_n = \{x | x\ is\ a\ multiple\ of\ n, and\ n = 1,2,3, \dots\}$. Then $A_1 = \{1,2,3, \dots\}$, $A_2 = \{2,4,6, \dots\}$, etc., are the indexed sets, and X is the indexed class of sets.

Let A be any set. The "power set" of A is defined to be the set composed of all the subsets of A, and it is denoted by $\wp(A)$, symbolically:

$$\wp(A) = \{B | B \subseteq A\}.$$

If a set A has n elements (where n is a finite number), then $\wp(A) = 2^n$, since each element has two possibilities, namely, present or absent (and, hence, the possible subsets are $2 \times 2 \times 2 \times \dots n$ times, that is, 2^n). For instance, if $A = \{x, y\}$, then $\wp(A) = 2^2 = 4$, specifically, $\wp(A) = \big\{\emptyset, \{x\}, \{y\}, \{x, y\}\big\}$.

2.1.2. Basic Operations on Sets

If A and B are two arbitrary sets, then we define their

i. "union": $A \cup B = \{every\ x\ such\ that\ x\ belongs\ to\ at\ least\ one\ of Aand\ B\}$;
ii. "intersection":
$A \cap B = \{every\ x\ such\ that\ x\ belongs\ to\ both\ A\ and\ B\}$;
iii. "difference":
$A - B = A \cap B^{\sim}$, where $B^{\sim}$ is the "complement" of B, that is: if $B\ belongs\ to\ the\ power\ set\ \wp(X)$ of a certain set X, then
$B^{\sim} = \{every\ x\ that\ belongs\ to\ X\ such\ that\ x\ does\ not\ belong\ to\ B\}$;
iv. "symmetric difference":
$A \triangle B = (A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

Two sets are called "(relatively) disjoint" if their intersection is the empty set. A class of sets is "pairwise disjoint" if the intersection of any two sets in the class is empty. A class $C(X)$ of subsets of a set X is called a "partition" of X if $C(X)$ is pairwise disjoint and the union of the sets in $C(X)$ is the set X; for instance, the class $\big\{\{4,8\}, \{2,6,10\}, \{12\}\big\}$ is a partition of the set $\{2,4,6,8,10,12\}$.

In mathematics, the following notation is used:

$\mathbb{N} = \{0,1,2,3, \dots, n, n+1, \dots\} = the\ set\ of\ all\ natural\ numbers$ (these are the so-called "counting numbers"); $\mathbb{N}^* \equiv \mathbb{N} - \{0\}$.

$\mathbb{Z} = \{\dots, -n-1, -n, \dots, -2, -1, 0, 1, 2, \dots, n, n+1, \dots\} = the\ set\ of\ all\ integers$; in this case, $\mathbb{Z}^* \equiv \mathbb{Z} - \{0\}$.

$\mathbb{Q} = \left\{\frac{p}{q} such\ that\ p, q \in \mathbb{Z}, q \neq 0, (p, q) = \pm 1\right\} = the\ set\ of\ all\ rational\ numbers.$

In this case, (p, q) denotes the greatest common divisor of two integers p and q;

since $(p, q) = \pm 1$, it follows that p and q are relatively prime integers (an integer x is called prime if its only divisors are ± 1 and $\pm x$); $\mathbb{Q}^* \equiv \mathbb{Q} - \{0\}$.

$\mathbb{Q}^{\sim} = the\ set\ of\ all\ irrational\ numbers,$

namely, the set of all numbers that cannot be written as the quotient of two relatively prime integers. For instance, we can prove that $\sqrt{2} \in \mathbb{Q}^{\sim}$ by *reductio ad absurdum* as follows: suppose that $\sqrt{2} = \frac{p}{q}$ where $p, q \in \mathbb{Z}, q \neq 0, and\ (p, q) = \pm 1$, so that

$$\sqrt{2} = \frac{p}{q} \Rightarrow 2 = \frac{p^2}{q^2} \Rightarrow p^2 = 2q^2 \Rightarrow p = 2k$$

for an appropriate integer k, and, therefore, $4k^2 = 2q^2 \Rightarrow q^2 = 2k^2$, meaning that $(p, q) = 2$, which contradicts the hypothesis. The history of the irrational numbers goes back to the Pythagorean mathematicians, who had demonstrated that there exist lengths incommensurable with a given unit of length; for instance, the diagonal of a square whose side is the unit length. Obviously, $\mathbb{Q}^{\sim}$ is the complement of $\mathbb{Q}$ in the set $\mathbb{R}$ of all real numbers. Hence,

$$\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^{\sim} = the\ set\ of\ all\ real\ numbers;\ \mathbb{R}^* \equiv \mathbb{R} - \{0\}.$$

Two important theorems related to set operations involving intersections, unions, and taking absolute complements are known as De Morgan's Laws (they are named after the nineteenth-century English mathematician Augustus De Morgan, even though these theorems were known to Aristotle and medieval logicians).

De Morgan's Laws[279]*:* For any two sets A and B such that $A \cup B \subseteq X$, the following complementation laws hold:

i. $(A \cup B)^{\sim} = A^{\sim} \cap B^{\sim}$,
ii. $(A \cap B)^{\sim} = A^{\sim} \cup B^{\sim}$.

Proof:

i. $x \in (A \cup B)^{\sim} \Leftrightarrow x \notin A \cup B (i.e., x \notin A \& x \notin B) \Leftrightarrow x \in A^{\sim} \cap B^{\sim}$.
   This theorem can be expressed as a rule of inference as follows: the negation of a disjunction is the conjunction of the negations. In terms of propositional logic (which I shall systematically study in Chapter 3), (i) can be expressed as follows:
   $not\ (A\ or\ B) = not\ A\ and\ not\ B.$
ii. It can be proved analogously to (i). This theorem can be expressed as a rule of inference as follows: the negation of a conjunction is the disjunction of the negations. In terms of propositional logic, (ii) can be expressed as follows:
   $not(AandB) = notAornotB.$■

Moreover, notice that $(A^{\sim})^{\sim} = A$ (i.e., a double negation implies an affirmation), and $A \subseteq B \Leftrightarrow B^{\sim} \subseteq A^{\sim}$.

[279] De Morgan, *Formal Logic.*

The algebra of sets is governed by the following laws, which were systematically studied by the English mathematician and philosopher George Boole in the nineteenth century[280]:

i. "Commutative Law":
$A \cup B = B \cup A$,
$A \cap B = B \cap A$.
ii. "Associative Law":
$(A \cup B) \cup C = A \cup (B \cup C)$,
$(A \cap B) \cap C = A \cap (B \cap C)$.
iii. "Distributive Law":
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Let $a \in A$ and $b \in B$. Then the "ordered pair" (a, b) is defined (according to the Polish mathematician and logician Kazimierz Kuratowski) as the set $\{\{a\}, \{a, b\}\}$, whereas the (unordered) "pair" is $\{a, b\} = \{a\} \cup \{b\}$. Moreover, $(a, a) = \{a\}$, the singleton of a.

By a "tuple," we mean a finite ordered list of elements. The 0-tuple (i.e., the empty tuple) is the empty set $\emptyset$. An n-tuple, where $n > 0$, namely, $(a_1, a_2, a_3, \dots, a_n)$ is a collection of n objects $a_1, a_2, a_3, \dots, a_n$ in which a_1 is the first element, a_2 is the second element, . . ., and $a_n$is the nth element, and it can be defined as an ordered pair of its first element and an $(n-1)$-tuple, namely, $(a_1, (a_2, a_3, \dots, a_n))$. This definition can be applied recursively to the $(n-1)$-tuple, so that we obtain

$$(a_1, a_2, a_3, \dots, a_n) = (a_1, \Big(a_2, \big(a_3, (\dots, (a_n, \emptyset) \dots)\big)\Big)).$$

The Fundamental Property of Ordered Pairs[281]*:* For any ordered pairs, (w, x) *and* (y, z), it holds that:

$$(w, x) = (y, z) \Leftrightarrow w = y \;\&\; x = z,$$

and then two ordered pairs are called "equal."

The "Cartesian product" (known also as the "direct product") $A \times B$ of two sets A and B is the set of all ordered pairs (a, b) such that $a \in A \; and \; b \in B$, symbolically:

$$A \times B = \{(a, b) | a \in A \;\&\; b \in B\}.$$

For instance, if $A = \{1,2\}$ and $B = \{1,3\}$, then the Cartesian product $A \times B$ is the set $\{(1,1), (1,3), (2,1), (2,3)\}$. In general, the Cartesian product of the sets $A_1, A_2, \dots, A_n$, denoted by $A_1 \times A_2 \times \dots \times A_n$ is the set of all ordered n-tuples of the form $(a_1, a_2, \dots, a_n)$, where a_i is an element of $A_i (i = 1,2, \dots, n)$.

Remark: It is easily checked that, for any sets A, B, and C, we have:

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[280] Boole, *An Investigation of the Laws of Thought*.
[281] Kuratowski, *Topology*, vol. 1.

$A \times (B \cup C) = (A \times B) \cup (A \times C)$,
$A \times (B \cap C) = (A \times B) \cap (A \times C)$.

If $A = \emptyset$ or $B = \emptyset$, then $A \times B = \emptyset$.

$A \times B = B \times A \Leftrightarrow A = B$.

Let $A \times B = \{(a, b) | a \,\&\, b \; are \; real \; numbers\}$. Then $A \times B$ is the set of all points in a plane whose coordinates are (a, b). Thus, $A \times B$ is the Cartesian plane

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$,

as shown, for instance, in Figure 2.1: in this case, each point P in the plane represents an ordered pair (a, b) of real numbers and vice versa. In other words, the vertical line through P meets the x-axis at a, and the horizontal line through P meets the y-axis at b. Thus, we can understand the relationship between set theory, mathematical analysis, and geometry.

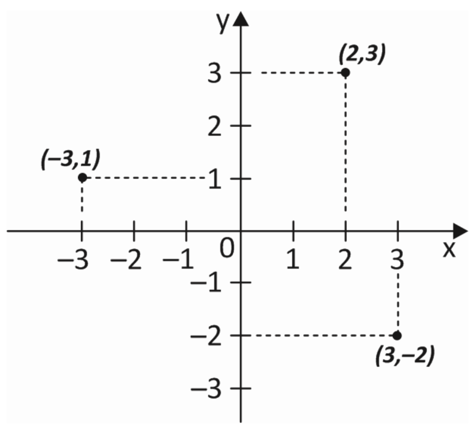

Figure 2.1. The Cartesian Plane.

Let A consist of all real numbers, and let B consist of all integers. Then $A \times B$ is the set consisting of all those points which lie on the straight line $y = m$, where $m \in B$.

Applications of Set Theory to Probability Theory

Probability theory is primarily concerned with the issue of uncertainty.[282] In fact, “probability” is a quantitative measure of uncertainty, and it is a number between 0 and 1, where 0 indicates impossibility and 1 indicates certainty. According to Chrystal’s formal definition of a probability, “if on taking any very large number N out of a series of cases in which an event A is in question, A happens on pN occasions, the probability of the event A is said to be p”[283] (the certainty of the corresponding proposition increases as the number N of specimen cases selected increases). Furthermore, according to Chrystal, the following corollaries and extensions may be added to the aforementioned definition of a probability: (i)

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[282] See: Gnedenko, *The Theory of Probability*.
[283] Chrystal, *Algebra*, vol. 2, p. 567.

"If the probability of an event be p, then out of N cases in which it is in question it will happen pN times, N being any very large number."[284] (ii) "If the probability of an event be p, the probability of its failing is $1-p$."[285]

Probability theory is based on set theory. By the term "experiment," we mean a process that leads to one of several possible outcomes. By the term "outcome," we mean an observation or measurement. The "sample space" is the set of all possible outcomes of an experiment. An "event" is a subset of a sample space, or, in other words, a set of basic outcomes. Thus, we say that the event "occurs" if the corresponding experiment gives rise to a basic outcome belonging to the event. Therefore, we obtain the following formula:

$$Probability\ of\ event\ A = \frac{n(A)}{n(S)},$$

where $n(A)$ is the number of elements in the set of the event A, and $n(S)$ is the number of elements in the sample space S. For instance, roulette as it is played in Las Vegas or Atlantic City consists of a wheel that has 36 numbers, numbered 1 through 36, and the number 0 as well as the number 00 (double zero). Therefore, in this case, the sample space, S, consists of 38 numbers, and the probability of winning a single number that you bet is $P = 1/38$.

When the sets corresponding to two events are disjoint (that is, their intersection is the empty set), then these events are called "mutually exclusive."

The axiomatic definition of probability is the following: Let E be a space of elementary events. Then the "probability of an event" $A \subseteq E$ is denoted by $P(A)$, and it is defined as a single number that corresponds to A and has the following properties:

(P1) $P(A) \geq 0$;
(P2) for each pair of mutually exclusive events, $A, B \subseteq E$, it holds that $P(A \cup B) = P(A) + P(B)$;
(P3) $P(E) = 1$.

Remark: For each $A, B \subseteq E$, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$; in case, A and B are mutually exclusive, $P(A \cap B) = 0$, and we obtain (P2).

By the term "conditional probability," we mean the probability of event A conditional upon the occurrence of event B. Assume that we investigate the probability of an event A given that we know that an event B has occurred and that event B influences the probability of event A. Then the probability of event A given the occurrence of event B is defined as the quotient of the probability of the intersection of A and B over the probability of event B; symbolically:

The "conditional probability" of event A given the occurrence of event B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)},$$

assuming that $P(B) \neq 0$. The aforementioned formula for the computation of conditional probability is known as Bayes's Law, since it was originally formulated by the eighteenth-

[284] Ibid, p. 569.
[285] Ibid.

century English statistician and philosopher Thomas Bayes.[286] According to Bayes's Law, two events A and B are independent of each other if and only if

$$P(A \cap B) = P(A)P(B).$$

Bayes's Law provides a method of revising existing predictions or theories (specifically, updating probabilities) given new additional evidence. As Matthew Large has pointed out, Bayes's Law implies that "the interpretation of any risk assessment must involve an estimate of the base rate," and "this base rate, which is never known with complete certainty at the time of the assessment, is a Bayesian 'prior probability.'"[287]

In general, probability theory underpins the scientific study of risk and uncertainty. One of the most important methods that are used to discover, describe, and explain "typical" behavior of mass data is the "arithmetic mean." The formula is

$$\bar{X} = \frac{\sum_{i=1}^{N} X_i}{N}$$

where $\bar{X}$ denotes the arithmetic mean, $\sum_{i=1}^{N} X_i$ denotes the summation of the values of the individual observations X_i under consideration ($i = 1,2, \dots, N$), and N is the total number of items in the series that has been summated.[288] It is worth noticing that arithmetic means are often "weighted" averages, in the sense that, when averaging values, it is sometimes logically necessary to assign more importance to some than to others (by multiplying each value with a suitable statistical weight), so that particular values may be more influential in determining the "typical" value than others.

One of the most important methods that are used to discover, describe, and explain "risk" or "uncertainty" is the "standard deviation," which is a quantity expressing by how much the members of a database (i.e., the data under consideration) differ from the arithmetic mean of the given database. The formula is:

$$\sigma = \sqrt{\frac{\sum_{i=1}^{N} x_i^2}{N}}$$

where: first, we calculate the arithmetic mean $\bar{X}$ of the values X_i ($i = 1,2, \dots, N$) under consideration; second, we record the deviation of each value X_i from the arithmetic mean, namely, $x_i = X_i - \bar{X}$; third, we square these deviations, namely, we compute x_i^2; fourth, we summate the squared deviations and divide by N (this is the "variance" of our data); and, fifth, we extract the square root to obtain σ.[289]

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[286] See: Moivre, *The Doctrine of Chances*.

[287] Large, "The Relevance of the Early History of Probability Theory to Current Risk Assessment Practices in Mental Health Care," p. 432.

[288] See: Neiswanger, *Elementary Statistical Methods*, pp. 256–57.

[289] Ibid, p. 311.

2.1.3. Relations

Let A and B be two arbitrary sets. Then a "relation" between A and B, denoted by R, is defined to be a subset of the Cartesian product $A \times B$, symbolically: $R \subseteq A \times B$. The "domain" of relation R is defined by $D_R = \{a|(a,b) \in R\}$, and the "range" of relation R is defined by $R_R = \{b|(a,b) \in R\}$. If R is a relation from A to B, then the relation from B to A is called the "inverse" of R, and it is defined by $R^{-1} = \{(b,a)|(a,b) \in R\}$. A relational proposition is often denoted by aRb, where R relates a term a to a term b. According to Bertrand Russell, "it is characteristic of a relation of two terms that it proceeds, so to speak, from one to the other."[290]

If R_1 is a relation from a set A to a set B, and if R_2 is a relation from B to a set C, then their "composition," denoted by $R_2 \circ R_1$, is a relation from A to C, symbolically:

$$R_2 \circ R_1 = \{(a,c)|for\ some\ b \in B, (a,b) \in R_1 \& (b,c) \in R_2\ with\ a \in A, c \in C\}.$$

If R_1 and R_2 are relations such that $R_1 \subseteq R_2$, then R_2 is said to be an "extension" of R_1, and R_1 is said to be a "restriction" of R_2.

A relation R on a set A is "reflexive" if (a,a) is an element of R for every $a \in A$; it is "symmetric" if (a,b) is an element of R whenever (b,a) is an element of R; and it is "transitive" if (a,c) is an element of R whenever (a,b) and (b,c) are elements of R. A relation R on a set A is "antisymmetric" if, whenever a and b are distinct, then (a,b) is an element of R only if (b,a) is not an element of R. For instance, if $A = \{u,v,w\}$ and R is a relation on A, then:

$R = \{(u,v),(v,u),(u,u),(v,v),(v,w),(w,w)\}$ is a reflexive relation on A;
$R = \{(u,v),(v,u),(w,w)\}$ is a symmetric relation on A;
$R = \{(u,v),(v,w)(u,w),(v,v)\}$ is a transitive relation on A;
$R = \{(u,w),(v,v),(u,v),(u,u)\}$ is an antisymmetric relation on A.

A relation R on a set A, namely, a subset of $A \times A$, is said to be an "equivalence relation," and it is denoted by $\sim$, if it is reflexive (i.e., $a \sim a\ \forall a \in A$), symmetric (i.e., $a \sim b$ implies that $b \sim a\ \forall a,b \in A$), and transitive (i.e., $a \sim b$ and $b \sim c$ imply that $a \sim c\ \forall a,b,c \in A$). For instance, since an integer a is said to be "congruent to an integer b modulo m," symbolically $a \equiv b(mod m)$, if m divides the difference $a - b$, it is evident that congruence is an equivalence relation on $\mathbb{Z}$. In general, an equivalence relation measures equality with regard to some attribute.

Let R be an equivalence relation on a non-empty set A. Then the "equivalence class" of any element $a \in A$ is denoted by $\bar{a}$ or $[a]$, and it is defined as the set of all elements of A to which a is related, namely:

$$\bar{a} = \{x \in A|(a,x) \in R\}.$$

[290] Russell, *The Principles of Mathematics*, p. 95.

Any element in an equivalence class is called a "representative" of that class. If R is an equivalence relation on a set A, then the set whose elements are the R-equivalent classes is called the "quotient set" of A by R, and it is denoted by A/R, namely:

$A/R = \{\bar{a}|a \in A\}$.

For instance, for the equivalence relation

$R = \{(x, y)|x \equiv y(mod2), where\ x, y \in \mathbb{Z}\}$,

there are two equivalence classes: the set of even numbers and the set of odd numbers (we assume that zero is an even number, because 0 is a multiple of 2, since $0 \times 2 = 0$, and, thus, 0 shares all the properties that characterize even numbers: 0 is neighbored on both sides by odd numbers in the set of all integers, positive and negative; any decimal integer has the same parity as its last digit, and, indeed, since 10 is even, 0 is even; if y is even, then $y + x$ has the same parity as x, and, indeed, x and $0 + x$ always have the same parity).

The Fundamental Theorem of Equivalence Relations[291]*:* If $\sim$ is an equivalence relation on a set A, then $A = \cup\, \bar{a}$, where this union runs over one element from each class, and $\overline{a_1} \neq \overline{a_2} \Rightarrow \overline{a_1} \cap \overline{a_2} = \emptyset$. In other words, an equivalence relation on a non-empty set A partitions A into equivalence classes, and, conversely, a partition of A induces an equivalence relation on A (the concept of a partition was defined in section 2.1.2).

Proof: Given that $a \in \bar{a}$, it holds that $\cup_{a \in A} \bar{a} = A$. The proof of the second assertion is also straightforward, because we can show that, $\overline{a_1} \neq \overline{a_2} \Rightarrow \overline{a_1} \cap \overline{a_2} = \emptyset$, or, equivalently, that $\overline{a_1} \cap \overline{a_2} \neq \emptyset \Rightarrow \overline{a_1} = \overline{a_2}$ as follows: Let $\overline{a_1} \cap \overline{a_2} \neq \emptyset$ and $c \in \overline{a_1} \cap \overline{a_2}$. By the definition of an equivalence class, $c \sim a$ because $c \in \bar{a}$, and $c \sim b$ because $c \in \bar{b}$. Therefore, due to the symmetry of $\sim$, $a \sim c$, and, because $a \sim c$ and $c \sim b$, it holds that $a \sim b$. Hence, $a \in \bar{b}$. If $x \in \bar{a}$, then $x \sim a$, and $a \sim b \Rightarrow x \sim b$, so that $x \in \bar{b}$. Therefore, $\bar{a} \subset \bar{b}$. Because the argument is symmetric in a and b, it also holds that $\bar{b} \subset \bar{a}$. Consequently, $\bar{a} = \bar{b}$, which proves the theorem.■

Let A and B be two arbitrary sets. A relation $f \subseteq A \times B$ is called a "function," or "mapping," or "transformation," denoted by $f: A \rightarrow B$, if it assigns to each element $a \in A$ exactly one element $b \in B$. The set A is called the "domain" of the function f and is denoted by D_f, while the set B is called the "codomain" of the function f. The set of all elements of B that are related to the elements of A via f is called the "range" of the function f, and it is denoted by R_f, meaning that the range of f is the image of A by f:

$f(A) = \{f(a)|a \in A\}$.

By the term "graph" of a function $f: A \rightarrow B$, we mean the set $\{x, f(x)\}$, where $x \in A$.

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[291] See: Herstein, *Abstract Algebra*, p. 67.

Two functions $f: A \to B$ and $g: A \to B$ are called "equal" if $f(x) = g(x), \forall x \in A$, and they are called "different" if there is at least one $x_0 \in A$ such that $f(x_0) \neq g(x_0)$.

If f is a function from X to Y, then, for any subset A of X, we have:

$A \neq \emptyset \Leftrightarrow f(A) \neq \emptyset$.
$f(\{x\}) = \{f(x)\}, \forall x \in X$.
$A \subseteq B \Leftrightarrow f(A) \subseteq f(B)$.
$f(A \cup B) = f(A) \cup f(B)$.

$f(A \cap B) \subseteq f(A) \cap f(B)$; since: if $y \in f(A \cap B)$, then, by definition, $y = f(x)$ for some $x \in A \cap B$, and, therefore, $f(x) \in A$ and $f(x) \in B$, so that $y = f(x) \in f(A) \cap f(B)$; for instance, given $f: \{1,2\} \to \{0\}$ with $A = \{1\}$ and $B = \{2\}$, it holds that $f(A \cap B) = f(\emptyset) = \emptyset$, and $f(A) \cap f(B) = \{0\}$.

A function f is said to be "odd" if $f(-x) = -f(x)$ for every x in the domain of f. A function f is said to be "even" if $f(-x) = f(x)$.

A function $f: X \to Y$ is called "one-to-one" (or "injective," or an "injection," or a "monomorphism") if

$f(x_1) = f(x_2) \Rightarrow x_1 = x_2, \forall x_1, x_2 \in X$.

If more than one elements of X have the same f-image in Y, then the function $f: X \to Y$ is said to be "many-to-one."

A function $f: X \to Y$ is called "into" if there exists at least one element of Y that is not the f-image of an element of X. In other words, for any into function $f: X \to Y$, the range set $f(X)$ is a proper subset of Y, symbolically, $f(X) \subset Y$.

If the range of a function f is the whole codomain of f, then f is said to be "onto" (or "surjective," or a "surjection," or an "epimorphism"). In other words, for any onto function $f: X \to Y, f(X) = Y$.

If a function is both one-to-one and onto, then it is called "bijective," or a "bijection," or an "one-to-one correspondence."

For instance:

i. If A is a subset of X, then the restriction to A of the identity mapping id_x, defined by $A \ni x \to x \in A$, is an injection j_A, called the "natural injection."
ii. The identity mapping of any set is bijective.
iii. The function $f: X \times Y \to Y \times X$ defined by $(x, y) \to (y, x)$, where $x \in X \, and \, y \in Y$, is bijective.
iv. The function $f(x) = x^2$, where $x \in \mathbb{R}$, is not injective, since $f(-x) = f(x) = x^2$, but the restriction to $\mathbb{R}^+$ (the set of all positive real numbers) of f is injective.
v. $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is an one-to-one and onto mapping, that is, a bijection from $\mathbb{R}$ to $\mathbb{R}$.

Let $f: X \to Y$ be a bijection. Because of the fact that f is surjective, it follows that, for every $y \in Y$, $\exists x \in X | y = f(x)$, and, since f is injective, this x is unique. Therefore, there exists an "inverse function" $f^{-1}: Y \to X$ such that $f^{-1}(f(x)) = x, \forall x \in X$, and $f(f^{-1}(y)) = y, \forall y \in Y$.

A function whose domain X and codomain Y are subsets of the set $\mathbb{R}$ of all real numbers is "strictly increasing" if $f(x_1) < f(x_2)\ whenever\ x_1 < x_2$ (in case $f(x_1) \le f(x_2)$, then it is simply "increasing"), and it is "strictly decreasing" if $f(x_1) > f(x_2)\ whenever\ x_1 < x_2$ (in case $f(x_1) \ge f(x_2)$, then it is simply "decreasing"). By definition, it follows that both strictly increasing functions and strictly decreasing functions are injective.

For any subset A' of Y, the subset of X defined by $f(x) \in A'$ is called the "inverse image" of A' by f and is denoted by $f^{-1}(A')$.

Let $g: X \to Y$ and $f: Y \to Z$ be two functions. The "composition" of f and g, denoted by $f \circ g$, is a function from X to Z defined by $(f \circ g)(x) = f(g(x))$. In other words, the function $f \circ g$ assigns to an element $x \in X$ that unique element assigned by f to $g(x)$. In case $g: X \to Y$ and $f: Y \to Z$ are two bijections, then $f \circ g: X \to Z$ is a bijection; and $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$, since:

$$x \in (f \circ g)^{-1}(A) \Leftrightarrow (f \circ g)(x) \in A \Leftrightarrow g(x) \in f^{-1}(A) \Leftrightarrow x \in g^{-1}(f^{-1}(A)), A \subseteq Z.$$

As I mentioned in section 2.1.1, an index set is a set whose members label, or index, members of another set. If A and X are two sets, then a function from A to X is sometimes also called a "family of elements of X having A as a set of indices," and it is written as $h \to x_h$, or $(x_h)_{h \in A}$, or simply (x_h) when no confusion is possible. A characteristic case of such a function is a "sequence," where A is a finite or an infinite subset of the set $\mathbb{N}$ of all natural numbers; in other words, a "sequence" is a function whose domain is a set of consecutive natural numbers. If $f: X \to Y$ is a sequence, then the image $f(i)$ of the natural number i is sometimes written as f_i and is called the "ith" term of the sequence."

Two functions $f: X \to Y$ and $g: X \to Y$ are defined to be "equal" if $f(x) = g(x)\ \forall x \in X$. A function $f: X \to Y$ is defined to be "constant" if $f(x) = c\ \forall x \in X$ where $c \in Y$.

According to the great German mathematician Richard Dedekind (1831–1916), a set A is "infinite" if and only if it is in one-to-to-one correspondence with at least one proper subset of it.[292] Equivalently, we can say that a set A is "infinite" if and only if there exists an one-to-one function $f: A \to A$ that is not onto.

Consider a non-empty family of non-empty sets, say $\hat{A} = \{A_i, i \in I\}$, where to each i corresponds a non-empty set A_i. Additionally, let pr be a function with domain I such that $pr(i) = a_i \in A_i$. The range of a function of this kind is a subset of the set $\cup_{i \in I} A_i$. The collection of all functions $pr: I \to \cup_{i \in I} A_i$ with $pr(i) = a_i \in A_i$ is the Cartesian product $X_i = \Pi_{i \in I} A_i$. If f is an arbitrary element of the product set X_i, and if j is an arbitrary index, then f_j is called the "jth coordinate" of f, and the set A_j is called the "jth factor set" (or the "jth coordinate set") of the product. For each index j, the "$j$thprojection" is defined by $pr_j: \Pi_{i \in I} A_i \to A_j$ with $pr_j(f) = f_j, \forall f \in \Pi_{i \in I} A_i$. For instance, if X and Y are non-empty sets, then the function $pr_1: X \times Y \to X$ defined by $(x, y) \to x$ is the projection onto the first coordinate, while $pr_2: X \times Y \to Y$ defined by $(x, y) \to y$ is the projection onto the second coordinate.

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[292] Dedekind, *Gesammelte mathematische Werke*.

2.1.4. Groups

As I explained in Chapter 1, "abstraction" means getting rid of what we consider unnecessary details (so that, after getting rid of unnecessary details, things that were different because of unnecessary details become identical), and, therefore, we have a non-trivial concept of "identity," on the basis of which we study the "sameness" of certain things, or we look at certain things as if they were the same. "Composition" means that we combine certain abstract objects into bigger abstract objects, so that, when we have to deal with complex problems, we need to be able to divide ("analyze") the bigger problem into smaller problems, solve them separately, and then combine the solutions together. These concepts underpin "operational structuralism," which, in turn, underpins "abstract algebra." This new attitude toward mathematics was initiated in the 1930s by the "Bourbaki school"[293] in France.

In the context of "abstract algebra," we start with a collection of objects, say S, and endow this collection with an algebraic structure by assuming that we can combine, in one or more ways (usually two), elements of S in order to obtain one or more elements of S. These ways of combining elements of S are called "operations" on S. Furthermore, we try to regulate the nature of S by imposing certain rules on the manner in which these operations behave on S, and these rules are called the "axioms" that define the particular structure on$S$. Thus, we obtain the basic algebraic structures, such as that of a group.[294] The abstract-algebraic structure that is called "group" can be used in several different settings, including number theory, geometry, computer science, etc.

Group theory is an outgrowth of number theory and the theory of equations, and it developed in the nineteenth century on the basis of the principle that groups control symmetries and associated geometries. The major pioneers of group theory were the French mathematician Camille Jordan (1838–1922), the German mathematician Felix Klein (1849–1925), and the Norwegian mathematician Sophus Lie (1842–99). From the perspective of group theory, any system that has the following attributes can be called a "group": (i) it contains a set of objects, called the elements of the system under consideration; (ii) it contains an operation, namely, a rule according to which we can combine any two elements of the given system; (iii) the corresponding set (the set of the elements of the system) is closed under the operation, meaning that, if you pick any two elements of the given set and combine them (according to the established operation), then you get another element of the given set; (iv) it has an identity (or neutral) element, namely, an element that has no effect when it is combined with other elements; (v) for every element, there exists an opposite element, which is called the inverse (so that, if you combine any element with its inverse according to the established operation, then you get the identity element); and (vi) it satisfies the associative property, according to which, when combining three elements, it does not matter how you group them. Therefore, the general, formal definition of a group is the following: A "group" is a non-empty set G with an operation (i.e., a law of composition), denoted by $*$, that associates with each pair $(g_1, g_2) \in G$ an element $g_1 * g_2 \in G$ so that:

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[293] "Nicolas Bourbaki" is the collective pseudonym of a group of mathematicians founded in the 1930s. The Bourbaki group's core founders were the prominent French mathematicians Henri Cartan, Claude Chevalley, Jean Delsarte, Jean Dieudonné, and André Weil.

[294] See: Gallian, *Contemporary Abstract Algebra*; Herstein, *Abstract Algebra*; Saracino, *Abstract Algebra*.

(i) composition is associative, namely:
$g_1 * (g_2 * g_3) = (g_1 * g_2) * g_3 \; \forall g_1, g_2, g_3 \in G$;

(ii) there exists an identity element e in G such that
$e * g = g * e = g \; \forall g \in G$;

(iii) to any $g \in G$ corresponds another element g^{-1} (the inverse of g) such that $g * g^{-1} = g^{-1} * g = e$.

A group structure is a formal expression of the degree of symmetry of the underlying object. A subset of a group G is a subgroup if it is a group under the operation defined on G. Every group has two standard subgroups: itself and the trivial group $\{e\}$, the singleton of its identity element.

If $g_1 * g_2 = g_2 * g_1 \; \forall g_1, g_2 \in G$, then G is said to be an "Abelian group," or a "commutative group" (in honor of the nineteenth-century Norwegian mathematician Niels Henrik Abel, who pioneered the study of such groups).

For instance:

i. The set $\mathbb{Z}$ of all integers forms a group under the operation of addition, and, in this case, the identity element is 0, and the inverse of an element is called its negative.
ii. The set $\mathbb{Q} - \{0\}$ of all non-zero rational numbers forms a group under the operation of multiplication, and, in this case, the identity element is 1, and the inverse of an element is called its reciprocal.
iii. Whereas the set $\mathbb{Q}^+$ of all positive rational numbers forms a group under multiplication, the set $\mathbb{Q}^-$ of all negative rational numbers does not form a group under multiplication (since it is not closed under multiplication, and it does not contain an identity element).
iv. The set $\mathbb{R}$ of all real numbers forms a group under addition, and the set $\mathbb{R} - \{0\}$ of all non-zero real numbers forms a group under multiplication.
v. The "Euclidean group": it consists of all the transformations of the plane that do not alter distances (I shall rigorously study the concept of distance in a subsequent section). If the distance between the transformed versions of two points ("image") is the same as the distance between the original two points ("pre-image"), then such a transformation is said to be an "isometry." The isometries of the Euclidean plane form a group under composition of transformations; this is the so-called "Euclidean group." The four major types of isometries are: translation (figure slides in any direction), reflection (figure flips over a line; i.e., a reflection in the plane moves an object into a new position that is a mirror image of the original position, and the "mirror" is a line called the axis of reflection), rotation (figure turns about a fixed point; i.e., a rotation keeps one point, called the center of the rotation, fixed, and it moves all other points a certain angle relative to the fixed point), and glide reflection (it consists of a translation followed by a reflection, and the axis of reflection must be parallel to the direction of the translation).

A group G is said to be a "finite group" if it has a finite number of elements. The number of elements in G is called the "order" of G, and it is denoted by $|G|$. If G is a group with an identity element e, then the "order of an element" $x \in G$ is the smallest positive integer n such that $x^n = e$, and it is denoted by $|x|$. If there is no such n, then x is said to have "infinite order.” For instance, $\mathbb{R}^* \equiv \mathbb{R} - \{0\}$ forms a group under multiplication (we omit zero, because it does not have a multiplicative inverse), and the identity element in this group is 1: the order of 1 is 1, symbolically, $|1| = 1$, since $1^1 = 1$, and, in general, the order of the identity element in any group is 1; the order of -1 is 2, symbolically, $|-1| = 2$, since $(-1)^2 = 1$ (except for 1 and -1, no other non-zero real number can be raised to a positive integral power to get 1, and, therefore, all other real numbers have infinite order in this group).

As I have already mentioned, when we work with groups, we typically use additive notation $(+)$ or multiplicative notation $(\times)$, and, when we use additive notation, the identity element is denoted by 0, whereas, when we use multiplicative notation, the identity element is denoted by 1. Let us consider an arbitrary group G with operation $\times$, and let us pick any element $x \in G$. Then we may study the following problem: what is the smallest subgroup of G that contains x? First, any subgroup of G that contains x must also contain the inverse of x, namely, x^{-1}; second, such a subgroup must contain the identity element, namely, 1; third, this subgroup must contain all powers of x, namely, $x, x^2, x^3, \dots$ (in order to be closed under the group operation); and, fourth, this subgroup must also contain all powers of the inverse of x, namely, $x^{-1}, x^{-2}, x^{-3}, \dots$ (again in order to be closed under the group operation). In fact, this set of all integral powers of x, namely,

$$\{\dots, x^{-3}, x^{-2}, x^{-1}, 1, x, x^2, x^3, \dots\},$$

is the smallest subgroup of G that contains x, it is called the group “generated by x,” and it is denoted by $\langle x \rangle$. If G contains an element x such that G equals the group generated by x, symbolically, $G = \langle x \rangle$, then G is said to be a “cyclic group.” The aforementioned definition can be reformulated using additive notation as follows: Let us consider an arbitrary group H with operation $+$, and let us pick any element $y \in H$. The group generated by y is the smallest subgroup of H containing y, and it must contain: y, its inverse, namely, $-y$, the identity element 0, as well as all positive and negative multiples of y, so that

$$\langle y \rangle = \{\dots, -3y, -2y, -y, 0, y, 2y, 3y, \dots\}.$$

If H can be generated by an element y, symbolically, $H = \langle y \rangle$, then H is said to be a “cyclic group.” For instance, the group of integers ($\mathbb{Z}$) under addition $(+)$ is a cyclic group, since the integers are generated by the number 1, symbolically, $\mathbb{Z} = \langle 1 \rangle$.

As I mentioned in section 2.1.3, $\sim$ denotes an equivalence relation (namely, a relation that satisfies the properties of reflexivity, symmetry, and transitivity). Then, given a group G, a subgroup H of G, and any elements $g_1, g_2 \in G$, we define the equivalence relation $g_1 \sim g_2$ if $g_1 * g_2^{-1} \in H$. Notice that, because $e \in H$ and $e = g * g^{-1}$, it holds that $g \sim g$. Moreover, if $g_1 * g_2^{-1} \in H$, then, because $H \leq G$, $(g_1 * g_2^{-1})^{-1} \in H$. But $(g_1 * g_2^{-1})^{-1} = (g_2^{-1})^{-1} * g_1^{-1} = g_2 * g_1^{-1}$, and, therefore, $g_2 * g_1^{-1} \in H$, which, in turn, implies that $g_2 \sim g_1$. Consequently, $g_1 \sim g_2$ implies that $g_2 \sim g_1$. Finally, if $g_1 \sim g_2$ and $g_2 \sim g_3$, then $g_1 * g_2^{-1} \in H$ and $g_2 * g_3^{-1} \in H$. But $(g_1 * g_2^{-1})(g_2 * g_3^{-1}) = g_1 * g_3^{-1}$, and, therefore, $g_1 * g_3^{-1} \in H$, which, in turn, implies that $g_1 \sim g_3$. As a conclusion, $\sim$ is an equivalence relation on G. Notice that, if G is the group of integers under addition, and if H is the subgroup consisting of all multiples of n, where $n > 1$ is a fixed integer, then $g_1 * g_2^{-1}$ can be interpreted as $g_1 \equiv$

$g_2(n)$, which means that congruence modulo n is a particular case of the aforementioned equivalence relation.

Let H be a subgroup of a group G, symbolically, $H \leq G$. Then, given an arbitrary element g of G, the "right coset" of H is the set $Hg = \{h * g | h \in H\}$, and the "left coset" of H is the set $gH = \{g * h | h \in H\}$.

Theorem[295]*:* Let H be a subgroup of a group G. Then the right cosets Hg form a partition of G.

Proof: Given that $e \in H$, $g = e * g \in Hg$, and, therefore, every element belongs to a coset. In fact, $g \in Hg$. Suppose that Hg_1 and Hg_2 are not disjoint, and that $k \in Hg_1 \cap Hg_2$. In order to prove the theorem, it suffices to prove that, in this case, $Hg_1 = Hg_2$. Because k belongs to both Hg_1 and Hg_2, it holds that $k = h_1 * g_1$ and $k = h_2 * g_2$, where $h_1, h_2 \in H$. Thus, $h_1 * g_1 = h_2 * g_2$, and we obtain $g_1 = h_1^{-1} * h_2 * g_2$. If $x \in Hg_1$, then $x = h_3 * g_1 = h_3 * h_1^{-1} * h_2 * g_2$, where $h_3 \in H$. Because $H \leq G$, $h_3 * h_1^{-1} * h_2 \in H$, and, hence, $x \in Hg_2$. Given that x was chosen to be an arbitrary element of Hg_1, it follows that $Hg_1 \leq Hg_2$. Similarly, it can be shown that $Hg_2 \leq Hg_1$. As a conclusion, $Hg_1 = Hg_2$, which proves the theorem.■

Theorem[296]*:* Let H be a finite subgroup of a group G. Then H and any coset Hg have the same number of elements.

Proof: Let $H = \{h_1, h_2, \dots, h_n\}$, where H has n elements. Then $Hg = \{h_1 * g, h_2 * g, \dots, h_n * g\}$. The fact that $h_i * g = h_j * g$ implies that $h_i = h_j$, and, therefore, the n elements listed in Hg are distinct.■

In general, we can study groups by analyzing them into subgroups, and this process is based on the analysis of cosets. The Italian-French mathematician Joseph-Louis Lagrange (1736–1813) has proved a theorem, known as "Lagrange's Theorem," which narrows down the possible list of subgroups into which a group can be analyzed.

Lagrange's Theorem[297]*:* If H is a subgroup of a finite group G, then the order of H divides the order of G (where the order of a group is the number of elements in the group), symbolically:

$$H \leq G \Rightarrow |H| \; divides \; |G|.$$

Proof: Suppose that H has r elements, i.e., $|H| = r$, and that there exist s distinct right cosets. As we have already proved, the cosets partition G, and the order of each coset (i.e., the number of its elements) is r. Therefore, $|G| = rs$, which proves that $|H| \; divides \; |G|$.■

Let us consider two arbitrary groups G_1 and G_2. In order to compare these two groups, that is, in order to determine how similar these groups are, and in order to clarify the meaning

[295] See: Herstein, *Abstract Algebra*, pp. 67–68.
[296] Ibid.
[297] Ibid.

of "similarity," we use a conceptual tool that is called a "homomorphism."[298] Let us denote the group operation in G_1 by $*$ and the group operation in G_2 by $\diamond$ (in both cases, the operation sign is pronounced "times," but it should not be confused with regular multiplication). Then a "homomorphism" is a function f from G_1 to G_2 such that $f(x * y) = f(x) \diamond f(y)\ \forall x, y \in G_1$;where the operation on the left is the group operation in G_1, and the group operation on the right is the group operation in G_2.

For instance:

i. Consider the function $f: \mathbb{Z} \to \mathbb{Z}$ from the set of all integers to itself defined by $f(x) = 2x$. Assume that, in this case, the group operation is regular addition $+$. Then f is a homomorphism, since $f(x + y) = 2(x + y) = 2x + 2y = f(x) + f(y)$.
ii. Let G be a group of integers under regular addition $+$, and $H = \{1, -1\}$ be the subgroup of the real numbers under multiplication. We define $f(x) = 1$ if x is even, and $f(x) = -1$ if x is odd. Then the statement that f is a homomorphism is equivalent to the statement that:
iii. $even + even = even, even + odd = odd, and\ odd + odd = even.$
iv. If G is the group of real numbers under addition, and if H is the group of positive real numbers under multiplication, then the function $f: G \to H$ defined by $f(x) = 2^x$ is a homomorphism, since $f(x + y) = 2^{x+y} = 2^x 2^y = f(x)f(y)$. Furthermore, because f is also bijective, f is an isomorphism.

A homomorphism $f: G_1 \to G_2$ is called a "monomorphism" if f is one-to-one. A monomorphism that is onto is called an "isomorphism" (i.e., of equal form). If two groups, G_1 and G_2, are isomorphic, then we write $G_1 \cong G_2$. An isomorphism from a mathematical object to itself is called an "automorphism," and it is, in some sense, a symmetry of a given object and a way of mapping a given object to itself while preserving its entire structure.

A homomorphism between two groups may not be an one-to-one function. If it is not one-to-one, then there exists a group that is associated with the given homomorphism and measures the degree to which the function is not one-to-one. This group is called the "kernel." Consider a group G_1 with the operation $*$ and a group G_2 with the operation $\diamond$. Suppose that the homomorphism $f: G_1 \to G_2$ is not one-to-one, and that, therefore, there exist more than one elements $x_1, x_2, \ldots$ of G_1 that map to the element y of G_2, so that

$$\begin{aligned} &f(x_1) = y \\ &f(x_2) = y \\ &\vdots \end{aligned}$$

If we multiply each one of the aforementioned equalities by $f(x_1^{-1})$, then we obtain:

$$\begin{aligned} &f(x_1) = y \Rightarrow f(x_1) \diamond f(x_1^{-1}) = y \diamond f(x_1^{-1}) = y \diamond y^{-1} \\ &f(x_2) = y \Rightarrow f(x_2) \diamond f(x_1^{-1}) = y \diamond f(x_1^{-1}) = y \diamond y^{-1} \\ &\vdots \end{aligned}$$

[298] See: Gallian, *Contemporary Abstract Algebra*; Herstein, *Abstract Algebra*; Saracino, *Abstract Algebra*.

since homomorphisms send inverses to inverses. We can simplify the right-hand side of the aforementioned equalities as follows:

$$f(x_1) = y \Rightarrow f(x_1) \diamond f(x_1^{-1}) = id_{G_2}$$
$$f(x_2) = y \Rightarrow f(x_2) \diamond f(x_1^{-1}) = id_{G_2}$$
$$\vdots$$

where $id_{G_2}$is the identity element in G_2. Moreover, given that f is a homomorphism, the left-hand side of the aforementioned equalities can be rewritten as follows:

$$f(x_1) = y \Rightarrow f(x_1 * x_1^{-1}) = id_{G_2}$$
$$f(x_2) = y \Rightarrow f(x_2 * x_1^{-1}) = id_{G_2}$$
$$\vdots$$

and, therefore, there are several elements of G_1 that all map to the identity element of G_2. These elements are called the “kernel” of the homomorphism $f: G_1 \to G_2$. The formal definition of a kernel for groups is the following: If f is a homomorphism from a group G_1 to a group G_2, then the “kernel” of f is defined by ker $(f) = \{x \in G_1 | f(x) = e_2\}$, where e_2 is the identity element of G_2. In other words, ker (f) measures the degree to which f fails to be one-to-one at one point, e_2. Moreover, notice that, if e_1 is the identity element of group G_1, and if e_2 is the identity element of group G_2, then $f(e_1) = e_2 \Rightarrow e_1 \in$ ker (f), and, given that, for any homomorphism

$f: G_1 \to G_2$, the identity element of G_1 maps to the identity element ofG_2, the kernel of f is never empty, since it contains at least one element, e_1. If the kernel contains only e_1, then f is one-to-one, symbolically, $\ker(f) = \{e_1\} \Rightarrow f is\ a\ monomorphism.$

2.2. NUMBER SYSTEMS, ALGEBRA, AND GEOMETRY

Numbers are abstract objects, concepts, and, simultaneously, they are intimately related to the world, since we organize the world with them (i.e., we count, we measure, and we form scientific theories with numbers). In order to understand the concept of a number, we have to keep in mind that what we count are not “things,” but “sets of things.” The German mathematician, logician, and philosopher Friedrich Ludwig Gottlob Frege (1848–1925) has explained that any number n can be used in order to count any n-membered set. For instance, the number two can be thought of as the set of all 2-membered sets, namely, as the set of all pairs, independently of the nature of the objects that constitute each pair. Similarly, the number three can be thought of as the set of all triples, the number four can be thought of as the set of all quadruples, etc.

In particular, in order to define the concept of a natural number, Frege defined, for every 2-place relation R, the concept “x is an ancestor of y in the R-series,” and this new relation is known as the “ancestor relation on R.”[299] The underlying idea can be easily grasped if we interpret Frege’s 2-place relation R as “x is the father of y in the R-series.” For instance, if

[299] Frege, *Begriffsschrift*, Section 26, Proposition 76.

$a$is the father of b, b is the father of c, and c is the father of d, then Frege's definition of "x is an ancestor of y in the fatherhood-series" ensures that a is an ancestor of b, c, and d, that b is an ancestor of c and d, and that c is an ancestor of d. More generally, given a series of facts of the form aRb, bRc, and cRd, Frege showed that we can define a relation R^* as "y follows x in the R-series." Thus, Frege formulated a rigorous definition of "precedes," and he concluded that a "natural number" is any number of the predecessor-series beginning with 0.

Using the concept of a "predecessor," the American mathematician John von Neumann (1903–57) has proposed an even more accurate definition of a "natural number." According to von Neumann, instead of defining a natural number n as the set of all n-membered sets, a natural number n should be defined as a particular n-membered set, namely, as the set of its predecessors.[300] For instance, the number two having two predecessors, namely, zero and one, we can think of the number two as the set $\{0,1\}$, where zero has no predecessor, and, therefore, it can be thought of as the empty set, denoted by $\emptyset$, and the number one has only one predecessor, namely, zero, and, therefore, we can think of the number one as $\{\emptyset\}$. Thus, von Neumann formulated the modern definition of "ordinal numbers." In particular, given the "successor operation," which is defined as

$$succesor(n) = n \cup \{n\},$$

the set of von Neumann natural numbers, namely, of the ordinal numbers, denoted by ω, is defined as follows:

i. $\emptyset \in \omega$.
ii. If $n \in \omega$, then $succesor(n) \in \omega$.
iii. Nothing belongs to ω unless it can be constructed using the preceding rules.

Thus, we obtain the following definitions:

$$0 = \emptyset.$$
$$1 = successor(0) = \emptyset \cup \{\emptyset\} = \{\emptyset\} = \{0\}.$$
$$2 = successor(1) = \{\emptyset\} \cup \big\{\{\emptyset\}\big\} = \big\{\emptyset, \{\emptyset\}\big\} = \{0,1\}.$$
$$3 = successor(2) = \big\{\emptyset, \{\emptyset\}\big\} \cup \Big\{\big\{\emptyset, \{\emptyset\}\big\}\Big\} = \Big\{\emptyset, \{\emptyset\}, \big\{\emptyset, \{\emptyset\}\big\}\Big\} = \{0,1,2\}.$$
$$\vdots$$

2.2.1. Axiomatic Number Theory

The System of Natural Numbers

By the expression "the system of natural numbers," we mean a structured set $(\mathbb{N}, 0, S) = (\mathbb{N}, (0, S))$ that satisfies the "Peano Axioms,"[301] namely:

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[300] Neumann, "Zur Einführung der transfiniten Zahlen."

[301] Peano, *Formulaire de Mathématiques*. In the aforementioned book, the Italian mathematician and glottologist Giuseppe Peano (1858–1932) expressed fundamental theorems of mathematics in a symbolic language developed by him.

i. 0 is a number, symbolically: $0 \in \mathbb{N}$.
ii. If n is a number, then the successor of n is also a number, namely, S is a function on $\mathbb{N}$, symbolically $S: \mathbb{N} \to \mathbb{N}$.
iii. If two numbers have the same successor, then the two numbers are identical, that is, S is a monomorphism, symbolically $Sn = Sm \Rightarrow n = m$.
iv. 0 is not the successor of any number, symbolically $Sn \neq 0 \; \forall n \in \mathbb{N}$.
v. "Induction Axiom": If X is a set containing both 0 and the successor of every number belonging to X, then every number belongs to X, symbolically: $(\forall X \subseteq \mathbb{N})\big[0 \in X \& \, (\forall n \in \mathbb{N})[n \in X \Rightarrow Sn \in X]\big] \Rightarrow X = \mathbb{N}$.

Remark: In $(\mathbb{N}, 0, S)$, we have:

i. $n \neq 0 \Rightarrow (\exists m \in \mathbb{N})[n = Sm]$.
ii. $(\forall n \in \mathbb{N})[Sn \neq n]$.

For (i), notice that, by the Induction Axiom, the set

$$X = \{n \in \mathbb{N} | n = 0 \, \& (\exists m \in \mathbb{N})[n = Sm]\}$$

is $\mathbb{N}$. Moreover, for (ii), notice that $S0 \neq 0$ (since $Sn \neq n \; \forall n \in \mathbb{N}$) and $Sn \neq n \Rightarrow SSn \neq Sn$ (since S is one-to-one).

Intimately related to the development of axiomatic number theory and logic is the development of algorithmic proof procedures. By the term "algorithm," we mean a step-by-step procedure that defines a set of instructions to be executed in a certain order to get the desired output. One of the most useful, elegant, and simple algorithmic proof procedures is "mathematical induction." The origins of this technique can be traced back to the era of classical Greece (and, in fact, Aristotle was one of its first rigorous exponents), but the term "induction" was coined by De Morgan in the nineteenth century.

Principle of Mathematical Induction[302]

Suppose that P is a proposition defined on the natural numbers $\mathbb{N}$, such that:

i. $P(1)$ is true.
ii. $P(n+1)$ is true whenever $P(n)$ is true.

Then P is true for every natural number. In this case, P is the "inductive hypothesis." By completing the aforementioned two steps of mathematical induction, we prove that P is true for every natural number. Another equivalent form of mathematical induction is the following:

Suppose that P is a proposition defined on the natural numbers $\mathbb{N}$, such that:

i. $P(1)$ is true.
ii. $P(n)$ is true whenever $P(k)$ is true for all $1 \leq k < n$.

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302 See: Balakrishnan, *Introductory Discrete Mathematics*; Kleene, *Introduction to Meta-Mathematics*; Moschovakis, *Notes on Set Theory*; Ram, *Discrete Mathematics*.

Then P is true for every natural number.

Remark: The aforementioned formulation of the principle of mathematical induction begins at $n_0 = 1$ and proves that $P(n)$ is true for all $n \geq 1$. Alternatively, one can begin at any natural number $n_0 = m$ and prove that $P(n)$ is true for all $n \geq m$.

Example 1: Let P be the proposition that the sum of the first n odd numbers is n^2, namely: $P(n) = 1 + 3 + 5 + \cdots + (2n - 1) = n^2$. We can prove that P is true for every natural number $n \in \mathbb{N}$ using mathematical induction as follows:

Basis step: $1 = 1^2$, and, thus, $P(1)$ is true.

Induction step: The nth odd number is $2n - 1$, and the next odd number is $2n + 1$. We assume that $P(n)$ is true, and we add $2n + 1$ to both sides of$P(n)$, obtaining

$$1 + 3 + 5 + \cdots + (2n - 1) + (2n + 1) = n^2 + (2n + 1) = (n + 1)^2,$$

which is $P(n + 1)$. Hence, $P(n + 1)$ is true whenever $P(n)$ is true. By the principle of mathematical induction, P is true for every natural number $n \in \mathbb{N}$.

Example 2: Let P be the proposition that the sum of the first n natural numbers is

$\frac{1}{2}n(n + 1)$, namely: $P(n) = 1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n + 1)$. We can prove that P is true for every natural number $n \in \mathbb{N}$ using mathematical induction as follows:

Basis step: The proposition holds for $n = 1$, because $1 = \frac{1}{2}(1)(1 + 1)$. Hence, $P(1)$ is true.

Induction step: We assume that $P(n)$ is true, and we add $n + 1$ to both sides of$P(n)$, obtaining

$$1 + 2 + 3 + \cdots + n + (n + 1) = \frac{1}{2}n(n + 1) + (n + 1) = \frac{1}{2}[n(n + 1) + 2(n + 1)] = \frac{1}{2}[(n + 1)(n + 2)],$$

which is $P(n + 1)$. Hence, $P(n + 1)$ is true whenever $P(n)$ is true. By the principle of mathematical induction, P is true for every natural number $n \in \mathbb{N}$.

Recursion

Before proceeding with the study of axiomatic number theory, we shall explain the meaning of recursion, which is parallel to the concept of induction, and it plays a very important role in computer science. In particular, V. K. Balakrishnan has clearly explained recursion as follows:

> [Recursion] is the process of solving a large problem by decomposing it into one or more subproblems such that each subproblem is identical in structure to the original problem but more or less simpler to solve. So in both situations, one must (1) decide a set of simple cases

> for which the proof or computation is easily handled, and (2) obtain an appropriate rule that can be applied repeatedly until the end. This concept underlying both induction and recursion can be used to justify the definition of some collection of objects in stages.[303]

For instance, let us consider the recursive definition of a set A of natural numbers divisible by the number 5. In general, a number a is said to be "divisible" by another number b when a third number k can be found such that $a = kb$; and, if this the case, then a is called a "multiple" of b, b is called a "divisor" of a, and k is called the "quotient" of a by b. The recursive definition of a set A of natural numbers divisible by the number 5 can be articulated as follows304:

a. Basis part: $5 \in A$.
b. Inductive/recursive part: $(n \in A) \Rightarrow (n + 5 \in A)$.
c. Closure part: for any object x, $x \in A$ if and only if it is obtained by a repeated application of (a) and (b).

Although Richard Dedekind was the first mathematician to put recursion in a rigorous setting, the first study of recursive definitions goes back to the German linguist and mathematician Hermann Grassmann (1809–77) and the American philosopher and mathematician Charles Sanders Peirce (1839–1914).

Properties of the System of Natural Numbers

Theorem[305]*:* The set $\mathbb{N}$ of all natural numbers is infinite.

Proof: As I have already mentioned, a set is infinite if and only if it is in one-to-to-one correspondence with at least one proper subset of it. Let us consider $\mathbb{N} - \{0\}$, which is a proper subset of $\mathbb{N} = \{0,1,2,3, \dots\}$. We define a function

$$f : \mathbb{N} \to \mathbb{N} - \{0\}$$

such that $f(x) = Sx \; \forall x \in \mathbb{N}$, where Sx denotes the successor of x.

Due to Peano's axiom III, if $x_1, x_2 \in \mathbb{N}$, then
$f(x_1) = f(x_2) \Rightarrow Sx_1 = Sx_2 \Rightarrow x_1 = x_2$. Thus, f is one-to-one.

Furthermore, due to Peano's axiom II, if $x_2 \in \mathbb{N} - \{0\}$, then x_2 must be a successor of some element $x_1 \in \mathbb{N}$. Thus, f is onto.

Because $f : \mathbb{N} \to \mathbb{N} - \{0\}$ is one-to-one and onto, $\mathbb{N}$ is an infinite set.■

The "addition function": The addition function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ given by $(m, n) \to m + n$ is recursively defined by

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[303] Balakrishnan, *Introductory Discrete Mathematics*, p. 20.
[304] Ibid, p. 21.
[305] Balakrishnan, *Introductory Discrete Mathematics*; Kleene, *Introduction to Meta-Mathematics*; Moschovakis, *Notes on Set Theory*; and Ram, *Discrete Mathematics*.

$m + 0 = m,$
$m + Sn = S(m + n).$

The “multiplication function”: The multiplication function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ given by $(m, n) \to m \cdot n$ is recursively defined by

$m \cdot 0 = 0,$
$m \cdot Sn = (m \cdot n) + m.$

The “exponentiation function”: The exponentiation function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ given by $(m, n) \to m^n$ is recursively defined by

$m^0 = 1,$
$m^{n+1} = m^n \cdot m.$

Dedekind[306] achieved the first explicit formulation of the elementary properties of the arithmetic operations from their recursive definitions. In particular, the following laws hold (which can be easily proved by the Induction Axiom):

Associative Law of Addition:$(x + y) + z = x + (y + z).$
Commutative Law of Addition:$x + y = y + x.$
Associative Law of Multiplication:$(x \cdot y) \cdot z = x \cdot (y \cdot z).$
Commutative Law of Multiplication:$x \cdot y = y \cdot x.$
Distributive Law of Multiplication:$x \cdot (y + z) = x \cdot y + x \cdot z.$

By the term “prime numbers,” we refer to those natural numbers with no factors or no divisors other than 1 and themselves (e.g., 2, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, etc. are prime numbers). In the books VII–IX of his seminal *Elements*, Euclid studied number theory in general and prime numbers in particular in a scientifically rigorous and systematic way.[307] In the ninth book of his *Elements* (Proposition 20), Euclid proved that the set of prime numbers is infinite by *reductio ad absurdum* as follows: Suppose that there exist only finitely many prime numbers, say $p_1, p_2, \dots, p_n$ for some natural number n. Consider a number a that is the product of all these prime numbers plus 1, namely, $a = p_1 \cdot p_2 \cdot \dots \cdot p_n + 1$. Then this number a is not divisible by any prime p_i, where $i = 1,2, \dots, n$, since, if you divide a by a prime number, say p_k, then you get a remainder of 1. Hence, there are two possibilities: The first possibility is that a is a prime number, which is impossible, since it contradicts the assumption that the list of all prime numbers is $p_1, p_2, \dots, p_n$. The other possibility is that a is a composite number, and, hence, it must have some prime factors itself, thus leading to a new contradiction, because such a prime factor is a new prime factor that is not included in the list $p_1, p_2, \dots, p_n$. Therefore, either way, we get some new prime numbers, and this fact proves that there are infinitely many prime numbers.

[306] Dedekind, *Gesammelte mathematische Werke*.
[307] Euclid, *The Thirteen Books of Elements*.

The Fundamental Theorem of Arithmetic (known also as the Unique Factorization Theorem)[308]: Every natural number greater than one is either a prime number (i.e., it cannot be exactly divided by any other number apart from 1 and itself) or can be written as a unique product of prime numbers (up to re-arrangement).

Proof: First, we have to prove that, for any natural number $n \geq 2$, there exists a representation as a product of powers of prime numbers, symbolically:

$$n = p_1^{a_1} \cdot \ldots \cdot p_k^{a_k},$$

and next we have to prove that this representation is unique up to reordering. We can prove the existence of such a representation by *reductio ad absurdum* as follows: Suppose that there is a natural number without such a representation. Let m be the smallest number that cannot be factored into a product of powers of prime numbers. Notice that m must be composite; otherwise, it can be represented as a product of powers of prime numbers. Therefore, by definition, $m = a \cdot b$, where $1 < a, b < m$. Since a and b are strictly smaller than m, we can write a and b in the form given, namely, as products of powers of prime numbers, and, consequently, m can be written as a product of powers of prime numbers, too, which is a contradiction. In other words, we have just proved that every natural number $n \geq 2$ can be written as a product of powers of prime numbers. Now, we shall show that this representation of a natural number $n \geq 2$ is unique up to reordering. Suppose that

$$p_1^{a_1} \cdot \ldots \cdot p_k^{a_k} = q_1^{b_1} \cdot \ldots \cdot q_l^{b_l}, \tag{$*$}$$

namely, that the same number can be represented as two products of powers of primes. In this case, we have to prove that $k = l$ and that $p_i = q_i$ for every i. Notice that, for every i, p_i divides the left-hand side of equation $(*)$, and, therefore, p_i divides the right-hand side of equation $(*)$, too. Hence, p_i divides $q_r^{b_r}$ for some r, and, then, p_i divides q_r, too. Given that, if two primes divide each other, they have to be the same, it follows that $p_i = q_r$. Consequently, $k = l$, and $p_i = q_r$ for some r. After some reordering and renaming, we can set $p_i = q_i$, which implies that

$$p_1^{a_1} \cdot \ldots \cdot p_k^{a_k} = p_1^{b_1} \cdot \ldots \cdot p_k^{b_k}. \tag{$**$}$$

By way of contradiction, suppose that $a_i \neq b_i$. Moreover, assume that $a_i > b_i$. Now, we shall divide each side of equation $(**)$ by $p_i^{b_i}$, which obviously divides the right-hand side of equation $(**)$, and, since $a_i \neq b_i$, it obviously divides the left-hand side of equation $(**)$, too, so that we obtain

$$p_1^{a_1} \cdot \ldots \cdot p_{i-1}^{a_{i-1}} \cdot p_i^{(a_i - b_i)} \cdot p_{i+1}^{a_{i+1}} \cdot \ldots \cdot p_k^{a_k} = p_1^{b_1} \cdot \ldots \cdot p_{i-1}^{b_{i-1}} \cdot p_{i+1}^{b_{i+1}} \cdot \ldots \cdot p_k^{b_k}. \tag{$***$}$$

[308] This theorem appeared in Euclid's *Elements* (approx. 300 B.C.). See: Mathews, *Theory of Numbers*, p.3.

Notice that p_i divides the left-hand side of equation $(***)$, since $a_i > b_i$, and, therefore, p_i divides the right-hand side of equation $(***)$, too. Consequently, p_i divides p_j for $i \neq j$, so that we have two primes, p_i and p_j, and p_i divides one of them that is not equal to itself, which is a contradiction. The aforementioned contradiction implies that $a_i = b_i$ (for every i), which, in turn, proves that the representation of any natural number $n \geq 2$ as a product of powers of prime numbers is unique.■

Enumeration

There is no doubt that numbers are applicable to sets. In case of a finite number, the individuals may be enumerated to make up the given number, and such a counting process takes place with no appeal to any set-theoretical concept. Since any finite collection of individuals forms a set, we obtain a number corresponding to that class. When this number is infinite, the individuals cannot be enumerated, but they are determined by means of some common property on the basis of which they are regarded as a "whole" (set), specifically, the number is a property of the given set. In the theory of infinity, it is highly important to determine the conditions under which two sets have the same cardinal number. Cantor was the first mathematician to state the basic definitions about the cardinality of sets.

One way of specifying a finite set is by listing its elements. Obviously, every finite set can be enumerated by putting its elements into a list, which has a beginning, and where each element of the list other than the first has a unique predecessor. Moreover, some infinite sets can also be enumerated, such as the set of $\mathbb{N}$ all natural numbers.

Two sets A and B are "equinumerous" or "have the same cardinality" if their elements can be correlated one-to-one in such a way that each element of either corresponds to exactly one of the other, namely, if there exists a bijection from A to B; then we write $A =_c B$.

Remark: By the above definition, Cantor proposed to accept the existence of an one-to-one correspondence as a characteristic property of equinumerosity, although the application of his intuitions about finite sets to infinite ones may seem to be questionable.[309] Thus, the set $\mathbb{N} = \{0,1,2, \dots\}$ of all natural numbers and its proper subset $\mathbb{N}^* = \mathbb{N} - \{0\} = \{1,2,3, \dots\}$ are equinumerous under the bijection $x \to x + 1$. Moreover, $\{x \in \mathbb{R} | 0 < x < 1\} =_c \{x \in \mathbb{R} | 0 < x < 2\}$ under the bijection $x \to 2x$.

Theorem[310]*:* For any sets A, B, and C,

i. $A =_c A$,
ii. $A =_c B \Rightarrow B =_c A$,
iii. $(A =_c B \& B =_c C) \Rightarrow \mathrm{A} =_c \mathrm{C}$.

Proof: (i) and (ii) are straightforward. For (iii), we can argue as follows: given the bijections $f: A \to B$ and $g: B \to C$, which show that $A =_c B$ and $B =_c C$, respectively, the bijection $g \circ f: A \to C$ shows that $A =_c C$.■

The cardinality of a set$A$ is "less than or equal to" that of a set B if A is equinumerous to a subset of B; then we write $A \leq_c B$.

[309] Cantor, *Gesammelte Abhandlungen*. Moreover, see: Johnstone, *Notes on Logic and Set Theory*; Moschovakis, *Notes on Set Theory*.
[310] Ibid.

Theorem[311]*:* $A \leq_c B$ if and only if there is a monomorphism (i.e., an one-to-one function) $f: A \to B$.

Proof: If $A =_c C \subseteq B$ so that $f: A \to C$ is a bijection, then f is a monomorphism from A to B. Conversely, if $f: A \to B$ is a monomorphism, then $A =_c f(A) \subseteq B$.■

A set A is "countable," or "denumerable," if it is finite or equinumerous to the set $\mathbb{N}$ of all natural numbers; otherwise, A is "uncountable." In particular, a set is said to be "countably infinite" if it can be put in an one-to-one correspondence with $\mathbb{N}$, and a set is said to be "countable" if it is either finite or countably infinite.

Theorem[312]*:* A set A is countable if and only if either $A = \emptyset$ or A accepts an "enumeration," namely, there exists an epimorphism (i.e., an onto function) $\varepsilon: \mathbb{N} \to A$ such that

$$A = \{\varepsilon(0), \varepsilon(1), \varepsilon(2), \dots\}.$$

Proof: Such a function ε determines an enumeration as defined above: $\varepsilon(0), \varepsilon(1), \varepsilon(2), \dots$ Since ε is surjective, every element of A is guaranteed to be the value of $\varepsilon(n)$ for some $n \in \mathbb{N}$. Hence, every element of A appears at some finite position in the list. Since the function may not be injective, the list may be redundant, but that is acceptable. On the other hand, given a list that enumerates all elements of A, we can define a surjective function $\varepsilon: \mathbb{N} \to A$ by letting $\varepsilon(n)$ be the nth element of the list that is not a gap, or the last element of the list if there is no nth element. There is one case in which this does not produce a surjective function, namely, if $A = \emptyset$, and, hence, the list is empty. Therefore, every non-empty list determines an epimorphism $\varepsilon: \mathbb{N} \to A$.■

Theorem[313]*:* The union of a countable collection of countable sets $A = \cup_n A_n$, where $n \in I \subseteq \mathbb{N}$, is a countable set.

Proof: Assume that I is infinite (if I is finite, then we work analogously), so that I can be replaced by $\mathbb{N}$. Then the given countable collection of countable sets may be designated by

$$A = \cup_{n=0}^{\infty} A_n = A_0 \cup A_1 \cup A_2 \cup \dots$$

Without loss of generality, assume that each A_n is non-empty. Then we can find an enumeration $\varepsilon^n: \mathbb{N} \to A_n$ for each A_n . Setting

$$a_i^n = \varepsilon^n(i),$$

we obtain

$$A_n = \{a_0^n, a_1^n, \dots\},$$

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[311] Ibid.
[312] Ibid.
[313] Ibid.

and we can construct a table containing every element of A as follows:

$$\begin{gathered} A_0: a_0^0 a_1^0 a_2^0 \ldots \\ A_1: a_0^1 a_1^1\, a_2^1 \ldots \\ A_2: a_0^2 a_1^2 a_2^2 \ldots \\ \vdots \end{gathered}$$

Therefore, collecting the aforementioned elements diagonally, we obtain

$A = \{a_0^0, a_0^1, a_1^0, a_2^0, a_1^1, \ldots\}$.■

Theorem[314]*:* If the sets $A_1, A_2, A_3, \ldots, A_n$ are countable, then their Cartesian product $A_1 \times A_2 \times \ldots \times A_n$ is a countable set.

Proof: By definition, if $A_i, i = 1,2, \ldots, n$, is empty, then the corresponding Cartesian product is empty. Otherwise, for two sets A and B, we have the enumeration of B given by

$B = \{b_0, b_1, b_2, \ldots\}$,

so that

$A \times B = \cup_{n=0}^{\infty} (A \times \{b_n\})$,

and each $A \times \{b_n\}$ is equinumerous to A (and, therefore, countable) by the correspondence $x \to (x, b_n)$.■

Theorem[315]*:* If A is an arbitrary set and $\wp(A)$ denotes the power set of A, then

$A <_c \wp(A)$.

Proof: We can prove this theorem by *reductio ad absurdum* as follows: First of all, the fact that $A \leq_c \wp(A)$ follows directly from the monomorphism

$A \ni x \to \{x\} \in \wp(A)$,

which assigns to each $x \in X$ the singleton $\{x\}$.

Assume that there exists a bijection

$\varepsilon: A \to \wp(A)$,

namely, that $A =_c \wp(A)$. Notice that, for any $x \in A$, $\varepsilon(x) \subseteq A$. Let

$B = \{x \in A | x \notin \varepsilon(x)\}$, so that $x \in B \Leftrightarrow x \notin \varepsilon(x)\ \forall x \in A$. $(*)$

[314] Ibid.
[315] Ibid.

Since B is a subset of A and ε is surjective, there must exist some $b \in A$ such that $B = \varepsilon(b)$. In $(*)$, we set $x = b$ and $B = \varepsilon(b)$, so that we obtain the contradiction that $b \in B \Leftrightarrow b \notin B$.■

Remark: Since there exists no epimorphism $\varepsilon: A \to \wp(A)$ for any set A, we realize that there exist many orders of infinity, namely:

$$\mathbb{N} <_c \wp(\mathbb{N}) <_c \wp\big(\wp(\mathbb{N})\big) <_c \ldots$$

The proof of the antisymmetry of the relation $\leq_c$ was a problem that attracted the interest of several mathematicians, including Cantor, Richard Dedekind, Felix Bernstein, and Ernst Schröder. The proof was given by Cantor's student Felix Bernstein in 1897 (a slightly simplified version of Bernstein's proof can be found in Émile Borel's book *Leçons sur la Théorie des Fonctions*, Paris: Gauthier-Villars, 1898). Bernstein's proof is based on the following lemma:

Lemma[316]*:* If $X \supseteq Y \supseteq X_1$ and $X =_c X_1$, then $X =_c Y$.

Proof: Because $X =_c X_1$, there exists an one-to-one correspondence $\alpha: X \to X_1$. But $X \supseteq Y$, so that the restriction to Y of α is one-to-one. Thus, Y is equinumerous to a subset, say Y_1, of X_1, where $X \supseteq Y \supseteq X_1 \supseteq Y_1$, and $\alpha: Y \to Y_1$ is one-to-one and onto. By analogy, we obtain $X \supseteq Y \supseteq X_1 \supseteq Y_1 \supseteq X_2$, where $X_1 =_c X_2$, and $\alpha: X_1 \to X_2$ is one-to-one and onto. Repeating the same process, we realize that there exist equinumerous sets $X_1, X_2, X_3, \ldots$ and equinumerous sets $Y_1, Y_2, Y_3, \ldots$ such that

$$X \supseteq Y \supseteq X_1 \supseteq Y_1 \supseteq X_2 \supseteq Y_2 \supseteq \cdots$$

Let $B = X \cap Y \cap X_1 \cap Y_1 \cap X_2 \cap Y_2 \cap \ldots$,

so that

$$X = (X - Y) \cup (Y - X_1) \cup (X_1 - Y_1) \cup \ldots \cup B,$$
$$Y = (Y - X_1) \cup (X_1 - Y_1) \cup (Y_1 - X_2) \cup \ldots \cup B.$$

Notice that

$$(X - Y) =_c (X_1 - Y_1) =_c (X_2 - Y_2) =_c \ldots$$

by the bijection $\alpha: (X_n - Y_n) \to (X_{n+1} - Y_{n+1})$.

If we define a function g such that

$$g(x) = \begin{cases} \alpha(x) \; if \; x \in X_i - Y_i \; or \; x \in X - Y \\ x \; if \; x \in Y_i - X_i \; or \; x \in B \end{cases},$$

[316] See: Borel, *Leçons sur la Théorie des Fonctions*.

then $X =_c Y\ by\ g.$■

Bernstein's Equinumerosity Theorem[317]*:* If $X \leq_c Y$ and $Y \leq_c X$, then $X =_c Y$. In other words, if there exist injections $f: X \to Y$ and $g: Y \to X$, then there exists a bijection $h: X \to Y$.

Proof: Let f and g be one-to-one functions from X to Y and from Y to X, respectively. If we let $f(X) = Y_1 \subseteq Y$, $g(Y) = X_1$, and $g(Y_1) = X_2$, then $X \supseteq X_1 \supseteq X_2$. Additionally, $g\big(f(X)\big) = X_2$, namely, $g \circ f$, is an one-to-one function from X onto X_2. Therefore, $X =_c X_2$, so that, by the aforementioned lemma, $X =_c X_1$. But, since g maps Y one-to-one onto X_1, it holds that $X_1 =_c Y$.■

Cantor has explained the "cardinal number" of a set M as the general concept emanating from M after a double abstraction: first, we ignore the special nature of the elements m of M, and, second, their order in M.[318] This double abstraction gives the cardinal number of M, denoted by $\bar{\bar{M}}$. Since each element m of M has become an abstract "individual," the cardinal number $\bar{\bar{M}}$ is a set consisting of individuals, and such a number can be understood as an intellectual projection of the set M.

According to the aforementioned reasoning, Cantor concluded that, for any sets A and B,

$A =_c \bar{\bar{A}}$,
$A =_c B \Leftrightarrow \bar{\bar{A}} = \bar{\bar{B}}$.

Moreover, Cantor has argued that, for every family $\mathbb{E}$ of sets, the class

$$\{\bar{\bar{X}} | X \in \mathbb{E}\}$$

is a set. According to Cantor, $\bar{\bar{A}}$ is a set of "monads" that is equinumerous to A.

In modern notation, Cantor's theory of cardinal numbers gives rise to the following problem: Define an operator $|A|$ on the class of all sets such that the following conditions are satisfied:

Condition (C1): $A =_c |A|$,
Condition (C2): $A =_c B \Leftrightarrow |A| = |B|$,
Condition (C3): for every set $\mathbb{E}$, $\{|X| | X \in \mathbb{E}\}$ is a set.

A rigorous solution to the above problem was given by John von Neumann. By a "weak cardinality operator," we mean any definite operator $|A|$ that satisfies the aforementioned conditions (C1) and (C3). Thus, "cardinal numbers" are the values of

$Card(\kappa) \Leftrightarrow \kappa \in Card \Leftrightarrow (\exists A)[\kappa = |A|]$.

[317] Ibid.
[318] Cantor, "Beiträge zur Begründung der transfiniten Mengenlehre," p. 481.

If, in addition, a cardinality operator $|A|$ satisfies the aforementioned condition (C2), namely, $(\forall\kappa \in Card)(\forall\lambda \in Card)[\kappa =_c \lambda \Leftrightarrow \kappa = \lambda]$, then $|A|$ is said to be a "strong cardinality operator."

Having a cardinality operator, we define the following operations on cardinal numbers:

$$\kappa + \lambda = |\kappa \uplus \lambda| =_c \kappa \uplus \lambda,$$
$$\kappa \cdot \lambda = |\kappa \times \lambda| =_c \kappa \times \lambda,$$
$$\kappa^\lambda = |(\lambda \to \kappa)| =_c (\lambda \to \kappa).$$

Thus, the following results of cardinal arithmetic are easily established[319]:

$$\kappa + 0 =_c \kappa, \kappa \cdot 0 =_c 0, \kappa \cdot 1 =_c \kappa,$$
$$\kappa + (\lambda + \mu) =_c (\kappa + \lambda) + \mu,$$
$$\kappa \cdot (\lambda \cdot \mu) =_c (\kappa \cdot \lambda) \cdot \mu,$$
$$\kappa \cdot \lambda =_c \lambda \cdot \kappa,$$
$$\kappa \cdot (\lambda + \mu) =_c \kappa \cdot \lambda + \kappa \cdot \mu,$$
$$|\wp(\kappa)| =_c 2^\kappa,$$
$$\kappa^0 =_c 1, \kappa^1 =_c \kappa, \kappa^2 =_c \kappa \cdot \kappa,$$
$$(\kappa \cdot \lambda)^\mu =_c \kappa^\mu \cdot \lambda^\mu,$$
$$\kappa^{\lambda+\mu} =_c \kappa^\lambda \cdot \kappa^\mu,$$
$$(\kappa^\lambda)^\mu =_c \kappa^{\lambda \cdot \mu},$$
$$\kappa \leq_c \mu \Rightarrow \kappa + \lambda \leq_c \mu + \lambda,$$
$$\kappa \leq_c \mu \Rightarrow \kappa \cdot \lambda \leq_c \mu \cdot \lambda,$$
$$\lambda \leq_c \mu \Rightarrow \kappa^\lambda \leq_c \kappa^\mu,$$
$$\kappa \leq_c \lambda \Rightarrow \kappa^\mu \leq_c \lambda^\mu.$$

Cantor denoted the cardinal number of $\mathbb{N}$ by the first letter of the Hebrew alphabet aleph-naught:

$$\aleph_0 = |\mathbb{N}|.$$

For every cardinal number κ and for every $n \in \mathbb{N}$,

$$\kappa^n = \left|\kappa^{(n)}\right|.$$

Remarks: $\mathbb{N}^n =_c \mathbb{N}^1$, that is, $\aleph_0^n = \aleph_0$, for all $n \in \mathbb{N}$. The cardinal number of the set of all the n-tuples of natural numbers is $\aleph_0$, because $m \to (m, 0, \dots, 0)$ is an one-to-one function from $\mathbb{N}$ to $\mathbb{N}^n$, and, if $2, 3, \dots, p_n$ are the first n prime numbers, then the Fundamental Theorem of Arithmetic implies that $(m_1, \dots, m_n) \to 2^{m_1} \cdot \dots \cdot p_n^{m_n}$ is an one-to-one function from $\mathbb{N}^n$ to $\mathbb{N}$.

[319] See: Schimmerling, *A Course on Set Theory*, Chapter 4; Suppes, *Axiomatic Set Theory*, Chapter 4.

Order in $\mathbb{N}$ ***and Ordinal Numbers***

If $a, b \in \mathbb{N}$, then we say that "a is less than b" if and only if there exists an $n \in \mathbb{N}$ such that $a + n = b$, and then we write $a < b$ or, equivalently, $b > a$, which is read "b is greater than a." According to the transitivity of order relation, if $a > b$ and $b > c$, then $a > c$. Moreover, according to the compatibility of order relation with addition and multiplication, $a > b \Rightarrow a + c > b + c$ and $a > b \Rightarrow ac > bc$, for any $a, b, c \in \mathbb{N}$.

The Law of Trichotomy: For any two natural numbers a and b, one and only one of the following holds:

i. $a = b$,
ii. $a < b$,
iii. $a > b$.

In general, "order" is one of the most significant concepts in set theory. In order to understand the meaning of order, we must study the manner in which order emerges; there exist two ways (in fact, the second way is reducible to the first one), namely[320]:

i. Given three terms, "ordinal elements," a, b, and c, one of them, say b, is "between" the other two. Then "between" is a relation of one term b to two others a and c, and it holds whenever there exists some relation from a to b and from b to c but not from b to a, nor from c to b, nor from c to a; that is, if b is between a and c, then it is impossible to have a between b and c or c between a and b.
ii. Given four terms, "ordinal elements," a, b, c, and d, then a and c are "separated" by b and d. In this case, there exists an asymmetrical relation that holds between a and b, b and c, c and d, or between a and d, d and c, c and b, or between c and d, d and a, a and b.

In order to establish an order, one may work as follows. Consider a finite or an infinite collection of terms in such a way that there exists a certain asymmetrical relation from each term (with the possible exception of exactly one term) to exactly one other term of the collection as well as a relation that is the inverse of the previous relation from every term (with one possible exception, different from the previous exceptional term) to exactly one other term of the collection. Let us denote such a relation by R and its inverse by R^{-1}. Assume that, if aRb and bRc, then cRa. Thus, with the two mentioned possible exceptions, every term of the collection has one relation to a second term, and the inverse relation to a third term, but these terms themselves do not have to each other either of these relations. Then the first term is between the second and the third terms. The term to which a given term has one of the two mentioned relations is called the ("immediate") "successor" of the given term, while the term to which the given term has the inverse relation is called the ("immediate") "predecessor" of the given term.

[320] See: Russell, *The Principles of Mathematics*.

Any arrangement of a set of n objects in a given order is called a "permutation" of the objects (taken all at a time). The number of permutations of n objects taken r at a time is usually denoted by

$$P(n, r),$$

and $P(n, r) = \frac{n!}{(n-r)!} = n(n-1)(n-2) \ldots (n-r+1)$,

where $n!$, called "n factorial," is the product of all natural numbers less than or equal to n (the value of 0! is 1). Obviously, there are $n!$ permutations of n objects taken all at a time.

As in the case of cardinal numbers, Cantor[321] has proposed another kind of abstraction in order to define "ordinal numbers." Each ordered set U has an ordinal number $\overline{U}$, which may be regarded as the general concept that emanates from U when we ignore the particular nature of the elements $u \in U$ and consider only their order. Thus, $\overline{U}$ is also an ordered set whose members are individuals that preserve the order of the members of U from which they have emerged by abstraction. Two ordered sets have the same ordinal number if they are similar, so that

$$U =_o V \Leftrightarrow \overline{U} = \overline{\mathrm{V}}.$$

Consider a "structured set" (or "space"), namely, a pair $U = (A, S)$ where $A = Field(U)$ is a set, the field of U, and S is any object, the structure of the space (A, S). A "well ordered set" (or "well ordered space") is a structured set

$$U = (Field(U), \leq_U),$$

where $\leq_U$ is the corresponding order on $Field(U)$, namely, a linear order such that every non-empty $X \subseteq Field(U)$ has a minimum element. For instance, the set $\mathbb{N}$ of all natural numbers is well order under its natural order. The fundamental problem of Cantor's theory of ordinal numbers is to assign a unique well ordered set $\overline{U}$ to each well ordered set U in such a way that

$$U =_o \overline{U} \; and \; U \leq_o V \Rightarrow \overline{U} \subseteq \overline{V}.$$

The answer given by John von Neumann[322] is based on defining $\overline{U}$ by recursively substituting each member of U with a set of its predecessors.

By the term "transfinite induction," we refer to an extension of mathematical induction (expounded in section 2.2.1) to well ordered sets (e.g., to sets of ordinal numbers or cardinal numbers).

Principle of Transfinite Induction[323]*:* For every well ordered set U and for every definite condition P in one variable,

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[321] Cantor, *Gesammelte Abhandlungen*.
[322] Neumann, "Zur Einführung der transfiniten Zahlen."

$$(\forall y \in U)[(\forall x < y)P(x) \Rightarrow P(y)] \Rightarrow (\forall y \in U)P(y).$$

Division

For any two natural numbers a and b, there exists a unique natural number n such that $a \cdot n = b$ if and only if a is a divisor of b, and then we write $n = b \div a$. The greatest common divisor (denoted by gcd) of two natural numbers a and b is the largest natural number that divides both a and b, and the Euclidean Algorithm for computing $gcd(a, b)$ is as follows:

i. If $a = 0$, then $gcd(a, b) = b$.
ii. If $b = 0$, then $gcd(a, b) = a$.
iii. If a and b are both non-zero natural numbers, then we write a in quotient remainder form, namely, $a = b \cdot q + r$, and, subsequently, we compute $gcd(b, r)$ using the Euclidean Algorithm since $gcd(a, b) = gcd(b, r)$. For instance, if $a = 280$ and $b = 120$, then we can compute $gcd(a, b)$ as follows: first, we use long division to find that $\frac{280}{120} = 2\, witharemainderof\ 40$, which can be written as $280 = 120 \times 2 + 40$; second, we compute $gcd(120{,}40) = 3\, witharemainderof\ 0$; and, therefore, $gcd(280{,}120) = 40$.

Let a and b be both non-zero natural numbers. Moreover, let $lcm(a, b)$ denote the least common multiple of a and b (i.e., $lcm(a, b)$ is the smallest natural number that is evenly divisible by both a and b). Then

$$gcd(a, b) = \frac{a \cdot b}{lcm(a, b)} \Leftrightarrow lcm(a, b) = \frac{a \cdot b}{gcd(a, b)}.$$

2.2.2. The Set of Integral Numbers

Let us consider the equation $x + 3 = 2$. This is an example of an equation that has no solution in $\mathbb{N}$ (in this case, $x = -1$). Therefore, we need a new kind of numbers, that is, we need to extend the set $\mathbb{N}$ of all natural numbers. The extended system will be the set $\mathbb{Z}$ of all integral numbers.[324] $\mathbb{N}$ is a proper subset of $\mathbb{Z}$. Moreover, $\mathbb{Z}$ is countable, meaning that both $\mathbb{N}$ and $\mathbb{Z}$ have the same order of infinity (cardinality). In fact, $\mathbb{Z}$ can be written as the union of two countable sets as follows:

$$\mathbb{Z} = \mathbb{N} \cup \{-1, -2, -3, \dots\},$$

and the set of all negative integers is countable under the bijection $x \to -(x + 1)$.

Let us consider the equivalence relation $(a, b) \sim (c, d)$ defined by $a + d = b + c$ in $\mathbb{N} \times \mathbb{N}$. Given this equivalence relation and the element$(a, b) \in \mathbb{N} \times \mathbb{N}$, we can define the corresponding equivalence class $\overline{(a, b)} = \{(x, y) \in \mathbb{N} \times \mathbb{N} | (x, y) \sim (a, b)\}$. Then the

[323] See: Campbell, *The Structure of Arithmetic*; Dummit and Foote, *Abstract Algebra*; Gallian, *Contemporary Abstract Algebra*; Mendelson, *Number Systems and the Foundations of Analysis*; Moschovakis, *Notes on Set Theory*.

[324] Ibid.

equivalence classes $\overline{(a,b)}, \overline{(c,d)}, \ldots$ are called "integral numbers," or simply "integers," and the quotient set $\mathbb{N} \times \mathbb{N}/\sim$ is called the set $\mathbb{Z}$ of all integers.

The "equality" of two integers is defined as follows: $\overline{(a,b)} = \overline{(c,d)}$ if $(a,b) \sim (c,d)$ or if $a + d = b + c$. The "sum" of two integers $\overline{(a,b)}$ and $\overline{(c,d)}$ is defined as follows: $\overline{(a,b)} + \overline{(c,d)} = \overline{(a+c, b+d)}$, and their "product" is defined by $\overline{(a,b)} \cdot \overline{(c,d)} = \overline{(a \cdot c + b \cdot d, a \cdot d + b \cdot c)}$, where $+$ and $\cdot$ denote the standard operations of addition and multiplication, respectively, in $\mathbb{N}$. The equivalence class $\overline{(a,a)}$ defines $0 \in \mathbb{Z}$, that is, $\overline{(a,a)} = 0 \ \forall a \in \mathbb{N}$.

Because $\overline{(a,b)} + \overline{(b,a)} = \overline{(a+b, b+a)} = \overline{(a+b, a+b)} = 0$, it follows that the equivalence class $\overline{(b,a)}$ is the "negative" of $\overline{(a,b)}$, that is, $\overline{(b,a)} = -\overline{(a,b)}$. The equivalence class $\overline{(n+b, b)}$ where n is a fixed natural number, and b is an arbitrary natural number denotes a "positive integer." The equivalence class $\overline{(a, n+a)}$ where n is a fixed natural number, and a is an arbitrary natural number denotes a "negative integer." Hence, $\overline{(b, n+b)} = -\overline{(n+b, b)}$.

We can easily verify that the set $\mathbb{Z}$ of all integers has the following main properties of operation ($x, y, z \in \mathbb{Z}$):

i. Addition in $\mathbb{Z}$ is
commutative: $x + y = y + x$ and
associative: $x + (y + z) = (x + y) + z$.
ii. Multiplication in $\mathbb{Z}$ is
commutative: $x \cdot y = y \cdot x$ and
associative: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.
iii. Multiplication in $\mathbb{Z}$ distributes over addition:
$x \cdot (y + z) = x \cdot y + x \cdot z$.
iv. Cancellation laws
for addition: $x + z = y + z \Rightarrow x = y$ and
for multiplication: if $z \neq 0$, then $x \cdot z = y \cdot z \Rightarrow x = y$.

Furthermore, it is worth pointing out that, if the operation $*$ stands for the ordinary addition, denoted by $+$, in $\mathbb{Z}$, then $\mathbb{Z}$ is a group under $*$ (see section 2.1.4). Indeed, the fact that $\mathbb{Z}$ is closed and associative under $*$ follows directly from the basic properties of integers. The identity element e of this group (under addition) is 0, since $a = a * e = a + e$. Moreover, in this case, the inverse element is $a^{-1} = -a$, since $e = 0 = a * a^{-1} = a + a^{-1}$, and $a * (-a) = a + (-a) = 0$. Obviously, $\mathbb{Z}$ is not a group under the ordinary multiplication.

Order in $\mathbb{Z}$: Let $\overline{(a,b)}$ and $\overline{(c,d)}$ be two arbitrary integers. Then the "order relation" in $\mathbb{Z}$ is denoted by $<$ (read "less than"), and it is defined as follows:

$$\overline{(a,b)} < \overline{(c,d)} \ if \ a + d < b + c,$$

where $<$ is the relation of "less than" as defined in $\mathbb{N}$. The "order relation" in $\mathbb{Z}$ satisfies the following properties:

i. $<$ is transitive in $\mathbb{Z}$.
ii. For any two integers a and b, one and only one of the following holds: $a = b$, or $a < b$, or $b < a$.
iii. $a < b \Rightarrow a + c < b + c$.
iv. If $0 < c$, then $a < b \Rightarrow a \cdot c < b \cdot c$; and,
if $c < 0$, then $a < b \Rightarrow b \cdot c < a \cdot c$.

In mathematics, by the term "embedding," we refer to one instance of some mathematical structure contained within another instance, such as a group that is a subgroup.[325] When an object X is said to be embedded in another object Y, then the embedding is defined by an injective, structure-preserving function $f: X \to Y$, and the precise meaning of "structure-preserving" depends on the kind of mathematical structure of which X and Y are instances (e.g., groups). The set $\mathbb{N}$ is embedded in the set $\mathbb{Z}$ due to the embedding $f: \mathbb{N} \to \mathbb{Z}$ defined by $f(x) = \overline{(x+1,1)}$. This function is injective, and it preserves the operations $+$ and $\cdot$ as well as the order relation $\leq$ (i.e., it is "structure-preserving").

2.2.3. The Set of Rational Numbers

Let us consider the equation $mx = n$ where $m, n \in \mathbb{Z}$. The solution of this equation, namely, $x = \frac{n}{m}$, may not belong to $\mathbb{Z}$ (e.g., in case $x = \frac{2}{3}$). Therefore, we need a new kind of numbers, that is, we need to extend the set $\mathbb{Z}$ of all integral numbers. The extended system will be the set $\mathbb{Q}$ of all rational numbers.[326] It is constructed from $\mathbb{Z}$ as follows: Let us consider the set $W = \mathbb{Z} \times \mathbb{Z} - \{0\} = \{(a,b) | a, b \in \mathbb{Z}, b \neq 0\}$. Then let us define the following equivalence relation in W: $(a,b) \sim (c,d)$ if $ad = bc$, where $(a,b), (c,d) \in W$. The equivalence relation $\sim$ partitions the set W into a set of equivalence classes $\left\{\overline{(a,b)}, \overline{(c,d)}, \ldots\right\}$ where $\overline{(a,b)} = \{(x,y) \in W | (x,y) \sim (a,b)\}$. The equivalence classes $\overline{(a,b)}, \overline{(c,d)}, \ldots$ are called "rational numbers," and the quotient set $W/\sim$ is called the set $\mathbb{Q}$ of all rational numbers.

The "equality" of two rational numbers is defined as follows: $\overline{(a,b)} = \overline{(c,d)}$ if $(a,b) \sim (c,d)$ or if $ad = bc$. The "sum" of two rational numbers $\overline{(a,b)}$ and $\overline{(c,d)}$ is defined as follows: $\overline{(a,b)} + \overline{(c,d)} = \overline{(ad+bc, bd)}$, and their "product" is defined as $\overline{(a,b)} \cdot \overline{(c,d)} = \overline{(a \cdot c, b \cdot d)}$. Because $\overline{(a,b)} + \overline{(0,n)} = \overline{(an,bn)} = \overline{(a,b)}$ and $\overline{(a,b)} \cdot \overline{(n,n)} = \overline{(an,bn)} = \overline{(a,b)}$, it follows that $\overline{(0,n)}$ is the additive identity element, and $\overline{(n,n)}$ is the multiplicative identity element.

Because $\overline{(a,b)} + \overline{(-a,b)} = \overline{(0,bb)} = \overline{(0,n)}$, namely, the additive identity element, and $\overline{(a,b)} \cdot \overline{(b,a)} = \overline{(ab,ba)} = \overline{(n,n)}$, namely, the multiplicative identity element, it follows that the additive inverse of $\overline{(a,b)}$ is $\overline{(-a,b)}$, and the multiplicative inverse of $\overline{(a,b)}$ is $\overline{(b,a)}$ with $a \neq 0$. The additive inverse of $\overline{(a,b)}$ is denoted by $-\overline{(a,b)}$, and the multiplicative inverse of $\overline{(a,b)}$ is denoted by $\overline{(a,b)}^{-1}$, and multiplicative inverse exists only in $\mathbb{Q} - \{0\}$.

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[325] Ibid.
[326] Ibid.

It can be easily verified that both addition and multiplication in $\mathbb{Q}$ are commutative and associative, and that multiplication in $\mathbb{Q}$ distributes over addition.

"Subtraction" in $\mathbb{Q}$ is defined as

$$x - y = x + (-y) \ \forall x, y \in \mathbb{Q}.$$

"Division" in $\mathbb{Q}$ is defined as

$$x \div y = x \cdot y^{-1} \ \forall x \in \mathbb{Q} \ and \ y \in \mathbb{Q} - \{0\}.$$

If the operation $*$ is the ordinary addition of rational numbers, then it can be easily verified that $\mathbb{Q}$ is a group under $*$. Notice that $\mathbb{Z} \subset \mathbb{Q}$, and that both $\mathbb{Z}$ and $\mathbb{Q}$ are groups under the same operation $*$. Moreover, if the operation $*$ is the ordinary multiplication of rational numbers, then it can be easily verified that $\mathbb{Q} - \{0\}$ is a group under $*$.

Order in $\mathbb{Q}$: Let us denote $\overline{(a,b)}$ by $\frac{a}{b}$ and $\overline{(c,d)}$ by $\frac{c}{d}$. Then the order relation in $\mathbb{Q}$ is defined as follows:

$\frac{a}{b} < \frac{c}{d} \ if \ ad < bc$ and
$\frac{a}{b} > \frac{c}{d} \ if \ ad > bc,$

where the relations $<$ and $>$ on the right-hand side are the order relations in $\mathbb{Z}$. For every rational number $\overline{(a,b)}$, namely $\frac{a}{b}$, one and only one of the following holds: $\frac{a}{b} = 0$, or $\frac{a}{b} > 0$, or $\frac{a}{b} < 0$.

The set $\mathbb{Z}$ is embedded in $\mathbb{Q}$ due to the embedding $f: \mathbb{Z} \to \mathbb{Q}$ defined by $f(x) = \overline{(x,1)}$. This function is injective, and it preserves the operations $+$ and $\cdot$ as well as the order relation $\leq$ (i.e., it is "structure-preserving").

The set $\mathbb{Q}$ of all rational numbers is countable. In fact, $\mathbb{Q}$ can be written as the union of two countable sets as follows:

$$\mathbb{Q} = \mathbb{Q}^- \cup \mathbb{Q}^+.$$

$\mathbb{Q}^+$ is countable, because $\mathbb{Q}^+ = \cup_{n=1}^{\infty} \left\{\frac{m}{n} | m \in \mathbb{N}\right\}$, and each $\left\{\frac{m}{n} | m \in \mathbb{N}\right\}$ is countable by $m \to \frac{m}{n}$; by analogy, we can prove that $\mathbb{Q}^-$ is countable, too.

2.2.4. The Set of Real Numbers

Let us consider the equation $x^2 = 2$. This is an example of an equation that has no solution in $\mathbb{Q}$ (in this case, $x = \pm\sqrt{2}$, and, as I showed in section 2.1.2, $\sqrt{2}$ is not a rational number). Therefore, we need a new kind of numbers, that is, we need to extend the set $\mathbb{Q}$ of all rational numbers. This process leads us to define the set $\mathbb{Q}^{\sim}$ of all "irrational numbers," and, thus, to extend $\mathbb{Q}$ to the set $\mathbb{R}$ of all "real numbers," where $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^{\sim}$.

Dedekind has defined a "real number" as a set of rational numbers that satisfy the following properties:

i. If x belongs to α, where α is a real number, and y is a rational number $< x$, then y belongs to α.
ii. $\alpha \neq \emptyset$.
iii. $\alpha \neq \mathbb{Q}$.
iv. There exists no maximal element in α, namely:
$x \in \alpha \Rightarrow (\exists y \in \alpha)[y > x]$.

For instance, $\{a \in \mathbb{Q} | a^2 < 2 \ or \ a < 0\}$ is the real number denoted by $\sqrt{2}$; since $\sqrt{2}$ partitions $\mathbb{Q}$into the following two infinite sets:

$A = \{a \in \mathbb{Q} | a^2 < 2 \ or \ a < 0\}$ and $B = \{b \in \mathbb{Q} | b^2 > 2 \ and \ b > 0\}$,

so that, according to Dedekind's terminology, the "cut" (A, B) defines $\sqrt{2}$. Actually, once we know A, the complement of A, namely, B, is determined, since $A \cup B = \mathbb{Q}$, and, therefore, the information contained in the pair (A, B) is, in a sense, also contained just in the set A, for which reason $\sqrt{2}$ can be more economically defined as $\{a \in \mathbb{Q} | a^2 < 2 \ or \ a < 0\}$.

Dedekind Algebra

By the term "Dedekind algebra," we mean a system (ω, s), where the elements of ω are called "natural numbers," and the function s is called the "successor function" on ω.[327]

Richard Dedekind,[328]in his *Stetigkeit und irrationale Zahlen* (1872), made an in-depth study of real numbers and continuity. He began with the following three properties of rational numbers:

i. If $a > b$ and $b > c$, then $a > c$.
ii. If a and c are two distinct (rational) numbers, then there exist infinitely many distinct numbers lying between a and c.
iii. If a is any definite (rational) number, then all numbers of the system $\mathbb{Q}$ fall into two classes, A_1 and A_2, each of which contains infinitely many individuals; A_1 contains all numbers a_1 that are $< a$, while A_2 contains all numbers a_2 that are $> a$; the number a itself may be assigned at pleasure to A_1 or A_2, being, respectively, the greatest number of A_1 or the least number of A_2.

Then Dedekind stated three properties of the points on a straight number line L:

i. If p lies to the right of q and q to the right of r, then p lies to the right of r; and q is said to lie between p and r.

[327] See: Potter, *Sets*, p. 68.
[328] Dedekind, *Gesammelte mathematische Werke*.

ii. If p and r are two distinct points, then there always exist infinitely many points lying between p and r.
iii. If p is a definite point in L, then all points in L fall into two classes, P_1 and P_2, each of which contains infinitely many individuals; P_1 contains all the points p_1 that lie to the left of p, while P_2 contains all the points p_2 that lie to the right of p; the point p itself may be assigned at pleasure to P_1 or P_2. In any case, every point of P_1 lies to the left of every point of P_2.

Each such division (or partition) of the set $\mathbb{Q}$ of all rational numbers defines a "cut," called the "Dedekind's cut."

However, after having observed that every rational number effects a "cut" in the set of rationals, Dedekind considered the inverse question, namely: if, by a given criterion, the set of rationals is divided into two subsets A and B so that every number in A is less than every number in B, is there always a greatest rational in A or a smallest rational in B? Dedekind immediately realized that the number line should be "continuous," or unbroken, in the intuitive sense, and, like Eudoxus and Cantor before him, he developed theoretical concepts for the purpose of filling the gaps in the ordered set of rationals so that the final geometric picture is a continuous, straight number line. However, the answer to the last question is in the negative: when A has no maximum rational and B has no minimum rational, there is, indeed, a gap in the rational series, that is, a puncture in the number line, which must be filled. In that case, the cut (A, B) is said to define (or to be) an irrational number.

Given a Dedekind's cut (A, B), let us consider the aforementioned four possibilities:

i. Let m be the greatest rational number in the left-hand class A, and n be the smallest rational number in the right-hand class B. Then either $m = n$ or $m < n$. But $m = n$ is not possible, because, according to the definition of a Dedekind's cut, every number in the left-hand class A is less than every number in the right-hand class B. Moreover, we cannot have $m < n$, because the rational number $\frac{1}{2}(m + n)$, which is greater than m, belongs to B and is less than n, and, therefore, it would also belong to A, which contradicts the definition of a Dedekind's cut (according to which, every rational number is in one class or the other). *Hence, there cannot be a greatest number in A and simultaneously a smallest number in B.*
ii. Assume that the left-hand class A contains the number $\frac{1}{3}$ and all rational numbers less than $\frac{1}{3}$, and that the right-hand class B contains all rational numbers greater than $\frac{1}{3}$. Then $\frac{1}{3}$ is the greatest number of A, and B has no smallest number. Obviously, the number $\frac{1}{3}$ can be replaced by any other rational number. *Hence, it is possible for A to have a largest number and for B to have no smallest number; and, in such a case, the cut defines a rational number.*
iii. Assume that the left-hand class A contains all rational numbers less than $\frac{1}{3}$, and that the right-hand class B contains the number $\frac{1}{3}$ and all rational numbers greater than $\frac{1}{3}$. Then A has no greatest number, and $\frac{1}{3}$ is the smallest number of B. Obviously, the

number $\frac{1}{3}$ can be replaced by any other rational number. *Hence, it is possible for A to have no largest number and for B to have a smallest number; and, in such a case, the cut defines a rational number.*

iv. Assume that the left-hand class A contains all negative rational numbers and all those positive rational numbers whose squares are less than 2, and that B contains all positive rational numbers whose squares are greater than 2. Then A has no greatest number, and B has no smallest number (e.g., if m is an arbitrary number of A, then a larger number always exists in A, and, if m is a rational number whose square is less than 2, then the number $m + \frac{2-m^2}{10}$ is a number greater than m and belonging to A). *Hence, it is possible for A to have no largest number and for B to have no smallest number; and, in such a case, the cut defines an irrational number.*

Therefore, the partition of the rational number system according to Dedekind's method defines two kinds of numbers: rationals and irrationals. The set of all rationals and all irrationals is the set $\mathbb{R}$ of all real numbers.

Dedekind observed that there exist infinitely many points in the straight number line L that correspond to no rational number. Thus, the domain of rational numbers is insufficient if we want to arithmetically follow up all phenomena on the straight line. Therefore, new numbers must be created in such a way that the domain of all numbers will gain the same "completeness" or "continuity" as the straight line. In fact, Dedekind observed that there exist infinitely many cuts that are not produced by rational numbers. For instance, construct a square $OABC$ on the unit segment OC (i.e., the length of OC is equal to one) and lay off in the positive direction a line segment OD equal in length to the diagonal OB, as shown in Figure 2.2; then it is clear that D is a point that does not correspond to any rational number, and, in fact, it corresponds to $\sqrt{2}$.

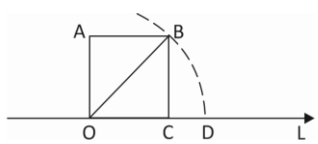

Figure 2.2: Irrational Numbers

In modern mathematical notation, the set of all real numbers x such that $a \leq x \leq b$ is said to be a "closed interval," denoted by $[a, b]$, of the real line $\mathbb{R}$, while the set of all real numbers x such that $a < x < b$ (which does not include its endpoints) is said to be an "open interval," denoted by (a, b), of the real line $\mathbb{R}$. The intervals $[a, b) = \{x \in \mathbb{R} | a \leq x < b\}$ and $(a, b] = \{x \in \mathbb{R} | a < x \leq b\}$ are neither open nor closed, but they are sometimes called "half-open" or "half-closed." Notice that $(a, a) = \emptyset$, and $[a, a] = \{a\}$. Moreover, we define the intervals:

$(a, \infty) = \{x \in \mathbb{R} | a < x\}$,
$[a, \infty) = \{x \in \mathbb{R} | a \leq x\}$,
$(-\infty, a) = \{x \in \mathbb{R} | x < a\}$,
$(-\infty, a] = \{x \in \mathbb{R} | x \leq a\}$.

In general, by the term "interval," we mean a set of points with the property that, if x and y are distinct points of the set, then every point between x and y is also a point of the set (if the points x and y are included, then the interval is closed; otherwise, it is open).

A real number b is said to be an "upper bound" of a non-empty subset S of $\mathbb{R}$ if every member of the set S is less than or equal to the number b, symbolically, if $x \leq b\ \forall x \in S$. If this is the case, then S is said to be "bounded from above." For instance, if $S = \{2,4,6,8,10\}$, then 10 is an upper bound of S, and every real number greater than 10 is also an upper bound of S. Notice that, if a set is bounded from above, then it has infinitely many upper bounds, and that an upper bound of such a set need not be a member of the given set. For instance, the number 10 is an upper bound of the open interval $(2,10)$, but $10 \notin (2,10)$. On the other hand, the set $\mathbb{N}$ of all natural numbers has no upper bound.

The least of all upper bounds of a set is said to be the "least upper bound" (often denoted by $l.u.b.$), or the "supremum" (often denoted by sup). Hence, a real number b is defined to be the $l.u.b.$ of a set S if b is an upper bound of S (i.e., $x \leq b\ \forall x \in S$), and if , given any other upper bound c of S, $b < c$; and then we write $\sup(S) = b$. For instance, if $S = \{2,4,6,8,10\}$, then $\sup(S) = 10$. On the other hand, the set $\mathbb{N}$ of all natural numbers has no supremum. The supremum, when it exists, is unique for a set.

A real number a is said to be a "lower bound" of a non-empty subset S of $\mathbb{R}$ if every member of the set S is greater than or equal to the number a, symbolically, if $x \geq a\ \forall x \in S$. If this is the case, then S is said to be "bounded from below." For instance, if $S = \{2,4,6,8,10\}$, then 2 is a lower bound of S, and every real number less than 2 is also a lower bound of S. Notice that, if a set is bounded from below, then it has infinitely many lower bounds, and that a lower bound of such a set need not be a member of the given set. For instance, the number 2 is a lower bound of the open interval $(2,10)$, but $2 \notin (2,10)$. On the other hand, the set $\mathbb{Z}$ of all integral numbers has no lower bound.

The greatest of all lower bounds of a set is said to be the "greatest lower bound" (often denoted by $g.l.b.$), or the "infimum" (often denoted by inf). Hence, a real number a is defined to be the $g.l.b.$ of a set S if a is a lower bound of S (i.e., $x \geq a\ \forall x \in S$), and if , given any other lower bound d of S, $a > d$; and then we write $\inf(S) = a$. For instance, if $S = \{2,4,6,8,10\}$, then $\inf(S) = 2$. On the other hand, the set $\mathbb{Z}$ of all integral numbers has no infimum. The infimum, when it exists, is unique for a set.

A set is said to be "bounded" if it is both bounded from above and bounded from below. In other words, a set S is bounded if there exist two real numbers a and b such that $a \leq x \leq b\ \forall x \in S$. If this is the case, then $x \in [a,b]\ \forall x \in S$, meaning that, for any bounded set S, there exist two real numbers a and b such that $S \subseteq [a,b]$.

Notice that the empty set, $\emptyset$, is a subset of every set, and, $\forall a,b \in \mathbb{R}$, $\emptyset \subseteq [a,b]$. Therefore, $\emptyset$ is a bounded set. Because of the fact that $\emptyset \subseteq [a,b]$ for any real numbers a and b, every real number is a lower bound of $\emptyset$, and every real number is an upper bound of $\emptyset$, meaning that $\emptyset$ does not have a supremum or an infimum.

Moreover, notice that, for an arbitrary singleton $A = \{x\}$, $\sup(A) = \inf(A) = x$. Thus, every singleton is a bounded set in which $supremum = infimum$.

It can be easily verified that the concept of a supremum and the concept of an infimum satisfy the following conditions:

i. If b is the supremum of a set A, then $-b$ is the infimum of the set $\{-x | x \in A\}$.
ii. If $A \subset B \subset \mathbb{R}$, then, if B is bounded, it follows that A is bounded, and

iii. $\inf(B) \leq \inf(A) \leq \sup(A) \leq \sup(B)$.
iv. If S is a non-empty bounded subset of $\mathbb{R}$, if A is the set of all the upper bounds of S, and if B is the set of all the lower bounds of S, then A has an infimum that belongs to A, and B has a supremum that belongs to B.

If the supremum of a set belongs to the given set, then it is said to be the "maximum element" of the given set. If the infimum of a set belongs to the given set, then it is said to be the "minimum element" of the given set. For instance, 5 is the maximum element of the set (closed interval) $[-3,5]$, and -3 is the minimum element of this set. However, the set (open interval) $(-3,5)$ does not have a maximum element or a minimum element.

The Completeness Axiom of $\mathbb{R}$*:* Every non-empty subset of $\mathbb{R}$ that is bounded from above has its supremum in $\mathbb{R}$. Equivalently, every non-empty subset of $\mathbb{R}$ that is bounded from below has its infimum in $\mathbb{R}$.

For instance, the set $\mathbb{Z}^-$ of all negative integers is a subset of $\mathbb{R}$ that is bounded from above, and its supremum is -1; and the set $\mathbb{Z}^+$ of all positive integers is a subset of $\mathbb{R}$ that is bounded from below, and its infimum is 1. On the other hand, the set $\mathbb{Q}$ of all rational numbers does not satisfy the Completeness Axiom, because, for instance, the supremum of the set $\{x \in \mathbb{Q} | 0 < x^2 < 2\}$ is $\sqrt{2}$, which does not belong to $\mathbb{Q}$. Therefore, $\mathbb{Q}$ is not complete.

$\mathbb{R}$ *as a Field*

As I mentioned in section 2.1.4, a group is an algebraic structure that has a single binary operation, usually called "multiplication," while sometimes it is called "addition," especially if the group is commutative. On the other hand, a "field" is an algebraic structure that has two binary operations, usually called "addition" and "multiplication," and both of them are always commutative. Whereas groups model symmetries (in the sense that the symmetries of an object can be constructed one after the other and then composed by the group operation), fields model number systems (since numbers can be added or multiplied, and, therefore, subtracted and divided, too, and various relationships hold true between them).[329] Hence, every field is a group, but not every group is a field.

A "field" is a structured set

$$(F, 0, 1, +, \cdot)$$

that satisfies the following properties:

(F1) $0,1 \in F, 0 \neq 1$, and $+$ and $\cdot$ are binary functions (operations) on F.
(F2) Addition $+$ satisfies the following identities:

i. $(x + y) + z = x + (y + z)$,
ii. $x + y = y + x$,
iii. $x + 0 = x$,

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[329] See: Campbell, *The Structure of Arithmetic*; Dummit and Foote, *Abstract Algebra*; Gallian, *Contemporary Abstract Algebra*; Mendelson, *Number Systems and the Foundations of Analysis*; Moschovakis, *Notes on Set Theory*.

and, for every x, there exists some x' such that $x + x' = 0$.

(F3) Multiplication $\cdot$ satisfies the following identities:

i. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,
ii. $x \cdot y = y \cdot x$,
iii. $x \cdot 1 = x$,

and, for every x, there exists some x'' such that $x \cdot x'' = 1$.

(F4) Both addition and multiplication satisfy the identity

$$x \cdot (y + z) = x \cdot y + x \cdot z.$$

Remark: The axioms of a field imply that any field F satisfies the following:

i. For every x, there exists a unique x' such that $x + x' = 0$; then $x' = -x$. Furthermore, for every $x \neq 0$, there exists a unique x'' such that $x \cdot x'' = 1$; then $x'' = x^{-1}$.
ii. $x \cdot 0 = 0$.
iii. $x \cdot y = 0 \Rightarrow x = 0\ or y = 0$.
iv. $(-x) \cdot y = -(x \cdot y)$.
v. A field is a set F that is closed under the operations of addition and multiplication such that: F is an Abelian group under addition, and $F - \{0\}$ (i.e., the set F without the additive identity element 0) is an Abelian group under multiplication.

Familiar examples of fields are the set $\mathbb{Q}$ of all rational numbers and the set $\mathbb{R}$ of all real numbers. Notice that the set $\mathbb{Z}$ of all integers is not a field, because not every element of $\mathbb{Z}$ has a multiplicative inverse (in fact, only 1 and -1 have multiplicative inverses in $\mathbb{Z}$). However, $\mathbb{Z}$ under addition is an Abelian group.

If a subset S of the elements of a field F satisfies the field axioms with the same operations of F, then S is called a “subfield” of F.

An “ordered field” is a structured set

$$(F, 0, 1, +, \cdot, \leq)$$

such that $(F, 0, 1, +, \cdot)$ is a field, the binary relation $\leq$ is a linear order on F, and the following conditions are satisfied by every $x, y, z \in F$:

$$x \leq y \Rightarrow x + z \leq y + z,$$
$$z > 0 \ \&\ x \leq y \Rightarrow z \cdot x \leq z \cdot y.$$

Remark: For every element x of an ordered field F, we have

$x \cdot x = x^2 \geq 0$, so that $0 < 1 \ \&\ x > 0 \Rightarrow x + 1 > 0$; because: $x = 0 \Rightarrow x^2 = 0 \geq 0$ and $x > 0 \Rightarrow x \cdot x \geq x \cdot 0 = 0$; if $x < 0$, then $x - x < 0 - x \Rightarrow 0 < -x \Rightarrow x < 0 \Rightarrow (-x) \cdot x < (-x) \cdot 0 \Rightarrow -(x^2) < 0 \Rightarrow (-x^2) + x^2 < x^2 \Rightarrow 0 < x^2$.

A “completely ordered field” is an ordered field wherein every non-empty set that is bounded from above has a supremum (least upper bound).

On the grounds of Dedekind’s definition of real numbers, given any real numbers x and y, the sum $x + y$ and the product $x \cdot y$ (more simply denoted by xy) are uniquely determined real numbers and satisfy the following properties:

i. Commutative law:
$x + y = y + x$ and $xy = yx$.
ii. Associative law:
$x + (y + z) = (x + y) + z$ and $x(yz) = (xy)z$.
iii. Distributive law:
$x(y + z) = xy + xz$.
iv. There exist distinct elements 0 and 1 in $\mathbb{R}$ such that
$x + 0 = x$ (0 is the additive identity element) and
$x1 = x$ (1 is the multiplicative identity element).
v. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} | x + y = 0$, and then y is called the additive inverse of x.
vi. $\forall x \in \mathbb{R} - \{0\}$, $\exists y \in \mathbb{R} | xy = 1$, and then y is called the multiplicative inverse of x.

Therefore, $\mathbb{R}$ is a field.

Order in $\mathbb{R}$*:* $\mathbb{R}$ is an ordered field, because, in addition to the aforementioned six properties, $\mathbb{R}$ satisfies the following properties:

i. $\mathbb{R}$ contains a set $\mathbb{R}^+$ of positive elements such that, $\forall x, y \in \mathbb{R}^+$, $x + y \in \mathbb{R}^+$ and $xy \in \mathbb{R}^+$.
ii. For any real number x, one and only one of the following holds: $x \in \mathbb{R}^+$, or $-x \in \mathbb{R}^+$, or $x = 0$ (Law of Trichotomy).

As a result of property (viii), $\mathbb{R} = \mathbb{R}^+ \cup \mathbb{R}^- \cup \{0\}$. For any two distinct real numbers x, y, it holds that either $x > y$ or $y > x$, and, therefore, the following properties hold:

i. $x > y \& y > z \Rightarrow x > z$, where z is an arbitrary real number (transitivity).
ii. $x > y \Rightarrow x + z > y + z$.
iii. $x > y \& z > 0 \Rightarrow xz > yz$.
iv. If we consider the possibility of $x = y$, then we write $x \geq y$ or $y \leq x$ (namely, we use the sign that means “less than or equal to”).

Given that $\mathbb{R}$ is an ordered field, and given the Completeness Axiom of $\mathbb{R}$, it follows that $\mathbb{R}$ is a completely ordered field.

The Dedekind–Cantor Axiom of Continuum: The system of the real numbers is called the “arithmetic continuum.” The graphical representation of the arithmetic continuum is a straight line that is called the “real line,” or the “geometric continuum,” or the “linear continuum”: each point on the real line corresponds to exactly one real number, and, conversely, each real number is represented by exactly one point on the real line. Hence, *there is an one-to-one* *correspondence between the system* $\mathbb{R}$ *of the real numbers and the system of points on the real line.* This statement is known as the Dedekind–Cantor Axiom of Continuum. In other words, from the perspective of Dedekind's theory of cuts, the set $\mathbb{R}$ of all real numbers can be rigorously founded as an ordered field $(F, \leq)$ that satisfies the following axiom of continuity: for every Dedekind cut (A, B) in F, there exists a ξ such that $a \leq \xi \leq b$ for every $a \in A$ and for every $b \in B$.

The Absolute Value of a Real Number

The "absolute value" (known also as the "modulus" or the "magnitude") of a real number x is denoted by $|x|$, and it is defined as follows:

$$|x| = \begin{cases} xifx \geq 0 \\ -x < 0 \end{cases}.$$

Therefore, the absolute value of any real number is always non-negative. The aforementioned definition implies the following[330]:

i. $|x|$ is the distance between the point x and zero (i.e., the "origin") on the real line. Hence, for instance, $|x| < 2$ means that the distance between x and the origin is less than 2, so that x lies between -2 and $+2$ on the real line, that is, $-2 < x < 2$.
ii. $|x| = |-x|$ ("evenness," namely, "reflection symmetry" of the graph).
iii. $|x| \geq x$ and $|x| \geq -x$.
iv. $|x| = |y|$ does not necessary imply that $x = y$.

Notice that, for instance, in order to convert the inequality $8 < x < 20$ into an absolute-value form, we add -14 to both sides, and we obtain $-6 < x < 6 \Rightarrow |x| < 6$, and, in order to convert the inequality $-3 < x < 5$ into an absolute-value form, we add -1 to both sides, and we obtain $-4 < x < 4 \Rightarrow |x| < 4$.

The concept of an absolute value was originally articulated by the French mathematician Jean-Robert Argand (1768–1822), who used the French term "module" (meaning "unit of measure"), which was borrowed into English as the Latin equivalent "modulus." The notation $|x|$ was introduced by the German mathematician Karl Weierstrass (1815–97).

Properties of the Absolute Value[331]*:* The absolute value of any real number has the following properties:

i. $|xy| = |x||y|$, and, generally, $|x_1 x_2 \dots x_n| = |x_1||x_2| \dots |x_n|$.
ii. $|x + y| \leq |x| + |y|$, and, generally, $|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|$ (subadditivity).
iii. $|x - y| \geq |x| - |y|$.
iv. $|x| - |y| \leq \big||x| - |y|\big| \leq |x - y|$ (triangle inequality).
v. $|x - y| < k \Rightarrow y - k < x < y + k$.

[330] Ibid.
[331] Ibid.

Proof:

i. $|xy|^2 = (xy)^2 = x^2y^2 = |x|^2|y|^2 \Rightarrow |xy| = |x||y|$.
Similarly, we can prove that $|x_1x_2 \ldots x_n| = |x_1||x_2| \ldots |x_n|$.

ii. $|x+y|^2 = (x+y)^2 = x^2 + y^2 + 2xy = |x|^2 + |y|^2 + 2xy$
$\leq |x|^2 + |y|^2 + 2|xy| \leq |x|^2 + |y|^2 + 2|x||y| \leq (|x| + |y|)^2$.
Thus, $|x+y| \leq |x| + |y|$, and, similarly, it can be verified that $|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|$, and that $|x - y| \leq |x| + |y|$.

iii. The proof is similar to the proof of (ii).

iv. If we set $z = |x| - |y|$, then, because, by definition, $z \leq |z|$, we obtain the required inequality: $|x| - |y| \leq \big||x| - |y|\big|$. Furthermore, in order to prove that $\big||x| - |y|\big| \leq |x - y|$, we work as follows:
$|x-y|^2 = (x-y)^2 = x^2 + y^2 - 2xy = |x|^2 + |y|^2 - 2xy$
$\geq |x|^2 + |y|^2 - 2|xy| \geq |x|^2 + |y|^2 - 2|x||y|$
$\geq (|x| - |y|)^2 = \big||x| - |y|\big|^2 \Rightarrow |x - y| \geq \big||x| - |y|\big|$.

v. The inequality $|x - y| < k$ implies that $(x - y) < k$ and $-(x - y) < k$, so that $x < y + k$ and $y - k < x$. Hence, $y - k < x < y + k$.■

Exponentiation and Logarithm

Let a be a real number. Then the product $a \cdot a \cdot a$ ... (n times) is denoted by a^n, where n is called the "exponent," and a is called the "base." Therefore, the following results hold $\forall a, b \in \mathbb{R}$[332]:

i. $a^n a^m = a^{n+m}$,
ii. $(a^n)^m = a^{nm}$,
iii. $\frac{a^n}{a^m} = a^{n-m}$,
iv. $a^0 = 1$, and
v. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$.

Intimately related to the concepts of an exponent and an index is the concept of a logarithm, which is the inverse function to exponentiation.[333] The "logarithm" of an arbitrary real number a is the exponent to which another fixed real number, the base b, must be raised to produce the real number a, symbolically:

$$log_b a = x \Leftrightarrow b^x = a.$$

For instance, $log_{10} 1{,}000 = 3$, since $10^3 = 1{,}000$, and $log_3 81 = 4$, since $3^4 = 81$.

The method of logarithms was originally developed by the Scottish mathematician, physicist, and astronomer John Napier (1550–1617), who published his book *Mirifici Logarithmorum Canonis Descriptio* (*Description of the Wonderful Rule of Logarithms*) in 1614.

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332 Ibid.
333 Ibid.

In case $b = e = \sum_{n=0}^{\infty} \frac{1}{n!} \approx 2.718$, which is known as Euler's number (in honor of the Swiss mathematician Leonhard Euler), then $log_e a$ is written as $ln a$, and it is said to be the "natural logarithm" of a. Euler's number e is irrational, and it was originally derived from the study of compound interest: if one places 1 USD into a deposit account at a banking institution with 100% interest, and the compounding period is n, as a fraction of a year, then the formula of the compound interest $(1 + \frac{r}{n})^n$, where, in our case, $r = 1$ (annual interest rate as a decimal), tends to e as n tends to infinity. However, the problem of compound interest was systematically investigated by the Swiss mathematician Jacob Bernoulli (1655–1705), who studied the following question: if an account starts with 1 USD and pays 100% interest per year, and if the interest is credited once, at the end of the year, then the value of the account at the year-end will be 2 USD, but what will happen if the interest is computed and credited more frequently during the year? In fact, Bernoulli noticed that, if there are n compounding intervals, then the interest for each interval will be $\frac{100\%}{n}$, and the value of the aforementioned account (which started with 1 USD) at the end of the year will be $1\, USD \times \left(1 + \frac{1}{n}\right)^n$. Furthermore, Bernoulli noticed that this sequence approaches a limit (the "force of interest"), specifically, the number e, as n increases, that is, as compounding intervals become smaller. For instance, compounding monthly (i.e., $n = 12$) yields approximately 2.613 USD, while compounding daily ($n = 365$) yields approximately 2.7146 USD. The limit as n tends to infinity is the number $e = \sum_{n=0}^{\infty} \frac{1}{n!} \approx 2.718$, meaning that, with continuous compounding, the value of the aforementioned account will reach approximately 2.718 USD. Leonhard Euler proved that the number e is irrational by showing that its simple continued fraction expansion is infinite (by a "continued fraction," we mean an expression obtained through an iterative process of representing a number as the sum of its integral part and the reciprocal of another number, then writing this other number as the sum of its integral part and another reciprocal, etc.).

The following properties of the logarithm can be easily verified[334]:

i. $log_b(xy) = log_b x + log_b y$,
ii. $log_b \left(\frac{x}{y}\right) = log_b x - log_b y$,
iii. $log_b x^k = k log_b x$,
iv. $log_b 1 = 0$,
v. $log_b b^x = x = b^{log_b x}$,
vi. $log_b x = \frac{log_a x}{log_a b}$ (change of base rule).
vii. If x, y, and b are positive real numbers with $b \neq 1$, then
$x = y \Rightarrow log_b x = log_b y$, and, conversely,
$log_b x = log_b y \Rightarrow x = y$. Hence, we can solve exponential equations (i.e., equations in which the unknown is in the exponent) by taking the logarithm of both sides of the equation.

[334] Ibid.

Properties of the System of the Real Numbers

Theorem[335]*:* The set $\mathbb{R}$ of all real numbers is uncountable.

Proof: We can prove this theorem by *reductio ad absurdum* as follows: For convenience, we shall show that the set S of all real numbers between 0 and 1 is uncountable. We regard the elements of S as infinite decimals, and, for definiteness, we agree not to use the version ending in 9s (e.g., 0.23999 ... and 0.24000 ... are regarded as the same number). Assume that S is a countable set, namely, that its elements can be listed as $r_1, r_2, r_3,$ ... We shall show that such an alleged enumeration of S is incomplete, namely, that there exists a missing number x that differs from r_1 in the first position after the decimal, differs from r_2 in the second position, etc. In particular, let the nth digit of x be 2 if the nth digit of r_n is 1 and let the nth digit of x be 1 otherwise. For instance, if

$$
\begin{aligned}
r_0 &= 0.1023 \ldots, \\
r_1 &= 0.1234 \ldots, \\
r_2 &= 0.3358 \ldots, \\
r_3 &= 0.9919 \ldots, \\
&\vdots
\end{aligned}
$$

then $x = 0.2111$ ... Obviously, x is missing from the list, since it differs from each r_i in at least one of its digits, which is a contradiction.■

Corollary[336]*:* The cardinality of the power set of the set $\mathbb{N}$ of all natural numbers is equal to the cardinality of the set $\mathbb{R}$ of all real numbers and to $2^{\mathbb{N}}$, where $2^{\mathbb{N}}$ denotes the set of all functions $\mathbb{N} \to \{0,1\}$; symbolically: $\wp(\mathbb{N}) =_c 2^{\mathbb{N}} =_c \mathbb{R}$. Notice that, usually, the cardinal number of the power set $\wp(\mathbb{N})$ of the set of all natural numbers is denoted by

$$c = |\wp(\mathbb{N})| =_c 2^{\aleph_0},$$

where c stand for the word "continuum."

Proof: First, we shall prove that $\wp(\mathbb{N}) =_c 2^{\mathbb{N}}$: This follows directly from the fact that $2^X = \wp(X)$ for any set X; this equality can be proved as follows: For each $Y \subseteq X$, define the function $\delta: X \to \{0,1\}$ as follows:

$$\delta_Y(a) = \begin{cases} 1 \; if a \in Y \\ 0 \; if \; a \notin Y \end{cases},$$

which is called the "characteristic function" of Y. If $f: \wp(X) \to 2^X$ is defined by $f: Y \to \delta_Y$, then it is easily verified that f is bijective (one-to-one and onto), and, therefore, $2^X = \wp(X)$.

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[335] Ibid.
[336] Ibid.

Second, we shall prove that $2^{\mathbb{N}} =_c \mathbb{R}$: First, notice that the unit interval $[0,1]$ can be divided into 2 parts, then into 4 parts, 8 parts, etc., and these parts can be designated with binary digits (e.g., the division into 8 parts includes .000, .001, .010, .011, .100, .101, .111, and 1.00). By continuing this process indefinitely, that is, by iterating the process up to the limit ordinal ω (i.e., ω is the smallest ordinal number greater than every natural number), one will obtain a representation of all the real numbers contained in $[0,1]$, and one will have produced 2^{ω} (and, thus, $2^{\mathbb{N}}$) points (since one doubles the number of points designated with each iteration of the process described); so that the reals between 0 and 1 have the cardinality of 2^{ω} (and, thus, of $2^{\mathbb{N}}$). Equivalently, yet more formally, we can argue as follows: Notice that each member of $I = \{x \in \mathbb{R} | 0 \leq x \leq 1\}$ has a dyadic expression $\sum_{n=1}^{\infty} 2^{-n} a_n$ with $a_n = 0 \; or \; 1$; the expansion is not unique, but the convention that, if any number has two expansions, we always choose the expansion with infinitely many $1s$ safeguards uniqueness. Let $A \subseteq 2^{\mathbb{N}}$ consist of all characteristic functions taking the value 1 infinitely often. If $g: 2^{\mathbb{N}} \to \mathbb{R}$ is defined by

$$g(\delta) = \begin{cases} \sum_{n=1}^{\infty} \frac{\delta(n)}{2^n} \; if \; \delta \in A \\ 2 + \sum_{n=1}^{\infty} \frac{\delta(n)}{2^n} \; if \; \delta \notin A \end{cases},$$

then g is one-to-one. But $(0,1] \leq_c g(2^{\mathbb{N}}) \leq_c \mathbb{R}$, and $(0,1] =_c \mathbb{R}$. Hence, $2^{\mathbb{N}} =_c \mathbb{R}$.■

Remarks:

i. $c \cdot c =_c 2^{\aleph_0} \cdot 2^{\aleph_0} =_c 2^{\aleph_0 + \aleph_0} =_c 2^{\aleph_0} =_c c$.
ii. $c =_c 2^{\aleph_0} \leq_c {\aleph_0}^{\aleph_0} \leq_c c^{\aleph_0} =_c (2^{\aleph_0})^{\aleph_0} =_c 2^{\aleph_0 \cdot \aleph_0} =_c 2^{\aleph_0} =_c c$, so that, by Bernstein's Equinumerosity Theorem, $c =_c {\aleph_0}^{\aleph_0} =_c c^{\aleph_0}$.
iii. The cardinal number of the family of all functions from $\mathbb{R}$ to $\mathbb{R}$ is $c^c =_c (2^{\aleph_0})^c =_c 2^{\aleph_0 \cdot c} =_c 2^c$.

Because the cardinal number of $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^{\sim}$ is uncountable while the cardinal number of $\mathbb{Q}$ is countably infinite, it follows that the set $\mathbb{Q}^{\sim}$of all irrational numbers is uncountable.

Theorem[337]*:* The "Archimedean property" of $\mathbb{R}$ asserts that, for every $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ such that $x < n$.

Proof: Let $x \in \mathbb{R}$, and let A denote the set of all natural numbers that are less than or equal to x. If A is empty, then the Archimedean property is trivially satisfied. On the other hand, if A is not empty, then A is bounded from above by x, and, therefore, because $\mathbb{R}$is a completely ordered field, there exists a least upper bound, say a, for A. Notice that $a - 1 < a$, so that $a - 1$ is not an upper bound for A. Hence, there exists an $m \in A$ such that $a - 1 < m$. Then $a < m + 1$, and, therefore, $m + 1$ is an element of $\mathbb{N}$ that is not in A, that is, $x < n = m + 1$, which proves the theorem.■

[337] Ibid.

Corollaries[338]*:* (i) The set $\mathbb{Q}$ of all rational numbers is "dense" in $\mathbb{R}$, which means that, between any two real numbers, there is a rational number. (ii) Moreover, the set $\mathbb{Q}^{\sim}$ of all irrational numbers is "dense" in $\mathbb{R}$, that is, between any two real numbers, there is an irrational number.

Proof: (i) Let $x, y \in \mathbb{R}$ with $x < y$. If $x \geq 0$, then $y - x > 0$ and $(y-x)^{-1} \in \mathbb{R}$, so that, by the Archimedean property of $\mathbb{R}$, it follows that there exists a number $m \in \mathbb{N}$ such that $(y-x)^{-1} < m \Leftrightarrow y - x > 1/m > 0$. Moreover, the Archimedean property of $\mathbb{R}$ implies that the set of positive integers k such that $y \leq k/m$ is not empty. Then, because every non-empty set of real numbers that is bounded from below has a greatest lower bound, the set has a smallest element, say n, so that

$$\frac{n-1}{m} < y \leq \frac{n}{m}.$$

Furthermore,

$$x = y - (y-x) < \frac{n}{m} - \frac{1}{m} = \frac{n-1}{m},$$

and, obviously, $x < r < y$ for $r = (n-1)/m \in \mathbb{Q}$.

If $x < 0$, then the Archimedean property of $\mathbb{R}$ implies that there exists a positive integer $k > -x$. If this is the case, then $k + x > 0$, and there exists a rational number r such that $k + x < r < k + y$. Hence, $r - k \in \mathbb{Q}$ lies between x and y.This completes the proof of Corollary (i).

(ii) Due to Corollary (i), there exists a rational number r between $x/\sqrt{2}$ and $y/\sqrt{2}$. Then the irrational number $r\sqrt{2}$ lies between x and y, which proves Corollary (ii).■

2.2.5. Matrices of Real Numbers and Vectors

Let F be the field of all real numbers $\mathbb{R}$ (F may be a field different from $\mathbb{R}$). Suppose that $a_{11}, a_{12}, a_{13}, \ldots, a_{mn}$ is a collection of mn elements in F. The rectangular array of these elements

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

consisting of m rows and n columns is called an " $m \times n$ matrix," usually denoted by $A = (a_{ij})$. In other words, a "matrix" is a function on the set of pairs of integers (i, j), where $1 \leq i \leq m, 1 \leq j \leq n$, with values in F in which a_{ij} designates the value of A at the pair (i, j).[339] Hence, the aforementioned array exhibits the range of the function A. The element a_{ij} is

[338] Ibid.

[339] See: Eves, *Elementary Matrix Theory*; Friedberg, Insel, and Spence, *Linear Algebra*; Householder, *The Theory of Matrices in Numerical Analysis*; Turnbull, *The Theory of Determinants, Matrices, and Invariants.*

called the (i, j) "entry" of A. The ith row of A is the sequence $a_{i1}, a_{i2}, a_{i3}, \dots, a_{in}$, and the jth column of A is the sequence $a_{1j}, a_{2j}, a_{3j}, \dots, a_{mj}$. The "main diagonal" of the matrix A is the collection of those entries that lie in the diagonal that runs from top left to bottom right. Since the a_{ij} are in the field F, we say that A is an $m \times n$ matrix over the field F. The totality of such matrices can be denoted by $M_{m,n}(F)$. If $m = n$, then the corresponding matrix A is said to be an "n-square" matrix, and the set of all n-square matrices over F is denoted by $M_n(F)$.

The term "matrix" was introduced by the nineteenth-century English mathematician James Sylvester, but it was his friend the mathematician Arthur Cayley who developed the algebra of matrices in the 1850s. The standard operations for matrices over F are defined as follows[340]:

i. Multiplication of $A = (a_{ij}) \in M_{m,n}(F)$ by an element ("scalar") $k \in F$: the product is defined as the matrix in $M_{m,n}(F)$ whose (i, j) entry is ka_{ij} and is denoted by kA.
ii. The sum of two matrices $A = (a_{ij})$ and $B = (b_{ij})$ in $M_{m,n}(F)$ is the matrix $C = (c_{ij}) \in M_{m,n}(F)$ whose (i, j) entry is $c_{ij} = a_{ij} + b_{ij}$.
   Obviously, the matrices A and B must be the same size in order that the sum be defined.
iii. The product of two matrices $A = (a_{ij}) \in M_{m,k}(F)$ and $B = (b_{ij}) \in M_{k,n}(F)$ *is the matrix* $C = (c_{ij}) \in M_{m,n}(F)$ *whose* (i, j) *entry is* $c_{ij} = \sum_{p=1}^{k} a_{ip}\, b_{pj}$, where $1 \leq i \leq m, 1 \leq j \leq n$.
   Obviously, the number of columns of A must be the same as the number of rows of B in order that the product $C = AB$ be defined. If $A \in M_n(F)$, namely, if A is an n-square matrix, then A^r denotes the rth power of A, and r is an arbitrary positive integer.

The rules connecting the aforementioned operations (given that the corresponding matrices are of appropriate sizes for the indicated operations to be defined) are the following[341]:

i. Commutativity: $A + B = B + A$. However, it may hold that $AB \neq BA$. If $AB = BA$, then the matrices A and B are said to "commute."
ii. Associativity: $A + (B + C) = (A + B) + C$, and $A(BC) = (AB)C$.
iii. Distributivity: $A(B + C) = AB + AC$, and $(B + C)A = BA + CA$.

An n-square matrix A is said to be "invertible" or "non-singular" if there exists an n-square matrix B with the following property:

$$AB = BA = I_n,$$

where I_n is the n-square identity matrix, namely, the $n \times n$ matrix with ones along the main diagonal and zeros elsewhere. If this is the case, then the matrix B is called the inverse of A, and the notation A^{-1} is used to designate B. If no such B exists, then A is said to be "singular."

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[340] Ibid.
[341] Ibid.

The "Kronecker delta" (named after the German mathematician Leopold Kronecker) is a function of two variables, usually non-negative integers, defined as follows:

$$\delta_{ij} = \begin{cases} 0 \ if i \neq j \\ 1 \ if i = j \end{cases},$$

and, therefore, the n-square identity matrix I_n has entries equal to the Kronecker delta:

$I_n = (\delta_{ij})$, where i and j take the values $1,2,3, \dots, n$.

The "transpose" of a matrix $A \in M_{m,n}(F)$ is denoted by A^T, and it is the matrix obtained by writing the rows of A, in order, as columns; namely, if $A = (a_{ij})$ is an $m \times n$ matrix, then $A^T = (a_{ij}^T)$ is the $n \times m$ matrix where $a_{ij}^T = a_{ji}$, for all i and j.

A matrix $A \in M_n(F)$ is called "symmetric" if $A^T = A$, and it is called "antisymmetric" (or "skew-symmetric") if $A^T = -A \Leftrightarrow a_{ji} = -a_{ij}$.

A matrix $A \in M_n(F)$ is called "orthogonal" if $A^T = A^{-1}$, that is, if $AA^T = A^T A = I$.

A matrix $D \in M_n(F)$ is called "diagonal," denoted by $D = diag(d_{11}, d_{22}, \dots, d_{nn})$, if its non-diagonal elements (i.e., the entries outside the main diagonal) are all zero. In particular, $A = (a_{ij})$ is said to be "upper triangular" (resp. "lower triangular") if its elements below (resp. above) the main diagonal are all zero, namely, if $a_{ij} = 0$ when $i < j$ (resp. when $i > j$).

Let $S(n)$ denote the totality of one-to-one functions, or "permutations," of the set $\{1,2, \dots, n\}$ onto itself. In other words, $S(n)$ denotes the "symmetric group" of degree n on the natural numbers $1,2, \dots, n$. Thus, the set $S(n)$ has $n!$ elements in it. A "cycle" in $S(n)$ is a permutation σ that has the following property: there exists a subset of $\{1,2, \dots, n\}$, say $\{i_1, i_2, \dots, i_k\}$, such that $\sigma(i_1) = i_2, \sigma(i_2) = i_3, \dots, \sigma(i_{k-1}) = i_k, \sigma(i_k) = i_1$, and $\sigma(j) = j$ for $j \neq i_p, p = 1,2, \dots, k$. The integer k is called the "length" of the cycle, and the cycles of length 2 are called "transpositions." In the case of transpositions, any $\sigma \in S(n)$ is the product of transpositions. Any permutation σ is a product of cycles acting on disjoint subsets of $\{1,2, \dots, n\}$, namely, on disjoint cycles; and this factorization is unique to within order. If the lengths of these cycles are $\lambda_1, \lambda_2, \dots, \lambda_m$, then σ is said to have the "cycle structure" $[\lambda_1, \lambda_2, \dots, \lambda_m]$, where some of the λ_i may be 1. The factorization into a product of transpositions is not unique, but any two such factorizations of the same permutation σ must both have an even or both have an odd number of transpositions, and, hence, σ is called "even" or "odd," respectively. The "sign" of σ is defined by

$$sgn(\sigma) = \begin{cases} 1 \ if \sigma is even \\ -1 \ if \sigma is odd \end{cases}.$$

If $A \in M_n(F)$ and $\sigma \in S(n)$, then the sequence of elements $a_{1\sigma(1)}, \dots, a_{n\sigma(n)}$ is said to be the "diagonal" of A corresponding to σ. If σ is the "identity permutation," namely,$\sigma(j) = j$, for $j = 1,2, \dots, n$, then the diagonal corresponding to σ, namely, $a_{11}, \dots a_{nn}$, is the "main diagonal" of A. A matrix $A \in M_n(F)$ such that $a_{i\sigma(i)} = 1$ $(i = 1,2, \dots, n)$ and $a_{ij} = 0$ otherwise, is called a "permutation matrix" (namely, a "permutation matrix" is a matrix obtained by permuting the rows of an n-square identity matrixaccording to some permutation of the numbers 1 to n, so that every row and every column contain precisely a single 1 with 0s everywhere else, and there are $n!$ permutation matrices of size n).

The determinant of an n-square matrix $A = (a_{ij}) \in M_n(F)$ is denoted by $\det(A)$, and it is defined as follows:

$$\det(\mathrm{A}) = \sum_{\sigma \in S(n)} sign\sigma \prod_{i=1}^{n} a_{\sigma(i)i}.$$

In other words, the determinant is the sum of the products of the elements in all $n!$ diagonals each weighted with ± 1 according as the diagonal corresponds to an even or an odd permutation $\sigma \in S(n)$.

Properties of determinants[342]:

i. $\det(A^{-1}) = (\det(A))^{-1}$.
ii. $\det(A^T) = \det(A)$.
iii. $\det(kA) = k^n \det(A)$, for any scalar n.
iv. $\det(AB) = \det(A)\det(B)$, for any two n-square matrices A and B.

Let us consider a system of 2 linear equations with 2 unknowns:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = c_1 \\ a_{21}x_1 + a_{22}x_2 = c_2 \end{cases},$$

which gives rise to the following three matrices:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, B = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \text{ and } X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Thus, the original system of linear equations can be reformulated as follows:

$$A \cdot X = B,$$

where A is the matrix of the system's coefficients, X is the matrix of the system's unknowns, and B is the matrix of the system's constant terms. The system has a unique solution if and only if the determinant $\det(A) = a_{11}a_{22} - a_{12}a_{21} \neq 0$, and that solution is:

$$x_1 = \frac{B_{x_1}}{\det(A)} = \frac{\begin{vmatrix} c_1 & a_{12} \\ c_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{c_1a_{22} - a_{12}c_2}{a_{11}a_{22} - a_{12}a_{21}},$$

and

$$x_2 = \frac{B_{x_2}}{\det(A)} = \frac{\begin{vmatrix} a_{11} & c_1 \\ a_{21} & c_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{a_{11}c_2 - c_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}},$$

[342] Ibid.

where the numerators B_{x_1} and B_{x_2} are obtained by substituting the column of constant terms in place of the column of coefficients of the corresponding unknown in the matrix of coefficients.

In case $\det(A) = 0$, then the system has either no solution or an infinite number of solutions.

Consider the 3-square matrix

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}.$$

The determinant of A is

$$\det(A) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_1c_2b_3 - b_1a_2c_3 - c_1b_2a_3.$$

Moreover, it can be easily shown that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

Let us consider a system of 3 linear equations with 3 unknowns:

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2. \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

The aforementioned system has a unique solution if and only if the determinant of the matrix of coefficients is not zero:

$$\det(A) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0.$$

In this case, the unique solution of the given system can be expressed as quotients of determinants as follows:

$$x = \frac{B_x}{\det(A)},$$
$$y = \frac{B_y}{\det(A)},$$
$$z = \frac{B_z}{\det(A)},$$

where the numerators B_x, B_y, and B_z are obtained by substituting the column of constant terms for the column of coefficients of the corresponding unknown in the matrix of coefficients, so that:

$$B_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, B_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \text{and } B_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d \end{vmatrix}.$$

In case $\det(A) = 0$, then the system has either no solution or an infinite number of solutions.

General Order Determinants: The expansion of a third order determinant in terms of second order determinants, which was originally developed by the French mathematician, engineer, and philosopher Pierre-Simon de Laplace (1749–1827), can be generalized. In particular, any determinant can be similarly expressed as a linear combination of determinants of lower order using any row or any column as coefficients, with the coefficient a_{ij} multiplied by $(-1)^{i+j}$.

The elementary row operation $(kR_i + R_j) \to (R_j)$, which adds k times row i to row j, does not change the value of the determinant. A similar result holds for the elementary column operation $(kC_i + C_j) \to (C_j)$.

If the n-square matrix A is triangular, then its determinant is equal to the product of its diagonal elements.

Let $A = (a_{ij})$ be a non-zero n-square matrix with $n > 1$. The following algorithm reduces the determinant of A to a determinant of order $n - 1$:

Step 1: We choose an element $a_{ij} = 1$ or, if lacking, $a_{ij} \neq 0$.

Step 2: We use a_{ij} as a pivot, and we apply elementary row (resp. column) operations in order to put zeros in all the other positions in column j (resp. row i).

Step 3: We expand the determinant using the column (resp. row) containing a_{ij}.

In particular, "Chiò's Condensation Method" is a method for evaluating an $n \times n$ determinant in terms of $(n-1) \times (n-1)$ determinants as follows[343]:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \frac{1}{a_{11}^{n-2}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{31} & a_{3n} \end{vmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{n1} & a_{n2} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{n1} & a_{n3} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{vmatrix} \end{vmatrix}.$$

[343] Ibid. Felice Chiò was an Italian mathematician and politician closely associated with Amedeo Avogadro.

Vectors[344]

A "scalar" is a quantity that can be specified by determining only its magnitude. However, the quantities that are specified by determining both magnitude and direction are called "vectors." In other words, a "vector" is a quantity that has both a direction and a magnitude of length, and, therefore, it is graphically denoted by an oriented line segment ("arrow"). In a more abstract way, a "vector" can be defined as an element of a "vector space," which, in turn, can be defined as follows: Let U be a set with two operations defined in the following way:

$+$: $U \times U \to U$ defined by $(u, v) \in U \times U \to u + v \in U$ for all $u, v \in U$, that is, U is "closed under addition";

: $k \times U \to U$ defined by $(k, u) \in K \times U \to k \cdot u \in U$ for every $k \in K$ (where K is a field, such as $\mathbb{R}$) and for every $u \in U$, that is, U is "closed under scalar multiplication." Of course, $0 \in U$, since, for every $u \in U$, $(-1)u \in U$, and, therefore, $u - u \in U \Rightarrow 0 \in U$. As a result of the aforementioned definition, we say that U under the operations of $+$ (addition) and $\cdot$ (scalar multiplication) forms a "vector space" over the field K, and, therefore, a "vector" can be defined as an element of such a U. Furthermore, let V be a vector space over the field K. Let U be a subset of V. Then U is a "subspace" of V if and only if U is a vector space itself under the operations defined in V. Using the definition of a vector space, it can be easily verified that, if V is a vector space over a field K, and if U_1 and U_2 are two subspaces of V, then $U_1 \cap U_2$ is a subspace of V; but $U_1 \cup U_2$ is not always a subspace of V, unless $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$ (in order to show that $U_1 \cup U_2$ is not always a subspace of V, consider the following example: let V be the xy-plane, which is a vector space over $\mathbb{R}$, let U_1, namely, the first subspace of V, be the x-axis, and let U_2, namely, the second subspace of V, be the y-axis; then, for $v_1 = (1,0) \in U_1 \cup U_2$ and $v_2 = (0,1) \in U_1 \cup U_2$, we obtain $v_1 + v_2 = (1,1) \notin U_1 \cup U_2$).

Examples:

i. If $V = \{ax^2 + bx + c | a, b, c \in \mathbb{R}\}$, then V is a vector space over $\mathbb{R}$. Proof:
Step 1: $0 = 0x^2 + 0x + 0 \in V$.
Step 2: Let
$$\begin{cases} v_1 = a_1x^2 + b_1x + c_1 \\ v_2 = a_2x^2 + b_2x + c_2 \end{cases},$$
so that $v_1 + v_2 = (a_1 + a_2)x^2 + (b_1 + b_2)x + (c_1 + c_2) \in V$.
Step 3: Let $v = ax^2 + bx + c$ with $a, b, c \in \mathbb{R}$. Then
$kv = (ka)x^2 + (kb)x + (kc) \in V$.
Therefore, V is a vector space over $\mathbb{R}$.

ii. A sphere S is not a vector space. Proof: Let v be a vector belonging to the sphere S. If we multiply v by an adequate number k, then kv does not belong to S any more. Hence, a sphere is not a vector space. This example helps us to understand why every bounded set is never a vector space.

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[344] See: Eves, *Elementary Matrix Theory*; Friedberg, Insel, and Spence, *Linear Algebra*; Householder, *The Theory of Matrices in Numerical Analysis*; Turnbull, *The Theory of Determinants, Matrices, and Invariants*.

Let V be a vector space over K, U be a subspace of V, and v be a vector belonging to V. Then the set defined by $v + U = \{v + u | u \in U\}$ is said to be the "coset" of U represented by v. For instance, if $V = \mathbb{R}^2$, and if U is some straight line through the origin, then the cosets of U are all the translates of that straight line (i.e., we have a partition of the real plane by separating it into a bunch of parallel lines); and, similarly, if $V = \mathbb{R}^3$, and if U is some plane through the origin, then the cosets of U are all the parallel planes (i.e., we fill up $\mathbb{R}^3$ with a stack of planes). The set of all cosets $v + U$ is denoted by$\frac{V}{U}$, and it is called the "quotient vector space."

If $\{v_1, v_2, \dots, v_n\}$ is a finite non-empty set of vectors belonging to the vector space V, then the vector $v = k_1 v_1 + k_2 v_2 + \cdots + k_n v_n$ is called a "linear combination" of $v_1, v_2, \dots, v_n$, and every subspace of V is a non-empty subset of V closed under linear combinations.

Linearly Independent Vectors: Let V be a vector space over K. The vectors $v_1, v_2, \dots, v_n$ of V are "linearly independent" if and only if every time

$$k_1 v_1 + k_2 v_2 + \cdots + k_n v_n = 0 \Rightarrow k_1 = k_2 = \cdots = k_n = 0.$$

For instance, the vectors $v_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $v_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are linearly independent, since

$$\begin{aligned} & k_1 v_1 + k_2 v_2 + \cdots + k_n v_n = 0 \\ & \qquad \Rightarrow \begin{pmatrix} k_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & k_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ k_3 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & k_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ & \Rightarrow \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow k_1 = k_2 = k_3 = k_4 = 0. \end{aligned}$$

Linearly Dependent Vectors: Let V be a vector space over K. The vectors $v_1, v_2, \dots, v_n$ of V are "linearly dependent" if and only if $k_1 v_1 + k_2 v_2 + \cdots + k_n v_n = 0$ for some $k_i \neq 0$, where $i = 1,2, \dots, n$.

For instance, the vectors $v_1 = (0,1)$, $v_2 = (1,0)$, and$v_3 = (1,1)$ are linearly dependent.

Basis: Let V be a vector space over K. The vectors $v_1, v_2, \dots, v_n$ form a "basis" of V if and only if these vectors are linearly independent and generate (or span) V, that is, every vector of V must be expressed in terms of $v_1, v_2, \dots, v_n$. For instance, if $V = \{a + bx + cx^2 | a, b, c \in \mathbb{R}\}$, then $v_1 = 1$, $v_2 = x$, and $v_3 = x^2$ form a basis of V, because: (i) v_1, v_2, and v_3 are linearly independent, since no vector from $\{1, x, x^2\}$ can be written in terms of the other vectors; (ii) $\{1, x, x^2\}$ generate V, since, for any $v \in V$, it holds that $v = k + lx + mx^2 = k \cdot 1 + lx + mx^2$. Every single-element set that contains a non-zero vector can form a basis if it is adequately enlarged, and, therefore, every (non-zero) vector space over a field K has at least one basis (actually, it has many different bases). However, every vector space V has an invariant property, namely: the number of vectors in every basis of V remains the same; and the "dimension" of a vector space V is the number of elements of any of its bases.[345]

Consider the totality $M_{m,n}(F)$ of all $m \times n$ matrices over a field F. A "row vector" (resp. a "column vector") over F is just an element of $M_{1,n}(F)$ (resp. of $M_{m,1}(F)$). In general, a "k-vector $\vec{v}$ over F" is an ordered k-tuple of elements of F, $(a_1, \dots, a_k)$, where a_i is called the ith "coordinate" of $\vec{v}$.

Vectors indicate the manner in which determinants are related to area and volume. Let $\overrightarrow{v_1}, \overrightarrow{v_2}, \dots, \overrightarrow{v_n}$ be vectors in $\mathbb{R}^n$. Assume that P is the parallelepiped formed by these vectors. Let A be the matrix with rows $\overrightarrow{v_1}, \overrightarrow{v_2}, \dots, \overrightarrow{v_n}$. Then, for $n = 3$, the volume of P is denoted by $V(P)$, and it is equal to the absolute value of the determinant of A. Similarly, when $n = 2$, the area of a parallelogram formed by the given vectors is equal to the absolute value of the determinant of the matrix whose rows are the given vectors. For instance, given two vectors $\overrightarrow{v_1} = (1,2)$ and $\overrightarrow{v_2} = (3,5)$ in $\mathbb{R}^2$, we can define a parallelogram P as follows: we draw the vectors (arrows) from the origin $O(0,0)$ of the coordinate system to the points $P_1(1,2)$ and $P_2(3,5)$ in the plane $\mathbb{R}^2$, and then we complete the parallelogram P by drawing parallels to $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$. The determinant of the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix},$$

whose rows are $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$, is $\det(A) = 5 - 6 = -1$. Therefore, the area of the parallelogram P is $|\det(A)| = |-1| = 1$.

Let V_1, V_2 be two vector spaces over a field K. A mapping $T: V_1 \to V_2$ is linear if and only if:

i. $T(v_1 + v_2) = T(v_1) + T(v_2)\ \forall v_1, v_2 \in V_1$, and
ii. $T(av) = aT(v)\ \forall a \in K, \forall v \in V_1$.

If (i) is satisfied by a mapping $T: V_1 \to V_2$, then T is said to be "additive." If (ii) is satisfied by a mapping $T: V_1 \to V_2$, then T is said to be a "homogeneous mapping." It is important to mention that a linear mapping preserves the origin and negatives, namely, $T(0) = T(0 \cdot 0) = 0T(0) = 0$, and $T(-v) = (-1)T(v) = -T(v)$. If a mapping is not linear, then it is called "nonlinear." A nonlinear mapping never satisfies the property of homogeneity.

If a mapping T between two algebraic structures of the same kind, say U and V (such as groups, vector spaces, etc.), preserves the operations of the algebraic structures, then T is a "homomorphism." Homomorphisms of vector spaces are also called linear mappings, that is, linear mappings are homomorphisms of vector spaces.

Let us consider an input–output system such that the inputs and the outputs are vectors, and the system effects a linear transformation represented by some square matrix A. In particular, let us assume that the input vector is $\vec{v}$, and that the output vector is $A\vec{v}$. Often, the direction of the output $A\vec{v}$ is different from the direction of the input $\vec{v}$. In order to understand such a system better, we have to find the input vectors that do not change direction when they pass through (i.e., when they are transformed by) the system (namely, we have to find those

345 Ibid.

vectors whose magnitude may change, but whose direction remains invariant). More formally stated, we have to find the $\vec{v}$'s for which $A\vec{v}$ is just a scalar multiple of $\vec{v}$. These $\vec{v}$'s are called eigenvectors; and the eigenvalue is the scaling factor. A non-zero vector $\vec{v}$ and a number λ are called, respectively, an "eigenvector" and an "eigenvalue" of a square matrix A if they satisfy the following equation:

$$A\vec{v} = \lambda\vec{v}.$$

In other words, eigenvectors inform us about the direction of spread of data, and eigenvalues inform us about the intensity of spread (i.e., about the magnitude of a distortion due to a linear transformation) in a particular direction (i.e., in the direction of the corresponding eigenvectors).

We can find the eigenvalues as follows: if I is an identity matrix, then

$$A\vec{v} = \lambda\vec{v} \Rightarrow A\vec{v} = \lambda I\vec{v} \Rightarrow A\vec{v} - \lambda I\vec{v} = 0 \Rightarrow (A - \lambda I)\vec{v} = 0,$$

and, if $\vec{v}$ is non-zero, then we can solve for λ using only the determinant:

$$det(A - \lambda I) = 0;$$

the above equation is known as the "characteristic equation."

We can find the eigenvectors as follows: for each eigenvalue λ_i found according to the aforementioned method, we solve the system

$$(A - \lambda_i I)\vec{v} = \vec{0}$$

for the corresponding $\vec{v}$. The set of eigenvectors for a given λ is called its "eigenspace."

Some Applications of Matrices

In this section, I provide a few basic examples of applications of matrix theory to the study of empirical problems, such as input–output analysis, linear programming, and game theory.[346]

Input–Output Analysis

As the renowned Russian/Soviet scientist and philosopher Alexander A. Bogdanov (1873–1928) has pointedly observed, "nature is what people call the endlessly unfolding field of their labor-experience,"[347] "labor, as a whole, is the activity of all humanity in the historical interconnectedness of all its generations,"[348] and "human beings change the correlation of the elements of nature so that they conform to their needs and desires, so that they serve their interests."[349] The major economic tasks that every society must accomplish pertain to decision-making about an economy's inputs and outputs. In economics, the term

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[346] Ibid.
[347] Bogdanov, *The Philosophy of Living Experience*, p. 42.
[348] Ibid.
[349] Ibid.

"input" refers to commodities or services used by firms in their production processes. Thus, by means of its technology, an economy combines inputs to produce outputs. In economics, the term "output" refers to the various useful goods or services that are either employed in further production or consumed. In the context of political economy, people as a social order have to figure out three things: first, what to produce; second, how to produce it; and, third, how to distribute the output.

The acknowledged founder of "input-output analysis" is the Russian-American economist Wassily Leontief, who won the Nobel Prize in Economics in 1973.[350] An input–output matrix is a square matrix, say $A = (a_{ij})$, whose entries a_{ij} represent the amount of input i required per unit of output j. A column of such a matrix depicts the inputs needed for the achievement of a specific output, and, therefore, from the perspective of economics, it can be considered as a "production technique." Hence, an input–output matrix is a "constellation" of production techniques. If the list of inputs is complete, including factor inputs, then the input–output matrix contains techniques for the production of the factor services as well. Input–output is an integral part of general equilibrium analysis. As the American economist Campbell R. McConnell has pointed out, the economy is "an interlocking network of prices wherein changes in one market are likely to elicit numerous and significant changes in other markets," and, therefore, economists need to study "the price system as a whole" and shift their analysis from equilibrium models pertaining to particular economic industries to "general equilibrium analysis," in order to determine the optimal level of production of the whole economy.[351]

For instance, let us consider a small economic network that consists of two interdependent industries A and B (e.g., A may represent the final goods industry, and B may represent the energy industry). Obviously, this method can be generalized to any number of industries. We assume that, for each dollar's worth of goods/services produced by A, A needs to consume a quantity of A's output and a quantity of B's output, and, for each dollar's worth of goods/services produced by B, B needs to consume a quantity of B's output and a quantity of A's output. In particular: the production of each dollar's worth of A requires $\$q_{11}$ worth of A and $\$q_{21}$ worth of B; and the production of each dollar's worth of B requires $\$q_{12}$ worth of A and $\$q_{22}$ worth of B. Therefore, both industries sell to each other and buy from each other. In addition, assume that there is an external demand for A and B; specifically, let the final demand from the outside sector of the economy be $\$d_1$ million for A and $\$d_2$ million for B. Let x_1 and x_2 represent the total output from A and B, respectively. Then we formulate the following equation:

$$X = QX + D \Rightarrow X - QX = D \Rightarrow IX - QX = D \Rightarrow (I - Q)X = D,$$

where: $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$, and $D = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$,

[350] Leontief, "Quantitative Input and Output Relations in the Economic System of the United States." Moreover, see: Raa, *The Economics of Input–Output Analysis*.

[351] McConnell, *Economics*, p. 579.

and X is the "output matrix" (i.e., X is a column matrix representing the equilibrium output levels in industry A and industry B), Q is the "technology matrix," D is the "final demand matrix," and I is the identity matrix. If $I - Q$ is invertible, then the solution for X is given by

$$X = (I - Q)^{-1}D,$$

which is the optimum level of production for the given economic network, meaning that the given economic network must produce x_1 million dollars of A (e.g., final goods) and x_2 million dollars of B (e.g., energy) in order meet both the internal demand and the external demand for A and B (and, thus, avoid both oversupplying and undersupplying the market with the corresponding commodities).

Linear Programming

By the term "linear programming," we mean a method to achieve the best outcome (e.g., to maximize profit, minimize cost, etc.) in a mathematical model whose requirements are represented by linear functions. The first contributions to linear programming are due to the Soviet mathematician and economist Leonid Vitaliyevich Kantorovich (1912–86), who won the Nobel Prize in Economics in 1975. Moreover, one of the acknowledged founders of linear programming is the American mathematician George Bernard Dantzig (1914–2005), who managed to make significant contributions to industrial engineering, operations research, economics, statistics, and computer science.[352] In fact, input–output analysis is a special, very important case of linear programming.

The "canonical form" of linear programming is the following: given a system of m linear constraints, namely, linear inequalities, with n variables, we wish to find non-negative values (i.e., ≥ 0) of these variables that will satisfy the constraints and will maximize a function of these variables; symbolically:

Given m linear inequalities and/or equalities

$$\sum_j a_{ij}\, x_j \leq b_i, i = 1,2, \dots, m, and\ j = 1,2, \dots, n, \quad (*)$$

we wish to find those values of x_j which satisfy the constraints $(*)$ and the condition that $x_j \geq 0$ (for $j = 1,2, \dots, n$) and simultaneously maximize the linear function

$$z = \sum c_j x_j\, , j = 1,2, \dots, n. \quad (**)$$

For instance, consider a problem where we wish to maximize the gross profit of an industry (or of a firm offering several product lines) that produces n commodities, and, thus, it has n sectors of production. Then $(*)$ and $(**)$ can be interpreted as follows: z is the value of overall performance measure, specifically, total gross profit; x_j is the level of activity j $(j = 1,2, \dots, n)$, specifically, the output of the jth sector of production (i.e., the produced quantity of the jth commodity); c_j is the performance measure coefficient for activity j, specifically, the gross profit per unit of output in the jth sector of production (so that the total

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[352] Dantzig, *Linear Programming and Extensions*. Moreover, see: Hadley, *Linear Programming*.

gross profit in the jth sector of production is $c_j x_j$); b_i is the amount of resource i available ($i = 1,2, \dots, m$); and a_{ij} is the amount of resource i consumed by each unit of activity j.

In matrix form, the constrained maximization problem ($**$) can be rewritten as follows:

$$z_{max} = (c_1 \quad c_2 \quad \cdots \quad c_n) \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

under the constraints

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \leq \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix},$$

and

$$x_j \geq 0 \text{ for } j = 1,2, \dots, n.$$

More simply, given the above concepts, we can write:

$$\left.\begin{matrix} maxz = cx \\ under\ the\ constraints \\ Ax \leq b \\ x_j \geq 0 \end{matrix}\right\}. \qquad (***)$$

Regarding the geometric significance of ($***$), notice that the constraints $Ax \leq b$ and $x_j \geq 0$ define a convex polyhedron P_n in $\mathbb{R}^n$, and such P_n is called the "feasible region" of the corresponding model, namely, the region of all the feasible solutions of the corresponding problem. In general, a polyhedron P_n in $\mathbb{R}^n$ is the set of all points $x \in \mathbb{R}^n$ that satisfy a finite set of linear inequalities. Moreover, a set Q in $\mathbb{R}^n$ is called "convex" (or "concave up") if, for any two points x and y in Q, the line segment joining them is also in Q; symbolically: $\forall x, y \in Q$, the "convex combination" $kx + (1-k)y \in Q$ for any k such that $0 \leq k \leq 1$. The goal of constrained maximization in the context of linear programming is to choose that feasible combination $(x_1, x_2, \dots, x_n)$ of actions that maximize a given function $z = cx$. This occurs at the maximum (most extreme) point $(x_1^*, x_2^*, \dots, x_n^*)$ of the feasible region.

The constrained maximization problem ($***$) is known as the "primal problem," while the so-called "dual problem" is the corresponding constrained minimization problem, where, given a system of m linear constraints, namely, linear inequalities, with n variables, we wish to find non-negative values (i.e., ≥ 0) of these variables that will satisfy the constraints and will minimize a function (e.g., a cost function) of these variables; symbolically:

$$\left.\begin{matrix} minz = cx \\ under\ the\ constraints \\ Ax \geq b \\ x_j \geq 0 \end{matrix}\right\}. \qquad (****)$$

For instance, using the "dual problem," we can create models of constrained cost minimization in economics and business management. Firms seek to minimize cost subject to the constraint that they produce at least b units of output, so that the firm's cost minimization problem is given by $(****)$.

Game Theory

Game theory is the study of mathematical models of strategic interaction between rational decision-makers. However, some important game-theoretical insights can be found in Plato's *Laches* and *Symposium*. Moreover, analyzing strategic thinking, Plato, in his *Republic*, refers to an ancient Greek strategy game that was called "petteia" (translated as "pebbles," or "pawns").

In general, "rationality" means that social behavior can be seen in terms of actors pursuing goals. The "rationality postulate" implies the following: (i) actors have well ordered preference systems over the set of outcomes (of alternative actions), namely, for all pairs c_i and c_j, there is a preference relation R such that either $c_i R c_j$ (the actor prefers c_i to c_j), or $c_j R c_i$ (the actor prefers c_j to c_i), or both (the actor is indifferent); (ii) each actor's preference system is substantially independent of the other social variables; (iii) each actor acts to maximize one's utility index. In particular, one can formulate a decreasing sequence of numbers (these numbers are called "utilities," u_n) where the largest number is assigned to the most preferred outcome, the second largest number to the next outcome in the preference order, etc. The function that maps consequences to numbers representing an actor's preference over those outcomes is said to be a "utility function." The most well-known utility function is the von Neumann–Morgenstern utility function, which is defined as follows: the actor considers a set of all conceivable states of the world and assesses the likelihood of each state S by assigning a probability $p(S)$ to it, so that the expected utility $U_e(A)$ for an action A can be calculated by multiplying the probability $p(S)$ of each state's occurring by the utility $u\big(C(S,A)\big)$ of the outcome that results from the given state of the world and the given action, and then summing these products over all the possible states:

$$U_e(A) = \sum_{all\ S} p(S) u\big(C(S,A)\big);$$

the actor chooses A such that $U_e(A)$ is maximized.

Game theory methods and models are the bases of modern general equilibrium theory in economics. The American economists Paul A. Samuelson and William D. Nordhaus have written that "game theory analyzes the way that two or more players or parties choose actions or strategies that jointly affect each participant."[353] In fact, the underlying reasoning of linear programming is similar to that of game theory. The first general mathematical formulation of game theory was given in the 1940s by the mathematician John von Neumann and the economist Oskar Morgenstern.[354] Moreover, several great Russian/Soviet mathematicians, such as Nikolai N. Vorob'ev,[355] Elena Borisovna Yanovskaya,[356] and Joseph Vladimirovich

353 Samuelson and Nordhaus, *Economics*, p. 205.
354 Neumann and Morgenstern, *Theory of Games and Economic Behavior.* Moreover, see: Luce and Raiffa, *Games and Decisions.*
355 Vorob'ev, *Game Theory.*
356 Yanovskaya, "First All-Union Conference on Game Theory."

Romanovsky,[357] have made foundational contributions to game theory, and they have proved advanced theorems in this field.

Every game is a model of a social situation, and, in every such model, there exist:

- n players (decision-makers), where $n \geq 2$;
- rules that determine the series of available actions (strategies or policies) and are known to the players;
- a well-defined set of outcomes; and
- pay-offs that are related to each outcome and are known to the players.

There are different kinds of games. In particular, with regard to the number of players, games can be distinguished into 2-player games and games that involve more than two players. With regard to the nature of the pay-offs, games can be distinguished into those that are zero-sum and those that are not. By a "zero-sum" game, we mean (in the case of a 2-player game) a game in which one player can only be made better off by making the other player worse off. With regard to the communication between the players, games can be distinguished into cooperative games and non-cooperative games. In the context of "non-cooperative" games, each player is assumed to maximize one's own utility function while treating the other players' expected strategic responses as constrains. In the context of "cooperative" games, players are assumed to have already, by some process, agreed on a series of strategies and, thus, on an outcome (for instance, this is naturally the case in parliamentary coalitions).

The most common type of competitive situations are 2-player zero-sum games, whose general form is shown in Table 2.1.

Table 2.1. The Pay-Off Matrix of a 2-Player Zero-Sum Game

Player B							
B_1	B_2	. . .	B_j	. . .	B_n		
c_{11}	c_{12}	. . .	c_{1j}	. . .	c_{1n}	A_1	Player A
c_{21}	c_{22}	. . .	c_{2j}	. . .	c_{2n}	A_2	
. . .	. . .	. . .	. . .	. . .	. . .	$\vdots$	
c_{i1}	c_{i2}	. . .	c_{ij}	. . .	c_{in}	A_i	
. . .	. . .	. . .	. . .	. . .	. . .	$\vdots$	
c_{m1}	c_{m2}	. . .	c_{mj}	. . .	c_{mn}	A_m	

In the above table, which is called the "pay-off matrix" of the corresponding game, the set $\{A_i, i = 1,2, \dots, m\}$ represents the strategies that are available to player A, and the set $\{B_j, j = 1,2, \dots, n\}$ represents the strategies of player B. If we assume that the pay-offs c_{ij} are expressed in terms of A's gains (and, thus, B's losses), then the quantity c_{ij} represents player A's benefit (and, thus, player B's loss) when the two players choose the pair of strategies (A_i, B_j).

In 2-player zero-sum games, there are several criteria ("decision rules") by which each player chooses one's strategy. The most important such criteria are the following:

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[357] Romanovsky, "Reduction of a Game with Perfect Recall to a Constrained Matrix Game."

i. *Wald's maximin criterion* (developed by the Hungarian-American mathematician Abraham Wald in the 1940s): According to this criterion, the player should select the alternative that provides the best of the worst possible outcomes, namely, the player should choose the strategy that maximizes the minimum possible outcome. In particular, given an $m \times n$-matrix (where m is the number of rows or the strategies available to A, and n is the number of columns or the strategies available to B), A takes the minimum pay-off, say k_i, from each row, namely, $k_i = minc_{ij}$, where $i = 1,2, \dots, m$, and then chooses the maximum k_i, namely, $max(k_1, k_2, \dots, k_m)$. A's strategy is the one which corresponds to the row that contains the maximum k_i. Although Wald's maximin criterion focuses on the most pessimistic outcome for each decision alternative, it should not be dismissed; for, it implicitly assumes a very strong aversion to risk, and, therefore, it is appropriate for decisions involving the possibility of catastrophic outcomes.

   Alternatively, Wald's maximax criterion states that the player should find the best possible (maximum) outcome for each decision alternative and then choose the option whose best outcome provides the highest (maximum) pay-off. The maximax criterion implicitly assumes that the player focuses on expected returns and disregards the dispersion of returns (i.e., risks) or that, in the given environment of action, the level of uncertainty is very low.

ii. *Laplace criterion:* Given an $m \times n$-matrix, player A assigns the probability $p = \frac{1}{n}$ to each strategy of B, since B has n alternative strategies, and A assumes that they are equally probable. Then A calculates the expected pay-off e_i of each row, given by
$e_i = \sum_{j=1}^{n} pc_{ij} = \frac{1}{n}(c_{i1} + \cdots + c_{in})$, $i = 1,2, \dots, m$.
Finally, A chooses the maximum e_i, namely, $max(e_1, e_2, \dots, e_m)$. Therefore, A's strategy is the one which corresponds to the row that contains the maximum e_i.

iii. *Hurwicz's optimism–pessimism criterion* (originally presented by the Polish-American economist and mathematician Leonid Hurwicz in 1951): According to this criterion, an index α such that $0 \leq \alpha \leq 1$, called the optimism index of player A, is assigned to the maximum pay-off of each row. Moreover, the index $1 - \alpha$, called the pessimism index of player A, is assigned to the minimum pay-off of each row. Finally, player A calculates the expected pay-off of each row as the weighted sum of the minimum and the maximum pay-offs of the corresponding row. Therefore, A's strategy is the one which corresponds to the row that contains the maximum expected pay-off. In particular, if x_i and y_i are the maximum and the minimum pay-offs of row i, respectively, namely,

   $x_i = maxc_{ij}$, and

   $y_i = minc_{ij}$,

where $i = 1,2, \dots, m$, then player A calculates the expected pay-off s_i of each row by the formula

   $s_i = \alpha x_i + (1 - \alpha) y_i$, $i = 1,2, \dots, m$,

and, finally, A finds the maximum such weighted sum, namely,

   max $(s_1, s_2, \dots, s_m)$.

   Hence, the strategy that player A must choose is the one which corresponds to the row that contains the maximum s_i. Notice that the index α is mostly subjective, and it is determined by the corresponding player's perception of one's historical environment as well as by the corresponding player's knowledge of history and policy-making.

iv. *Regret (or Minimax) criterion:* According to this criterion, the player should minimize the maximum possible regret (opportunity loss) associated with a wrong decision after the fact. In other words, this criterion states that the player should minimize the difference between possible outcomes and the best outcome for each state of the world. It should be mentioned that "opportunity loss" is defined to be the difference between a given pay-off and the highest possible pay-off for the resulting state of the world. Opportunity losses result because returns actually received under conditions of uncertainty are usually lower than the maximum return that would have been possible if the player had perfect knowledge beforehand. In mathematical terms, the regret or minimax criterion states that, from the original pay-off matrix, player A constructs a new matrix, called regret matrix, and then applies Wald's maximin criterion. The regret matrix is constructed by subtracting the maximum element of each column of the original pay-off matrix from each element of this column, namely, the elements of the regret matrix will be the r_{ij}'s which, for the column j (where $j = 1,2,\dots,n$), are given by

$$r_{ij} = c_{ij} - max c_{ij},$$

where $i = 1,2,\dots,m$.

A significant part of advanced game-theoretical research works emphasizes various forms of dynamic analysis. Within a given game, analysis concentrates on the modification of strategies as the game unfolds, instead of assuming that strategies are chosen once-and-for-all (and players may play mixed strategies, too). Moreover, multi-level games consider series of linked games, and, within the framework of a multi-level game, the outcome of each stage determines which game is to be played next. Hypergame analysis[358] goes further by starting from the assumption that the players may perceive the game in quite different terms. The basic model of hypergame analysis is not a single game perceived by all the players but a set of subjective games, each expressing one player's view of the situation. Thus, in hypergame terms, a situation in which both players correctly perceive the same game is said to be a level-zero hypergame; a situation in which both players believe that they are playing the same game while at least one player misperceives the game is said to be a level-one hypergame; a situation in which at least one perceives the other player's (assumed) misperceptions is said to be a level-two hypergame. The analysis of hypergames of level higher than two is a very arduous task, because these hypergames require long mental recursions of the type "I think he thinks I think he thinks, and so on."

2.2.6. Analytic Geometry and the Abstract Concept of a Distance

Geometry is the scientific study of the quantitative and the qualitative properties of spatial forms and relations (the criteria for equality of triangles provide instances of qualitative geometric knowledge, and the computation of lengths, areas, and volumes

[358] See: Takahashi, Fraser, and Hipel, "A Procedure for Analyzing Hypergames."

exemplifies quantitative geometric knowledge). In geometry, the abstraction of a straight line can be attributed to mathematical intuition. According to the ancient Greek mathematician Euclid, an arbitrary straight line can be construed as a "length without breadth" that is perceived as a whole. Furthermore, as I explained in section 2.2, there are points on every straight line, each point on the straight line corresponds to a real number, and the straight line is complete, for which reason it is known as the arithmetic or the geometric continuum. In fact, the ancient Greek mathematicians' awareness of the existence of real numbers was developed with reference to geometric processes, in the sense that they understood a real number either as a completed process of combining units/monads (that is, as a rational number) or as an incomplete process of measuring non-commensurate quantities (that is, as an irrational number).

Analytic geometry signifies the introduction of coordinates into geometry in a systematic way, specifically, by unifying aspects of algebra and aspects of geometry. In fact, the development of analytic geometry set the stage for the development of infinitesimal calculus. The first pioneers of analytic geometry were the second-century B.C. Greek astronomer and mathematician Hipparchus of Nicaea, who introduced coordinates for the sphere (in the context of his studies of the night sky), and the third-century B.C. Greek geometer Apollonius of Perga, who introduced coordinates for the study of conic sections. In the Middle Ages, the use of coordinates in mathematics and analytic geometry was further analyzed and developed by the fourteenth-century French philosopher and mathematician Nicolas d'Oresme.

By the term "locus," we mean a set of all the points that satisfy a specific rule. Moreover, the path drawn by a point moving according to a given rule is called the "locus of the point."The seventeenth-century French jurist and mathematician Pierre de Fermat has pointed out that, "whenever two unknown magnitudes appear in a final equation, we have a locus, the extremity of one of the unknown magnitudes describing a straight line or curve," and, whereas "the straight line is simple and unique," it is easily understood that "the classes of curves are indefinitely many—circle, parabola, hyperbola, ellipse, etc."[359] Thus, according to Fermat's terminology, "when the extremity of the unknown magnitude which traces the locus, follows a straight line or a circle, the locus is said to be plane; when the extremity describes a parabola, a hyperbola, or an ellipse, the locus is said to be solid."[360] Thus, we can study geometric problems through algebra.

One of the most important geometric theorems is the Pythagorean Theorem, which states that, in every right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides. This theorem can be proved in an algebraic way, in the spirit of Fermat's aforementioned observations, as follows.

Pythagorean Theorem[361]*:* Consider a right-angled triangle ΔABC, whose hypotenuse is c, and whose other two sides are a and b, as shown in Figure 2.3. Then $a^2 + b^2 = c^2$.

Proof: Given the triangle shown in Figure 2.3, we create four triangles identical to it, and we use them in order to form a square with side lengths $a + b$ as shown in Figure 2.4. The area of this square is

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[359] Fermat, "On Analytic Geometry," p. 389.
[360] Ibid.
[361] See: Maor, *The Pythagorean Theorem*.

$$A = (a + b)(a + b). \quad (*)$$

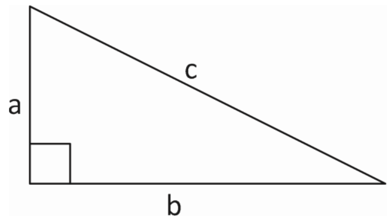

Figure 2.3. Pythagorean Theorem.

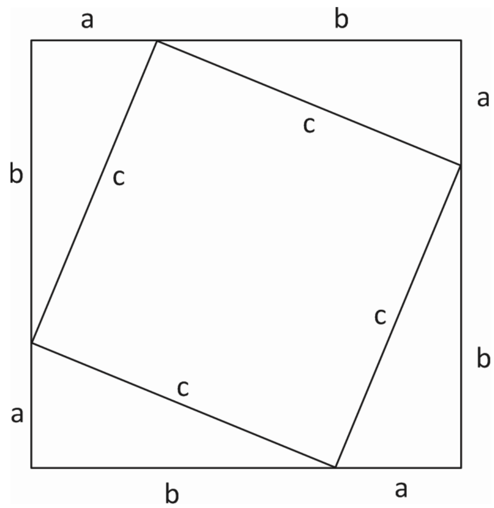

Figure 2.4. Proof of the Pythagorean Theorem.

In Figure 2.4, inside the big square, the hypotenuses of the four identical triangles form another smaller square, whose area is equal to c^2. Each of the four triangles has an area of $\frac{ab}{2}$. In general, notice that, given an arbitrary rectangle $ABCD$ whose height is h, and whose base is b, its area is equal to hb, and, therefore, if we draw a diagonal from one vertex, say diagonal AC, it will break the rectangle into two congruent, or equal, triangles, and the area of each of these triangles is half the area of the rectangle, that is, $\frac{hb}{2}$.

The area of all four of the triangles that are shown in Figure 2.4 is equal to $\frac{4ab}{2} = 2ab$. Adding up the areas of the smaller square and of the four triangles, we obtain

$$A = c^2 + 2ab. \quad (**)$$

Hence, given $(*)$ and $(**)$,

$$(a + b)(a + b) = c^2 + 2ab \Leftrightarrow a^2 + b^2 = c^2. \blacksquare$$

Remark: The Pythagorean Theorem is often a quadratic equation, whenever one of the lengths is unknown. The general form of the quadratic equation is

$$ax^2 + bx + c = 0, a \neq 0, a, b, c \in \mathbb{R},$$

and its roots are given by the formula

$$x = \frac{-b \pm (b^2 - 4ac)^{1/2}}{2a}.$$

However, the quadratic equation describes parabolic curves, whereas the Pythagorean Theorem describes right-angled triangles.

Analytic geometry is a branch of mathematics that studies geometric problems through algebra. As I have already mentioned, the roots of analytic geometry can be traced back to ancient Greek mathematicians, such as the third-century B.C. Greek geometer and astronomer Apollonius of Perga, who is famous for his work on "conic sections." Indeed, ancient Greek mathematicians had observed that circles, ellipses, hyperbolas, and parabolas result from the intersection of a cone by an adequate plane. A circle is produced when the cone is cut by a plane that is parallel to the base of the cone. An ellipse is produced when the cone is cut by a plane that is not parallel to the base of the cone or the side of the cone, and it cuts only one nappe of the cone. A hyperbola is produced when the intersecting plane cuts both nappes of the cone. A parabola is produced when the oblique section of the cone is parallel to the slant height (i.e., the height of a cone from the vertex to the periphery (rather than the center) of the base).

In analytic geometry, we put traditional (Euclidean) geometry on the Cartesian plane. René Descartes has pointed out that "any problem in geometry can easily be reduced to such terms that knowledge of lengths of certain straight lines is sufficient for its construction."[362] In particular, according to Descartes, "just as arithmetic consists of only four or five operations, namely, addition, subtraction, multiplication, division, and the extraction of roots, which may be considered a kind of division, so in geometry," we can find required lines by merely adding or subtracting other lines; or else, by working as follows:

> . . . taking one line which I shall call unity in order to relate it as closely as possible to numbers, and which can in general be chosen arbitrarily, and having given two other lines, to find a fourth line which shall be to one of the given lines as the other is to unity (which is the same as multiplication); or, again, to find a fourth line which is to one of the given lines as unity is to the other (which is equivalent to division); or, finally, to find one, two, or several mean proportionals between unity and some other line (which is the same as extracting the square root, cube root, etc., of the given line).[363]

Consider two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ on the xy-plane and connect them with a straight line segment as shown in Figure 2.5.

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362 Descartes, "On Analytic Geometry," p. 397.
363 Ibid, p. 397–98.

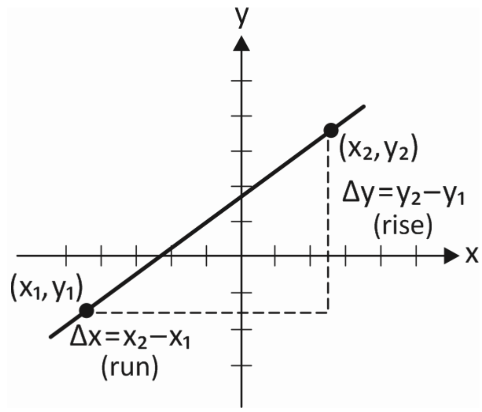

Figure 2.5. Slope and Distance.

The x-coordinate of point P is x_1, the x-coordinate of point Q is x_2, and the distance between x_1 and x_2 is $x_2 - x_1$; and, in order to avoid the use of plus and minus signs, we can use the absolute value $|x_2 - x_1|$. The y-coordinate of point P is y_1, the y-coordinate of point Q is y_2, and the distance between y_2 and y_1 is $y_2 - y_1$; and, in order to avoid the use of plus and minus signs, we can use the absolute value $|y_2 - y_1|$. Therefore, the horizontal distance between points P and Q is $x_2 - x_1$, and the vertical distance between points P and Q is $y_2 - y_1$. Now, consider the right-angled triangle that is defined by the points $P(x_1, y_1)$, $Q(x_2, y_2)$, and the point R (the intersection between the horizontal side and the vertical side): the three sides of this right-angled triangle are the hypotenuse PQ, the horizontal side, which is $x_2 - x_1$, and the vertical side, which is $y_2 - y_1$. The "slope," or "gradient," of the straight line segment PQ, denoted by m_{PQ}, is the quotient of the "rise" over the "run," comparing how much one travels vertically ("up and down") versus how much one travels horizontally, and, thus, it relates the steepness or inclination of the straight line segment PQ to the coordinates; symbolically:

$$m_{PQ} = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1}.$$

In Figure 2.5, the distance between points P and Q, denoted by d_{PQ}, is given by (and, indeed, is a version of) the Pythagorean Theorem. Therefore, in Figure 2.5,

$$\left(d_{PQ}\right)^2 = (run)^2 + (rise)^2$$
$$\Leftrightarrow d_{PQ} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

It can be easily verified that the midpoint of the straight line segment joining points (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

All points (x, y) in $\mathbb{R}^2$ satisfying the equation $y = mx + b$ form a straight line, and m is the slope of the straight line. For the slope m of the straight line passing through the points (x_1, y_1) and (x_2, y_2), we have:

i. If $x_1 = x_2$, m is undefined (the line is vertical).

ii. If $x_1 \neq x_2$, then $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$.

Two straight lines y_1 and y_2 with slopes m_1 and m_2, respectively, are parallel if and only if $m_1 = m_2$, and they are perpendicular if and only if $m_2 = -\frac{1}{m_1}$.

To find the equation of a non-vertical straight line, we work as follows:

i. we find a point (x_1, y_1) on the line;
ii. we find the slope m of the line;
iii. we write the equation of the line as follows: $y - y_1 = m(x - x_1)$; this equation is called the "point-slope" form of the equation of a line.

For instance, let us find the equation of the straight line passing through the points $(5, -0.5)$ and $(10, 9.5)$. First, we define the point $(x_1, y_1) = (5, -0.5)$. Second, we find the slope of the required line: $m = \frac{9.5-(-0.5)}{10-5} = 2$. Third, we find the equation of the required line: $y - y_1 = m(x - x_1) \Rightarrow y - (-0.5) = 2(x - 5) \Rightarrow y = 2x - 10.5$.

Circle

As we can see in Figure 2.6, a circle with center $O(v, w)$ and radius r is the set of all points in the xy-plane whose distance from O is r (in Figure 2.6, $O(v, w) = O(2, -1)$, and $r = 3$).

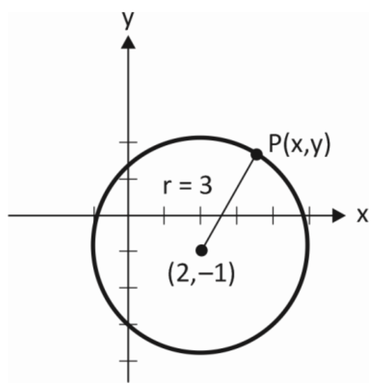

Figure 2.6. Circle.

If (x, y) is a point on the circle with center $O(v, w)$ and radius r, then the distance formula implies that

$$r = \sqrt{(x - v)^2 + (y - w)^2} \Leftrightarrow r^2 = (x - v)^2 + (y - w)^2,$$

which is the standard form of the equation of a circle with center (v, w) and radius r. The circumference of a circle of radius r is $C = 2\pi r$, and the area of a circle of radius r is $A = \pi r^2$, where $\pi \approx 3.14$ is Archimedes's constant (the ratio of the circle's circumference to its diameter).[364] It is worth mentioning that the degenerate possibilities for a circle are the following: a point or no graph at all.

The study of the circle underpins trigonometry. The term “trigonometry” appeared for the first time in the book *Trigonometria* by Bartholomaeus Pitiscus (1561–1613) in 1595, and it literary means measuring (and, more broadly, studying) “trigons” (“trigon” being the Latin word for “triangle”). The acknowledged founder of trigonometry is the ancient Greek astronomer and mathematician Hipparchus of Nicaea (ca. 190–ca. 120 B.C.). Moreover, around 100 A.D., another Greek mathematician, Menelaus of Alexandria, published a series of treatises on chords.

Trigonometric Functions[365]

In the context of analytic geometry, we can also study the basic trigonometric functions on the unit circle (specifically, on a circle whose center is $(0,0)$ and whose radius $r = 1$).

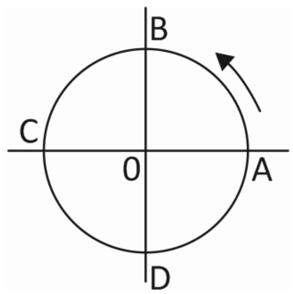

Figure 2.7. The Number Circle.

Consider a circle of unit radius, as shown in Figure 2.7, and let point A (the right-hand end point of the horizontal diameter) be a reference point. Let an anti-clockwise motion round the circle be a positive direction, and a clockwise motion be a negative direction. A circle of unit radius with a reference point and the direction of tracing specified is called the “number circle.” Given an arbitrary point P of the number circle, there are infinitely many arcs beginning at the point A and terminating at the point P. One of these arcs is the shortest arc connecting the points A and P, and all the other arcs are obtained from the shortest arc by adding or subtracting an integral number of complete revolutions. Hence, every point P of the number circle is associated with an infinite set of numbers that consists of the values of all the arcs beginning at the point A and terminating at the point P (the lengths of the arcs are taken with the plus or minus sign according as the motion from the point A to the point P is anti-clockwise or clockwise, respectively). The circumference of the circle of unit radius is equal to 2π, and, therefore, the lengths of all the arcs terminating at the given point P differ from one another by an integer number multiple of 2π, so that the general form of these quantities

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[364] See: Swokowski and Cole, *Algebra and Trigonometry with Analytic Geometry*; Wildberger, *Divine Proportions*. Archimedes approximated π by using the fact that the circumference of a circle is bounded by the perimeter of an inscribed polygon and the perimeter of a circumscribed polygon. In particular, he used a 96-sided inscribed polygon and a 96-sided circumscribed polygon to find the following approximation:
$3 + \frac{10}{71} < \pi < 3 + \frac{10}{70}$.
Moreover, regarding Archimedes and his scientific legacy, see: Rassias, ed., *Geometry, Analysis and Mechanics*.

[365] See: Swokowski and Cole, *Algebra and Trigonometry with Analytic Geometry*; Wildberger, *Divine Proportions*.

is $x + 2\pi a$, where $a \in \mathbb{Z}$, and x is the length of the shortest arc connecting the points A and P. Thus, for every real number x, there is a point $P(x)$ of the number circle such that the length of the arc AP is x, and every point P of the circle corresponds to an infinite set of numbers of the form $x + 2\pi a$, where $a \in \mathbb{Z}$, and x is the length of one of the arcs connecting the points A and P.

Assume that the center of the number circle coincides with the origin $O(0,0)$ of the rectangular coordinate system XOY, as shown in Figure 2.8. Let x be an arbitrary real number. Then, on the number circle, we find the point $P(x)$ that corresponds to x. The ordinate of the point $P(x)$ is called the "sine" of the number x (denoted by $sinx$), the abscissa of the point $P(x)$ is called the "cosine" of the number x (denoted by $cosx$), the ratio $\frac{sinx}{cosx}$ is called the "tangent" of the number x (denoted by $tanx$), and the ratio $\frac{cosx}{sinx}$ is called the "cotangent" of the number x (denoted by $cotx$).

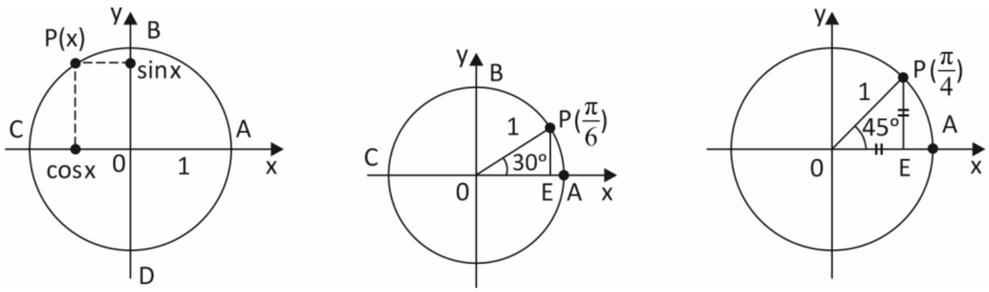

Figure 2.8. Trigonometric Functions.

Notice that the reference point A on the number circle corresponds to the number 0, since the abscissa and the ordinate of this point are 1 and 0, respectively, and we have $cos0 = 1$, $sin0 = 0$, and $tan0 = \frac{sin0}{cos0} = 0$. The point B of intersection of the circle and the positive ray of the axis OY corresponds to the number $\pi/2$. Since the abscissa and the ordinate of the point B are 0 and 1, respectively, we have $\cos\left(\frac{\pi}{2}\right) = 0$ and $\sin\left(\frac{\pi}{2}\right) = 1$, whereas $\tan\left(\frac{\pi}{2}\right)$ is not defined. Similarly, as shown in Figure 2.8, given the coordinates of the points C and D, we realize that $cos\pi = -1$, $sin\pi = 0$, $tan\pi = 0$, $\cos\left(\frac{3\pi}{2}\right) = 0$, $sin\left(\frac{3\pi}{2}\right) = -1$, and $tan\left(\frac{3\pi}{2}\right)$ is not defined.

We can summarize the basic definitions and the basic formulas of trigonometry as follows:

$Sine$: $\sin\theta = \frac{opposite\ side}{hypotenuse}$,

$Cosine$: $cos\theta = \frac{adjacent\ side}{hypotenuse}$,

$Tangent$: $tan\theta = \frac{opposite\ side}{adjacent\ side}$,

$Cosecant$: $csc\theta = \frac{hypotenuse}{opposite\ side}$,

$Secant$: $sec\theta = \frac{hypotenuse}{adjacent\ side}$,

$Cotagngent$: $cot\theta = \frac{adjacent\ side}{opposite\ side}$, and the basic trigonometric identities:

$sin^2a + cos^2a = 1$,
$\sin(-a) = -sina$,
$\cos(-a) = cosa$,
$\sin(a \pm b) = sina \cdot cosb \pm cosa \cdot sinb$,
$\cos(a \pm b) = cosa \cdot cosb \mp sina \cdot sinb$,
$sina + sinb = 2sin\frac{1}{2}(a+b) \cdot cos\frac{1}{2}(a-b)$,
$cosa + cosb = 2cos\frac{1}{2}(a+b) \cdot cos\frac{1}{2}(a-b)$,
$sin\frac{1}{2}a = \sqrt{\frac{1-cosa}{2}}$,
$cos\frac{1}{2}a = \sqrt{\frac{1+cosa}{2}}$,
$sin2a = 2sina \cdot cosa$,
$cos2a = cos^2a - sin^2a = 2cos^2a - 1$,
$tana = \frac{sina}{cosa}$,
$\sin(a \pm \frac{\pi}{2}) = \pm cosa$,
$\cos(a \pm \frac{\pi}{2}) = \mp sina$.

The inverse trigonometric functions are denoted as follows: $arcsinx \equiv sin^{-1}x$ ($y = arcsinx \Leftrightarrow x = siny$), $arccosx \equiv cos^{-1}x$ ($y = arccosx \Leftrightarrow x = cosy$), and $arctanx = tan^{-1}x$ ($y = arctanx \Leftrightarrow x = tany$).

Ellipse

As we can see in Figure 2.9, an "ellipse" is the set of all points in a plane the sum of whose distances from two fixed points ("foci") is constant.[366] Notice that, if the two foci coincide, then we receive a circle.

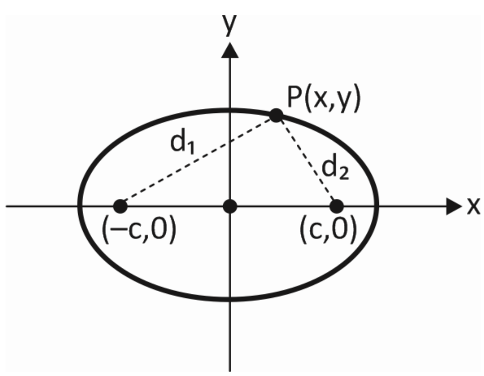

Figure 2.9. Ellipse.

[366] Ibid.

Let the two foci of an ellipse be the points $(-c, 0)$ and $(c, 0)$, so that $d_1 + d_2 = 2a$. Then a point $P(x, y)$ is on this ellipse if and only if

$$\sqrt{[x-(-c)]^2+(y-0)^2}+\sqrt{(x-c)^2+(y-0)^2}=2a$$
$$\Rightarrow \sqrt{(x+c)^2+(y-0)^2}+\sqrt{(x-c)^2+(y-0)^2}=2a$$
$$\Rightarrow \sqrt{(x+c)^2+y^2}=2a-\sqrt{(x-c)^2+y^2}$$
$$\Rightarrow (x+c)^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2$$
$$\Rightarrow xc=a^2-a\sqrt{(x-c)^2+y^2}$$
$$\Rightarrow a\sqrt{(x-c)^2+y^2=}\,a^2-xc$$
$$\Rightarrow a^2[(x-c)^2+y^2]=a^4-2a^2cx+c^2x^2$$
$$\Rightarrow a^2x^2-2a^2cx+a^2c^2+a^2y^2=a^4-2a^2cx+c^2x^2$$
$$\Rightarrow (a^2-c^2)x^2+a^2y^2=a^2(a^2-c^2)$$
$$\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1.$$

Notice that, because $d_1 + d_2$ is greater than the distance between the foci, it holds that $a > c$ and $a^2 - c^2 > 0$. If $b^2 = a^2 - c^2 \Leftrightarrow a^2 = b^2 + c^2$, then we receive the standard form of the equation of an ellipse with center at the origin and foci on the x-axis, namely,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

If we set $y = 0$, then we find the x-intercepts of the ellipse, say $(-a, 0)$ and $(a, 0)$; and, if we set $x = 0$, then we find the y-intercepts of the ellipse, say $(0, -b)$ and $(0, b)$. In Figure 2.9, the larger segment from $(-a, 0)$ to $(a, 0)$ is called the “major axis,” and the “minor axis” is the segment from $(0, -b)$ to $(0, b)$. The endpoints of the major axis are called the “vertices of the ellipse.” Hence, we have:

foci: $(-c, 0)$ and $(c, 0)$;
vertices: $(-a, 0)$ and $(a, 0)$.

In general, if the ellipse is centered at the point (u, v), and if the major axis is parallel to the x-axis, then the standard form of the equation of an ellipse is

$$\frac{(x-u)^2}{a^2}+\frac{(y-v)^2}{b^2}=1.$$

Similarly, we can show that, if the foci of an ellipse are two points $(0, -c)$ and $(0, c)$ on the y-axis, then the standard form of the equation of an ellipse becomes

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1,$$

and, then, the major axis is along the y-axis, so than then we have:

foci: $(0, c)$ and $(0, -c)$;

vertices: $(0, a)$ and $(0, -a)$.

In general, if the ellipse is centered at the point (u, v), and if the major axis is parallel to the y-axis, then the standard form of the equation of an ellipse is

$$\frac{(x-u)^2}{b^2} + \frac{(y-v)^2}{a^2} = 1.$$

Given the definition of an ellipse, the degenerate possibilities for an ellipse are the following: a point or no graph at all.

In our solar system, many bodies revolve in elliptical orbits around a larger body that is located at one focus. In the seventeenth century, Johannes Kepler, based on Apollonius's mathematical study of the ellipse, articulated a rigorous explanation of planetary motions.

Moreover, regarding the ellipse, it should be mentioned that it has a reflection property that causes any ray or wave that originates at one focus to strike the ellipse and pass through the other focus. In terms of acoustics, the aforementioned property implies that, in a room with an elliptical ceiling, even a slight noise made at one focus can be heard at the other focus, but, if people are standing between the foci, then they hear nothing. Such rooms are known as whispering galleries.

Hyperbola

As we can see in Figure 2.10, a "hyperbola" is the set of all points in a plane the difference of whose distances from two fixed points ("foci") is a positive constant.[367]

Let the two foci of a hyperbola be the points $(-c, 0)$ and $(c, 0)$, so that $|d_1 - d_2| = 2a$, or $d_1 - d_2 = \pm 2a$ (according as $d_1 > d_2$ or $d_1 < d_2$). Then a point $P(x, y)$ is on this hyperbola if and only if

$$\sqrt{(x+c)^2 + (y-0)^2} - \sqrt{(x-c)^2 + (y-0)^2} = \pm 2a$$
$$\Rightarrow \sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm 2a$$
$$\Rightarrow \sqrt{(x+c)^2 + y^2} = \pm 2a + \sqrt{(x-c)^2 + y^2}$$
$$\Rightarrow (x+c)^2 + y^2 = 4a^2 \pm 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2$$
$$\Rightarrow cx - a^2 = \pm a\sqrt{(x-c)^2 + y^2}$$
$$\Rightarrow c^2x^2 - 2cxa^2 + a^4 = a^2[(x-c)^2 + y^2]$$
$$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1.$$

For $c > 0 \Rightarrow a^2 - c^2 < 0$, so that, setting $b = \sqrt{c^2 - a^2}$, it follows that the standard form of the equation of the given hyperbola becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

If we set $y = 0$, then we find the x-intercepts of the hyperbola, say $(-a, 0)$ and $(a, 0)$. The segment of the x-axis joining $(-a, 0)$ and $(a, 0)$ is called the "transverse axis," and the

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[367] Ibid.

endpoints of the transverse axis are called the "vertices of the hyperbola." If we set $x = 0$, then we find that there are no y-intercepts. Notice that

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow y = \frac{\pm bx}{a}\sqrt{1 - \frac{a^2}{x^2}},$$

and, as $|x| \to \infty$, $1 - \frac{a^2}{x^2} \to 1$, so that the graph of the hyperbola approaches the lines

$$y = \pm\frac{b}{a}x.$$

Thus, we have just found the "asymptotes of the hyperbola" (which can be construed as the diagonals of rectangle of dimensions $2a$ by $2b$). In general, if the hyperbola is centered at the point (u, v), and if the transverse axis is parallel to the x-axis, then the standard form of the equation of a hyperbola is

$$\frac{(x-u)^2}{a^2} - \frac{(y-v)^2}{b^2} = 1.$$

Similarly, if the foci of a hyperbola are the points $(0, -c)$ and $(0, c)$ on the y-axis, then the standard equation of the hyperbola is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1,$$

and its asymptotes are the lines

$$y = \pm\frac{a}{b}x\ .$$

In general, if the hyperbola is centered at the point (u, v), and if the transverse axis is parallel to the y-axis, then the standard form of the equation of a hyperbola is

$$\frac{(y-v)^2}{a^2} - \frac{(x-u)^2}{b^2} = 1.$$

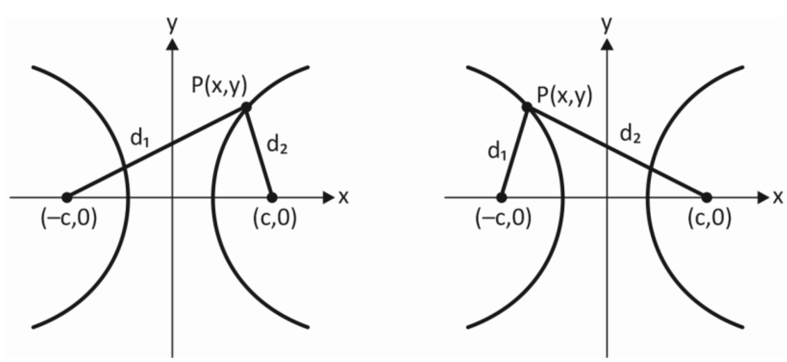

Figure 2.10. Hyperbola.

Given the definition of a hyperbola, the degenerate possibilities for a hyperbola are two intersecting straight lines.

Hyperbolic Functions[368]*:* The trigonometric or circular functions are sin, cos, tan, cot, sec, and csc, and they are defined on the trigonometric circle, where the y-axis is the sine axis, and the x-axis is the cosine axis. The trigonometric circle emerges while we parametrize the algebraic equation of the unit circle $x^2 + y^2 = 1$ setting $x = cos\theta$ and $y = sin\theta$, that is, $cos^2 + sin^2\theta = 1$. The hyperbolic functions are defined in the following way:

Hyperbolic sine of θ: $sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2}$.
Hyperbolic cosine of θ: $cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2}$.
Hyperbolic tangent of θ: $tanh\theta = \frac{sinh\theta}{cosh\theta} = \frac{e^{\theta}-e^{-\theta}}{e^{\theta}+e^{-\theta}}$.
Hyperbolic cotangent of θ: $coth\theta = \frac{cosh\theta}{sinh\theta} = \frac{e^{\theta}+e^{-\theta}}{e^{\theta}-e^{-\theta}}$.
Hyperbolic secant of θ: $sech\theta = \frac{1}{cosh\theta} = \frac{2}{e^{\theta}+e^{-\theta}}$.
Hyperbolic cosecant of θ, for $\theta \neq 0$: $csch\theta = \frac{1}{sinh\theta} = \frac{2}{e^{\theta}-e^{-\theta}}$.

Given that the standard form of a hyperbola with center at the origin and foci on the x-axis is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the right branch of a hyperbola can be parametrized by

$x = acosh\theta$ and $y = bsinh\theta$

and the left branch can be parametrized by

$x = -acosh\theta$ and $y = bsinh\theta$, so that
$cosh^2\theta - sinh^2\theta = 1$, and
$cosh^2\theta + sinh^2\theta = cosh2\theta$.

Parabola

As we can see in Figure 2.11, a "parabola" is the set of all points in a plane whose distances from a fixed line ("directrix") and a fixed point ("focus") that does not belong to the given line (i.e., to the directrix) are equal.[369]

For simplicity, as shown in Figure 2.11, let the directrix be the line $x = -p$, and let the focus of the parabola be the point $(p, 0)$. Then $P(x, y)$ designates an arbitrary point on the parabola. Because the points P_1 and P have the same y-coordinate, the distance d_1 is given by

$d_1 = |x - (-p)| = |x + p|$.

[368] Ibid.
[369] Ibid.

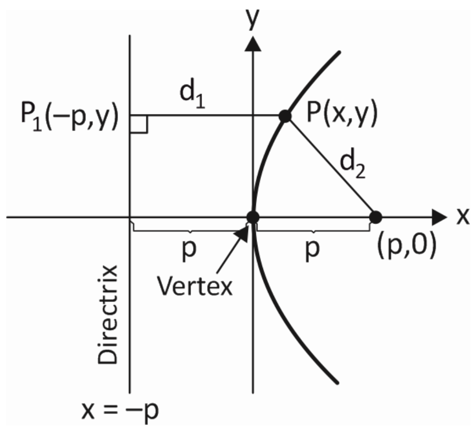

Figure 2.11. Parabola.

The distance from $P(x, y)$ to $(p, 0)$ is given by

$$d_2 = \sqrt{(x-p)^2 + (y-0)^2}.$$

By the definition of a parabola,

$$d_1 = d_2 \Rightarrow |x+p| = \sqrt{(x-p)^2 + y^2} \Rightarrow (x+p)^2 = (x-p)^2 + y^2.$$

Thus,

$$4px = y^2,$$

which is the standard form of the equation of a parabola with directrix $x = -p$ and focus at $(p, 0)$. The x-axis is the axis of symmetry of the given parabola (and the parabola opens to the right). If the axis of symmetry is the x-axis, but the parabola opens to the left, then the parabola's standard form is given by the equation

$$y^2 = -4px.$$

In case the axis of symmetry is the y-axis, then the parabola's standard form is $x^2 = 4py$ if it opens upward, or $x^2 = -4py$ if it opens downward. Given the definition of a parabola, the degenerate possibilities for a parabola are the following: a line, a pair of parallel lines, or no graph at all.

As regards the parabola, in general, it should be mentioned that it has a reflection property that causes any ray or wave that originates at the focus and strikes the parabola to be reflected parallel to the axis of symmetry. Thus, for instance, flashlights and searchlights use a parabolic reflector with the bulb located at the focus. Additionally, due to the reflection property of a parabola, any ray or wave that comes into a parabolic reflector parallel to the axis of symmetry is directed to the focus point. For this reason, radars, radio antennas, and reflecting telescopes operate according to this principle. Finally, due to their great strength, parabolic arches are used extensively in bridges, cathedrals, and elsewhere in architecture and engineering, especially in case we have equally spaced load.

Analytic Geometry of Space

Consider three coordinate axes Ox, Oy, and Oz that pass through the same point $O(0,0,0)$ and do not lie on the same plane. The x-axis, or Ox, is called the axis of the "abscissas," or the "abscissa axis"; the y-axis, or Oy, is called the axis of the "ordinates," or the "ordinate axis"; and the z-axis, or Oz, is called the axis of the "applicates," or the "applicate axis." The point $O(0,0,0)$ is called the origin of the xyz-coordinate system. The three planes xOy, yOz, and zOx, which are determined by the three coordinate axes, are called coordinate planes, and they are perpendicular to each other (thus, this coordinate system is called orthogonal).

Consider a vector $\overrightarrow{P_1P_2}$ such that the coordinates of the point P_1 are (x_1, y_1, z_1) and the coordinates of the point P_2 are (x_2, y_2, z_2). We project $\overrightarrow{P_1P_2}$ onto each of the three coordinate axes Ox, Oy, and Oz, and, each time, it is parallel to the coordinate planes that are determined by the other two coordinate axes. Then the coordinate projections of $\overrightarrow{P_1P_2}$ are $x_2 - x_1$ on the x-axis, $y_2 - y_1$ on the y-axis, and $z_2 - z_1$ on the z-axis.

Consider two points $P_1(x_1, y_1, z_1)$ with radius vector r_1 (i.e., r_1 is the vector from the origin of the xyz-coordinate system to the current position of P_1) and $P_2(x_2, y_2, z_2)$ with radius vector r_2 (i.e., r_2 is the vector from the origin of the xyz-coordinate system to the current position of P_2). Let P be an arbitrary point situated on the straight line segment P_1P_2 with radius vector r. If k is the partial ratio of P_1, P_2, and P, namely, if $k = \frac{\overrightarrow{P_1P}}{\overrightarrow{PP_2}}$, then the position vector $\vec{r}$ of P is given by the following formula: $\vec{r} = \frac{\overrightarrow{r_1} + k\overrightarrow{r_2}}{1+k}$. If P is the middle point of the straight line segment P_1P_2, then $k = 1$, and the position vector of P is $\vec{r} = \frac{\overrightarrow{r_1} + \overrightarrow{r_2}}{2}$. The coordinates of an arbitrary point $P(x, y, z)$ situated on the straight line segment P_1P_2 are given by the following formulas:

$$x = \frac{x_1 + kx_2}{1+k}, y = \frac{y_1 + ky_2}{1+k}, and\ z = \frac{z_1 + kz_2}{1+k},$$

and, in case P is the middle point of the straight line segment P_1P_2, then the coordinates of P are given by the following formulas:

$$x = \frac{x_1 + x_2}{2}, y = \frac{y_1 + y_2}{2}, and\ z = \frac{z_1 + z_2}{2}.$$

Two vectors $\vec{u}$ and $\vec{v}$ are "collinear" if and only if there exist two non-negative scalars (numbers) k and l such that $k\vec{u} + l\vec{v} = 0$. Consider a vector $\vec{u}$ written in component form as $\langle u_1, v_1, w_1 \rangle$ and a vector $\vec{v}$ written in component form as $\langle u_2, v_2, w_2 \rangle$. If $\vec{u}$ and $\vec{v}$ are collinear, then $\frac{u_1}{u_2} = \frac{v_1}{v_2} = \frac{w_1}{w_2}$, that is, their coordinate projections are proportional.

Three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ are "coplanar" if and only if there exist three non-negative scalars (numbers) k, l, and m such that $k\vec{u} + l\vec{v} + m\vec{w} = 0$. Three arbitrary vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ whose component forms (i.e., coordinate projections) are$\langle u_1, v_1, w_1 \rangle$, $\langle u_2, v_2, w_2 \rangle$, and $\langle u_3, v_3, w_3 \rangle$, respectively, are coplanar if and only if the determinant

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} = 0.$$

Length (or magnitude) of a vector: If $\vec{u} = \langle u_1, u_2, u_3 \rangle$, then the length (or the magnitude) of $\vec{u}$, denoted by $|\vec{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}$, that is, it is equal to the square root of the sum of the squares of its coordinate projections.

Dot or Scalar or Inner Product of vectors: Consider two vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$. Then their "dot (or scalar or inner) product" is given by the following formula:

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3,$$

so that we obtain a scalar (instead of a vector). The geometric significance of this operation is that we multiply the length of $\vec{u}$ times the length of $\vec{v}$ times the cosine of the angle θ between $\vec{u}$ and $\vec{v}$. In other words,

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| cos\theta.$$

Thus, this operation gives us information about the lengths of the vectors under consideration as well as about the angle that is formed between these vectors. Notice that the sign of $\vec{u} \cdot \vec{v}$ is going to be positive if $\theta < 90°$, and it is going to be negative if $\theta > 90°$ ($\vec{u} \cdot \vec{v}$ is going to be equal to 0 if $\theta = 90°$).

Corollaries: The cosine of the angle θ between two vectors $\vec{u}$ and $\vec{v}$ is equal to $\frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$. Two vectors are perpendicular to each other if and only if their dot product is zero (since $cos 90° = 0$).

Norm of a vector: The "norm" of a vector $\vec{u} = \langle u_1, u_2, \dots, u_n \rangle$ in $\mathbb{R}^n$ is the distance of the vector from the origin, and it is denoted by $\|\vec{u}\|$. Hence, the Euclidean norm of $\vec{u} = \langle u_1, u_2, \dots, u_n \rangle$ is (as a consequence of the Pythagorean Theorem):

$$\|\vec{u}\| = \sqrt{u \cdot u} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}.$$

Cross Product of two vectors in a 3-dimensional space: Consider two vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$, and let $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$ be the unit vectors of the three coordinate axes, respectively. Then the cross product of $\vec{u}$ and $\vec{v}$ is a vector given by the following determinant:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \equiv \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \vec{\imath} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \vec{\jmath} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \vec{k}.$$

The geometric significance of this operation is that, if θ is the angle between $\vec{u}$ and $\vec{v}$ with $0 \leq \theta \leq \pi$, then

$$\vec{u} \times \vec{v} = |\vec{u}||\vec{v}|sin\theta.$$

Whereas the resultant of the dot product of two vectors $\vec{u}$ and $\vec{v}$ is a scalar quantity, the cross product of two vectors $\vec{u}$ and $\vec{v}$ is a third vector whose direction is perpendicular to both $\vec{u}$ and $\vec{v}$. Whereas the dot product is zero when the vectors are orthogonal, the cross product is maximum when the vectors are orthogonal. Two vectors are parallel to each other if and only if they are scalar multiples of each other.

The area of the parallelogram P spanned by two vectors $\vec{u}$ and $\vec{v}$ scales with $\vec{u}$ and $\vec{v}$, and the proportionality factor is determined by the sine of the angle θ between $\vec{u}$ and $\vec{v}$, so that the length (or magnitude) of the cross product, namely, $|\vec{u} \times \vec{v}|$, is the area of the parallelogram spanned by $\vec{u}$ and $\vec{v}$. Following the same reasoning, if $\vec{u}$ is the base of a triangle, and if $\vec{v}$ is the altitude of the given triangle, then the area of the given triangle is equal to $\frac{1}{2}|\vec{u} \times \vec{v}|$.

The volume of the parallelepiped Q spanned by $\vec{u}$, $\vec{v}$, and $\vec{x}$ as a function of $\vec{x}$ is proportional to the base parallelogram P, which is spanned by $\vec{u}$ and $\vec{v}$. In particular, the volume of Q is equal to the dot product between $\vec{x}$ and$\vec{u} \times \vec{v}$, symbolically,

$$\vec{x} \cdot (\vec{u} \times \vec{v}),$$

known as the "triple product" of the given 3-dimensional vectors.

The Abstract Concept of a Distance

We shall use the notation $\mathbb{R}^n$ for the real n-space, namely, the set of all ordered n-tuples of real numbers. As I have already explained, a set of n real independent variables $x_1, x_2, \dots, x_n$ can be considered as the coordinates of a given point in the n-dimensional space $\mathbb{R}^n$, in the sense that each set of values of the variables defines a point of $\mathbb{R}^n$.

In $\mathbb{R}^n$, or, equivalently, in an n-dimensional vector space V over the real field $\mathbb{R}$, we define, in accordance with the Pythagorean Theorem, a distance function between points $x = (a_1, a_2, \dots, a_n)$ and $y = (b_1, b_2, \dots, b_n)$ by

$$d_E(x, y) \equiv |x - y| = [\sum_{k=1}^{n}(a_k - b_k)^2]^{1/2},$$

and, thus, we obtain the n-dimensional Euclidean space denoted by $\mathbb{R}^n$.

The Euclidean distance d_E has the following basic properties[370]:

(i) $|x - y| \geq 0; |x - y| = 0 \Leftrightarrow x = y$;
(ii) $|x - y| = |y - x|$;
(iii) $|x - y| \leq |x - z| + |z - y|$.

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[370] See: Abbott, *Understanding Analysis*; Baum, *Elements of Point Set Topology*; Blackett, *Elementary Topology*; Courant, *Differential and Integral Calculus*; Dieudonné, *Treatise on Analysis*; Haaser and Sullivan, *Real Analysis*; Kaplansky, *Set Theory and Metric Spaces*; Kolmogorov and Fomin, *Introductory Real Analysis*; Mendelson, *Introduction to Topology*; Rudin, *Real and Complex Analysis*. In fact, these books are my major bibliographical sources for the study of metric spaces, topology, real analysis, and complex analysis.

Properties (i) and (ii) follow directly from the definition of Euclidean distance d_E. We can prove property (iii) as follows: Let r be an arbitrary real number. Then

$$0 \leq \sum_{k=1}^{n}(a_k - rb_k)^2 = \sum_{k=1}^{n} a_k^2 - 2r\sum_{k=1}^{n} a_k\, b_k + r^2\sum_{k=1}^{n} b_k^2.$$

If $\sum_{k=1}^{n} b_k^2 \neq 0$, and if we set $r = (\sum_{k=1}^{n} a_k b_k)(\sum_{k=1}^{n} b_k^2)^{-1}$, we obtain

$$(\sum_{k=1}^{n} a_k b_k)^2 \leq \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2,$$

which is the "Cauchy–Schwarz–Buniakowski Inequality."[371] In case $\sum_{k=1}^{n} b_k^2 = 0$, the Cauchy–Schwarz–Buniakowski Inequality holds trivially, in the sense that $0 = 0$. If we set $x - z = (a_1, a_2, \dots, a_n)$ and $z - y = (b_1, b_2, \dots, b_n)$, then, using the Cauchy–Schwarz–Buniakowski Inequality, we obtain:

$$|x-y|^2 = \sum_{k=1}^{n}(a_k + b_k)^2 = \sum_{k=1}^{n} a_k^2 + 2\sum_{k=1}^{n} a_k\, b_k + \sum_{k=1}^{n} b_k^2 \leq \sum_{k=1}^{n} a_k^2 + 2(\sum_{k=1}^{n} a_k^2)^{1/2}(\sum_{k=1}^{n} b_k^2)^{1/2} + \sum_{k=1}^{n} b_k^2 = |x-z|^2 + 2|x-z||z-y| + |z-y|^2 = (|x-z| + |z-y|)^2.$$

Therefore, $|x-y| \leq |x-z| + |z-y|$.

Remark: The Cauchy–Schwarz–Buniakowski Inequality implies the "Minkowski Inequality,"[372] according to which, if a_k and b_k are any real numbers ($k = 1,2, \dots, n$), then

$$[\sum_{k=1}^{n}(a_k + b_k)^2]^{1/2} \leq (\sum_{k=1}^{n} a_k^2)^{1/2} + (\sum_{k=1}^{n} b_k^2)^{1/2}.$$

For $\varepsilon > 0$, we can define an "ε-neighborhood" of point P in $\mathbb{R}^n$ as the set X of points in $\mathbb{R}^n$ such that the distance $d_E(X, P) < \varepsilon$. Hence, a neighborhood is an open set. Then we shall use the notation $N_\varepsilon(P)$ in order to denote this set of points, and we shall use the notation $N'_\varepsilon(P)$ in order to denote the "deleted neighborhood," consisting of $N_\varepsilon(P)$ with the point P deleted. In terms of the real line $\mathbb{R}$, a deleted neighborhood is an interval on $\mathbb{R}$ with the center point removed. Notice that, in the 2-dimensional Euclidean space $\mathbb{R}^2$, $N'_\varepsilon(P)$ is 2-dimensional open ball (that is, an open disc) with center P and radius ε, and, in the 3-dimensional Euclidean space $\mathbb{R}^3$, $N'_\varepsilon(P)$ is a 3-dimensional open ball with center P and radius ε.

Furthermore, we can observe the following: on the real line $\mathbb{R}$, an open ball with center P and radius ε is the open interval $(P - \varepsilon, P + \varepsilon)$, a closed ball with center P and radius ε is the closed interval $[P - \varepsilon, P + \varepsilon]$, and a sphere with center P and radius ε is the set $\{P - \varepsilon, P + \varepsilon\}$. In the n-dimensional Euclidean space $\mathbb{R}^n$, an open ball with center $(P_1, P_2, \dots, P_n)$ and radius ε is, according to the Pythagorean Theorem, analytically expressed by the condition

$$(Q_1 - P_1)^2 + \cdots + (Q_n - P_n)^2 < \varepsilon^2,$$

while the corresponding sphere is analytically expressed by the condition

[371] Ibid.
[372] Ibid.

$$(Q_1 - P_1)^2 + \cdots + (Q_n - P_n)^2 = \varepsilon^2.$$

Thus, an n-dimensional closed ball is analytically expressed by the condition

$$(Q_1 - P_1)^2 + \cdots + (Q_n - P_n)^2 \leq \varepsilon^2,$$

and it is determined by $n + 1$ (independent) variables: the n coordinates of its center and its radius.

In general, a point in an n-dimensional space is determined by n coordinates, and a figure in an n-dimensional space is a set of points of the given space that satisfy a specific condition. For instance, a 3-dimensional closed ball is the set of all the points in $\mathbb{R}^3$ whose distance from a given point in $\mathbb{R}^3$ is not greater than a given number, and, therefore, it is determined by four variables: the three coordinates of its center and its radius. Such a geometry of 3-dimensional closed balls may be considered as a 4-dimensional geometry, so that a 3-dimensional closed ball may be considered as a point in a 4-dimensional space.

It is worth pointing out that the Polish-American logician and mathematician Alfred Tarski (1901–83) has observed that, after defining "concentric spheres," points can be identified with equivalence classes of concentric spheres, and "equidistance" can be defined by arrangements of spheres. In particular, in 1929, Tarski showed that much of Euclidean solid geometry can be expressed in terms of a first-order theory whose individuals are spheres (a primitive notion), a single primitive binary relation "is contained in," congruence axioms, and betweenness axioms (these axioms imply, among others, that containment partially orders the spheres).

The concept of a "distance function," known also as a "metric," such as the Euclidean distance function d_E, allows us to compute the distance between arbitrary sets and not only between singletons (ordinary geometric points). Let A and B be two non-empty sets in the Euclidean space $\mathbb{R}^n$. Then the distance of A to B is given by

$$d_E(A, B) = inf\{d_E(x, y) | x \in A, y \in B\}.$$

The aforementioned definition of distance of a set A to a set B suggests that $d_E(A, B)$ can be considered as a "generalized straight line." Moreover, given a non-empty set A in the Euclidean space $\mathbb{R}^n$, the "diameter" of A is given by

$$diam(A) = sup\{d_E(x, y) | x \in A, y \in A\}.$$

A set A in the Euclidean space $\mathbb{R}^n$ is called "bounded" if $diam(A) < \infty$. Thus, points in a bounded space are all within some fixed distance of each other. If $A \subseteq B$, then $diam(A) \leq diam(B)$. If A contains only one element, then $diam(A) = 0$.

In general, a "metric space" (X, d) is a set X endowed with a metric (distance function) d defined on it. The formal definition of a metric is the following[373]: A "metric," or "distance function," on a set X is a real-valued function d defined on $X \times X$ that has the following properties for all x, y, and z:

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[373] Ibid.

(D1) $d(x,y) \geq 0; d(x,y) = 0 \Leftrightarrow x = y$;
(D2) $d(x,y) = d(y,x)$;
(D3) $d(x,y) \leq d(x,z) + d(z,y)$.

Properties (D1), (D2), and (D3) are known, respectively, as the "positive definite" property, the "symmetric property," and the "triangle inequality." In other words, a metric on X is a real-valued function that is positive definite and symmetric and satisfies the triangle inequality. The systematic study of metric spaces was initiated by the French mathematician Maurice Fréchet in 1905.

Remark:

(i) As a result of the symmetric property and the triangle inequality, the definition of a metric implies that, for any points x, y, z in a metric space,

$$|d(x,z) - d(y,z)| \leq d(x,y),$$

which is Euclid's triangle inequality (for any triangle, the sum of the lengths of any two sides must be greater than or equal to the length of the remaining side).

(ii) It is possible to define more than one metrics on the same set X, and, in general, different metrics define different metric spaces on X. Let $\mathbb{R}^2$ be the usual Euclidean plane, where the set of all ordered pairs are real numbers. Two typical points of $\mathbb{R}^2$ are $p = (x_1, y_1)$ and $q = (x_2, y_2)$. The Euclidean distance is given by

$$d_E(p,q) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}; \tag{a}$$

and, in the metric space $(\mathbb{R}^2, d_E)$, the open region (neighborhood) with center $(0,0)$ and radius 1 is a unit disk, as shown in Figure 2.12(a).

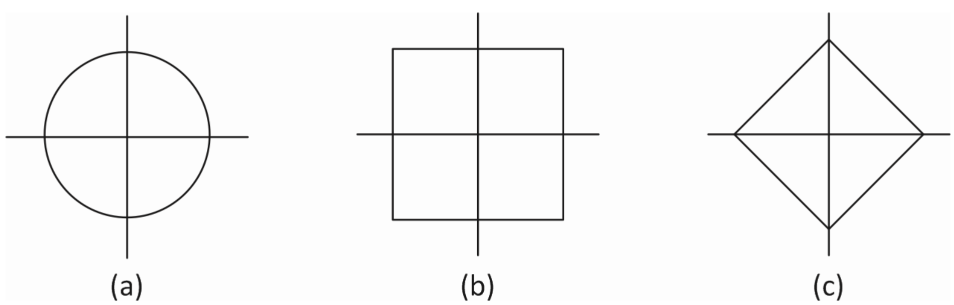

Figure 2.12. Different Metrics.

Two alternative distance functions in $\mathbb{R}^2$ are the following:

$$d_m(p,q) = \max\left(|x_1 - x_2|, |y_1 - y_2|\right); \tag{b}$$

and, in the metric space $(\mathbb{R}^2, d_m)$, the open region (neighborhood) with center $(0,0)$ and radius 1 is a unit square, as shown in Figure 2.12(b).

$$d_s(p,q) = |x_1 - x_2| + |y_1 - y_2|; \qquad \text{(c)}$$

and, in the metric space $(\mathbb{R}^2, d_s)$, the open region (neighborhood) with center $(0,0)$ and radius 1 is a unit rhombus, as shown in Figure 2.12(c).

Let (X_1, d_1) and (X_2, d_2) be two metric spaces. Then a function f from X_1 to X_2 is said to be an "isometry" if

$$d_2\big(f(x), f(y)\big) = d_1(x,y)\ \forall x, y \in X_1.$$

In other words, an isometry between two metric spaces preserves distance between points, and it is injective (if it were not injective, then it would contradict the property (D1) of the metric). An "isometric isomorphism" (known also as a "global isometry") between two metric spaces is a bijective isometry. Two metric spaces (X_1, d_1) and (X_2, d_2) are said to be "isometric" if there exists a bijective isometry (i.e., an isometric isomorphism) from X_1 to X_2.

Notice that a mapping $T: \mathbb{R}^n \to \mathbb{R}^n$ that maps every point $p \in \mathbb{R}^n$ to $p + a$ for a fixed $a \in \mathbb{R}^n$ is called a "translation." Moreover, notice that an orientation preserving linear mapping $T: \mathbb{R}^n \to \mathbb{R}^n$ that carries a set $\{e_1, e_2, \dots, e_n\}$ of orthogonal unit vectors at 0 to another set $\{e'_1, e'_2, \dots, e'_n\}$ of orthogonal unit vectors at 0 in such a way that $T(e_i) = e'_i$, where $i = 1,2, \dots, n$, is called a "rotation" (about 0). It is easily verified that translations and rotations are isometries. Moreover, since the product of two isometries $T_1 T_2$ on $\mathbb{R}^n$ is an isometry (in the sense that $d(T_2 \circ T_1(p), T_2 \circ T_1(q)) = d\big(T_1(p), T_1(q)\big) = d(p,q)$), the inverse $T^{-1}: \mathbb{R}^n \to \mathbb{R}^n$ of an isometry $T: \mathbb{R}^n \to \mathbb{R}^n$ is also an isometry, and the identity mapping is obviously an isometry, it follows that the collection of all isometric mappings of $\mathbb{R}^n$ forms a group, called the group of "Euclidean (rigid) motions."

If X and Y are metric spaces with metrics d_1 and d_2, respectively, then a function f from X to Y is said to be "continuous" at a point x_0 in X if, for each open ball (i.e., for each neighborhood) $B_\varepsilon\big(f(x_0)\big)$ centered at $f(x_0)$ with radius ε, there exists an open ball (i.e., a neighborhood) $B_\delta(x_0)$ centered at x_0 with radius δ such that $f\big(B_\delta(x_0)\big) \subseteq B_\varepsilon\big(f(x_0)\big)$. If f is continuous at each point in its domain, then it is said to be a "continuous mapping." It is easily verified that $f: X \to Y$ is continuous over X if and only if $f^{-1}(A)$ is open in X whenever A is open in Y (i.e., continuous functions associate open sets with open sets).

2.3. Topology of Real Numbers

Topology is a highly abstract kind of qualitative geometric knowledge, in the sense that it deals with the qualitative concept of nearness to spaces that might be conceptually close, without, however, using the quantitative concept of a distance function. Hence, intuitively, topology offers tools to model the concept of nearness in a set, just as, for instance, group theory offers tools to model the concept of symmetry. In the context of topology, instead of using a ruler, we can think of two points x and y as being near each other if there are many open sets that contain both x and y, whereas, if there are no open sets containing two given points, then these two points are far apart (of course, the whole space is considered to be an open set containing every point under consideration). It is conventional to call the qualitative properties "topological properties." In order to understand what we mean by the qualitative properties of geometric figures, one can imagine a sphere to be a rubber ball that can be stretched and shrunk in any manner without being torn or gluing any two of its points together. Such transformations of a sphere are called homeomorphisms, and the different replicas that can be obtained as a result of homeomorphisms are said to be homeomorphic to each other. In other words, "homeomorphisms" are isomorphisms in the category of topological spaces. Hence, the qualitative properties of the sphere are those that it shares with all its homeomorphic replicas, that is, those which are preserved under homeomorphisms. For instance, one of the qualitative ("topological") properties of the sphere is its integrity (namely, "connectedness"). Some of the most important pioneers and founders of topology are the French mathematician, epistemologist, and theoretical physicist Henri Poincaré (1854–1912), the German mathematician Felix Hausdorff (1868–1942), and the Russian/Soviet mathematicians Pavel Sergeyevich Alexandrov (1896–1982), and Andrey Nikolayevich Tikhonov (1906–93).

Topology is the weakest structure (that is, the most "economical" structure in terms of assumption) that can be established on a set and secure a good definition of continuity of mappings. By the term "topological space," we mean a set endowed with a topology defined on it. By the term "topology," we mean a collection of subsets of the given set that are declared to be open. However, it does not suffice to declare a set open, since we want our open sets to have additional qualities, and we want to be able to perform set operations on them to preserve the given sets' qualities. In fact, in $\mathbb{R}^n$, the union of any collection of open sets is an open set, and the intersection of a finite collection of open sets is an open set. Thus, with these conditions and with the declarations that the empty set and the whole set are open sets, we come up with the "Euclidean topology" $\mathcal{T}_E$ of $\mathbb{R}^n$. In general, a topology endows a set with a structure based on the concept of a neighborhood. The formal definition of a topology is the following[374]: A "topology" $\mathcal{T}$ on a non-empty set X is a collection of subsets of X, called open sets, such that:

(T1) the empty set, $\emptyset$, and X are open, symbolically, $\emptyset, X \in \mathcal{T}$;
(T2) the union of any collection of open sets is open, symbolically, if $U_a \in \mathcal{T}$ for $a \in \mathcal{A}$, then $\cup_{a\in\mathcal{A}} U_a \in \mathcal{T}$;
(T3) the intersection of a finite collection of open sets is open, symbolically, if $U_i \in \mathcal{T}$ for $i = 1,2, \dots, n$, then $\cap_{i=1}^{n} U_i \in \mathcal{T}$.

Then the pair $(X, \mathcal{T})$ is called a "topological space." Whereas the concept of a metric space is based on the concept of a distance (or, more specifically, on the concept of a distance function), the concept of a topological space is based on the more abstract concept of closeness (or, more specifically, on the concept of a neighborhood).

For instance, given the set $X = \{1,2,3,4,5\}$, the family $F_1 = \{\emptyset, X, \{1\}, \{1,2\}, \{1,3\}\}$ is not a topology on X, because $\{1,2\}$ and $\{1,3\}$ belong to F_1, but $\{1,2\} \cup \{1,3\} = \{1,2,3\} \notin F_1$, whereas the family $F_2 = \{\emptyset, X, \{1\}, \{1,2\}, \{1,3,4\}, \{1,2,3,4\}, \{1,2,5\}\}$ is a topology on X.

Given a metric space (X, d), the set of all open sets (as defined in section 2.2.6) is a topology on X, and it is called the "metric topology" on X. The open sets of the Euclidean

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374 Ibid.

topology $\mathcal{T}_E$ on $\mathbb{R}^n$ are given by arbitrary unions of the open balls $B_r(p)$, defined as $B_r(p) = \{x \in \mathbb{R}^n | d_E(p,x) < r\}$, for all $r > 0$ and for all $p \in \mathbb{R}^n$, where d_E is the Euclidean metric. In fact, the circle S^1 is a topological space, in the sense that all the points that are on the circle lie in the set S^1, and, by analogy, the sphere S^2, which is embedded in $\mathbb{R}^3$, and inherits the topology $\mathcal{T}_E$ from the embedding topological space $(\mathbb{R}^3, \mathcal{T}_E)$, is a topological space, too.

Notice that, given a non-empty set X, the collection $\{\emptyset, X\}$, consisting of the empty set and the whole set, is a topology on X, and it is known as the "trivial topology" on X. The power set $\wp(X)$ of X, consisting of all the subsets of X, is a topology on X, and it is called the "discrete topology" on X.

If X and Y are topological spaces, then a mapping f from X to Y is said to be a "continuous mapping" if $f^{-1}(A)$ is open in X whenever A is open in Y.

2.3.1. Neighborhoods

A subset N of $\mathbb{R}$ is said to be a "(topological) neighborhood" of a real number p if there exists an open interval (a,b) such that it contains p and is itself contained in N, symbolically:

$$p \in (a,b) \subseteq N. \qquad (*)$$

Thus, we symbolize a neighborhood of p by $N(p)$. According to $(*)$, the only subsets of $\mathbb{R}$ that can be neighborhoods are those which have open intervals as their subsets. By analogy, using the concept of an open ball (mentioned in section 2.2.6) instead of the concept of an open interval, we can think of a topological neighborhood in $\mathbb{R}^n$ (moreover, as I explained in section 2.2.6, for $\varepsilon > 0$, we can quantitatively define an "ε-neighborhood" of point p in $\mathbb{R}^n$ as the set X of points in $\mathbb{R}^n$ such that the distance $d_E(X,p) < \varepsilon$).

For instance, the closed interval $[1,3]$ is a neighborhood of point 2, since it contains, for instance, the open interval $\left(\frac{3}{2}, \frac{5}{2}\right)$, which contains 2. However, the closed interval $[1,3]$ is not a neighborhood of its endpoints 1 and 3, because, in both of these cases, there is not an open interval satisfying $(*)$. In general, any closed interval $[a,b]$ of the real line is a neighborhood of each of its elements except the endpoints a and b. However, any open interval (a,b) of the real line is a neighborhood of each of its elements.

Notice that none of the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{Q}^{\sim}$ is a neighborhood of any of its elements, whereas $\mathbb{R}$ is a neighborhood of each of its elements. Moreover, no non-empty finite set is a neighborhood of any of its elements, because no open interval of the real line can be a subset of a finite set. The empty set, $\emptyset$, is considered to be a neighborhood of each of its points, namely, of "emptiness," because it contains no elements, and there exists no element of $\emptyset$ of which $\emptyset$ it is not a neighborhood.

Assume that ε is a positive real number, that is, $\varepsilon > 0$. Then, in the expression $(*)$, let $N = (p-\varepsilon, p+\varepsilon)$. Hence, $p \in (a,b) \subseteq (p-\varepsilon, p+\varepsilon)$.If this is the case, then $(p-\varepsilon, p+\varepsilon)$ is called the ε-neighborhood of the point p, and it is denoted by $N_\varepsilon(p)$. In other words, the ε-neighborhood of a point p on the real line is the set of all those real numbers which are within an ε distance of p on either side of it; p is the midpoint or the center of $N_\varepsilon(p)$; ε is the radius of $N_\varepsilon(p)$. Hence, $x \in N_\varepsilon(p) \Leftrightarrow |x-p| < \varepsilon$. As I mentioned in section 2.2.6, a "deleted neighborhood” of a point p is a neighborhood of p from which p itself is deleted, and it is denoted by $N'_\varepsilon(p)$, symbolically, $N'_\varepsilon(p) = N_\varepsilon(p) - \{p\} = (p-\varepsilon, p) \cup (p, p+\varepsilon)$.

Theorem[375]*:* If N is a neighborhood of p, and $N \subseteq S$, then S is also a neighborhood of p.

Proof: Assume that N is a neighborhood of p, and, therefore,

$\exists \varepsilon > 0 | (p-\varepsilon, p+\varepsilon) \subseteq N$.

Moreover, $N \subseteq S \Rightarrow (p-\varepsilon, p+\varepsilon) \subseteq S$, which proves that S is also a neighborhood of p.■

Theorem[376]*:* If N_{ε_1} and N_{ε_2} are neighborhoods of p, then $N_{\varepsilon_1} \cap N_{\varepsilon_2}$ is also a neighborhood of p.

Proof: Given that N_{ε_1} and N_{ε_2} are neighborhoods of p,

$\exists \varepsilon_1 > 0 | (p-\varepsilon_1, p+\varepsilon_1) \subseteq N_{\varepsilon_1}$ and
$\exists \varepsilon_2 > 0 | (p-\varepsilon_2, p+\varepsilon_2) \subseteq N_{\varepsilon_2}$.

If ε is the smallest of the two ε_1 and ε_2, then

$(p-\varepsilon, p+\varepsilon) \subseteq (p-\varepsilon_1, p+\varepsilon_1) \subseteq N_{\varepsilon_1}$ and
$(p-\varepsilon, p+\varepsilon) \subseteq (p-\varepsilon_2, p+\varepsilon_2) \subseteq N_{\varepsilon_2}$.

Hence, $(p-\varepsilon, p+\varepsilon) \subseteq N_{\varepsilon_1} \cap N_{\varepsilon_2}$, meaning that $N_{\varepsilon_1} \cap N_{\varepsilon_2}$ is also a neighborhood of p.■

Given a set S, a real number p is said to be an “interior point” of S if S is a neighborhood of p, symbolically, if $p \in (a,b) \subseteq S$. Obviously, an interior point of a set S belongs to S. The set of all interior points of a given set S is called the “interior” of S, and it is denoted by $Int(S)$. In general, a point $p \in \mathbb{R}^n$ is said to be an “interior point” of U if some neighborhood (open ball)$N_\varepsilon(p)$ with center p is contained in U.

For instance, if $S = [2,5]$, then $\frac{7}{2}$ is an interior point of S, whereas neither 2 nor 5 is an interior point of S, because $[2,5]$ is not a neighborhood of 2 and 5. The interior of $[2,5]$ is $(2,5)$. In general, $Int([a,b]) = (a,b) = Int([a,b)) = Int((a,b])$. Notice that, because the set $\mathbb{N}$ of all natural numbers is not a neighborhood of any of its elements, $Int(\mathbb{N}) = \emptyset$, and, similarly, $Int(\mathbb{Z}) = \emptyset$, $Int(\mathbb{Q}) = \emptyset$, and $Int(\mathbb{Q}^\sim) = \emptyset$. However, because the set $\mathbb{R}$ of all real numbers is a neighborhood of each of its elements, $Int(\mathbb{R}) = \mathbb{R}$. For any subsets A and B of $\mathbb{R}$, it can be easily verified that $Int(A \cap B) = Int(A) \cap Int(B)$, and $Int(A \cup B) \supseteq Int(A) \cap Int(B)$. Moreover, $Int(\emptyset) = \emptyset$.

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[375] Ibid.
[376] Ibid.

Given a set S, a real number p is said to be an "exterior point" of S if there exists a neighborhood of p that contains only points that belong to the complement of S (as I mentioned in section 2.1.2, the complement of a set, denoted by $S^{\sim}$, is the set of all elements in the given universal set that do not belong to S). The set of all exterior points of S is called the "exterior" of S, and it is denoted by $Ext(S)$. Hence, $Ext(S) = Int(S^{\sim})$. For instance, $Ext(\mathbb{Q}) = Int(\mathbb{Q}^{\sim}) = \emptyset$. In general, a point $p \in \mathbb{R}^n$ is said to be an "exterior point" of U if some neighborhood (open ball)$N_\varepsilon(p)$ with center p is contained in $U^{\sim}$ (the complement of U), that is, if there is a neighborhood$N_\varepsilon(p)$ with center p such that $N_\varepsilon(p) \cap U = \emptyset$. Using De Morgan's Laws, it can be easily verified that $Ext(A \cup B) = Ext(A) \cap Ext(B)$.

Given a set S, if every neighborhood of a real number p contains points that belong to both the set S and the set $S^{\sim}$ (i.e., the complement of S), then p is said to be a "boundary point" of S. The set of all boundary points of S is called the 'boundary" of S, and it is denoted by $Bdy(S)$, or $\partial(S)$. A boundary point of U may belong to either U or $U^{\sim}$. Notice that $Int(S)$, $Ext(S)$, and $Bdy(S)$ constitute a "partition" of $\mathbb{R}$, in the sense that they are pairwise disjoint, and

$$\mathbb{R} = Int(S) \cup Ext(S) \cup Bdy(S).$$

Because every neighborhood of a real number contains rational and irrational numbers, it follows that each real number is a boundary point of the set $\mathbb{Q}$, symbolically, $Bdy(\mathbb{Q}) = \mathbb{R}$. Moreover, $S \subseteq Int(S) \cup Bdy(S)$. In general, a point $p \in \mathbb{R}^n$ is said to be a "boundary point" of U if every neighborhood (open ball) with center p contains at least one point of U and at least one point of $U^{\sim}$ (the complement of U), that is, if, for every$N_\varepsilon(p)$ with center p, $N_\varepsilon(p) \cap U \neq \emptyset$ and $N_\varepsilon(p) \cap U^{\sim} \neq \emptyset$.

A point $p \in S$ is said to be an "isolated point" of S if there exists a neighborhood of p that contains no point of S other than p itself. In general, a point $p \in \mathbb{R}^n$ is said to be an "isolated point" of U if there is a neighborhood (open ball)$N_\varepsilon(p)$ with center p such that $N_\varepsilon(p) \cap U = \{p\}$. If all the elements of a set are isolated points, then this set is said to be a "discrete set."

2.3.2. Open Sets

Theorem[377]*:* Let S be an arbitrary subset of $\mathbb{R}$. Then: (i) $Int(S)$ is an open subset of $\mathbb{R}$, and (ii) $Int(S)$ is the largest open set contained in S.

Proof:

(i) Suppose that p is an arbitrary element of $Int(S)$, so that S is a neighborhood of p, and $p \in (p - \varepsilon, p + \varepsilon) \subseteq S$ for some $\varepsilon > 0$. Because $(p - \varepsilon, p + \varepsilon)$ is an open interval, it is a neighborhood of each of its points, and, therefore, its superset S is also a neighborhood of each point of $(p - \varepsilon, p + \varepsilon)$. Consequently,

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[377] Ibid.

$(p-\varepsilon, p+\varepsilon) \subseteq Int(S) \Rightarrow p \in (p-\varepsilon, p+\varepsilon) \subseteq Int(S)$
$\Rightarrow Int(S)\ is\ a\ neighborhood\ of\ p$
$\Rightarrow Int(S)\ is\ a\ neighborhood\ of\ each\ of\ its\ points.$

This means that $Int(S)$ is an open subset of $\mathbb{R}$.

(ii) Suppose that A is an arbitrary open subset of S and that $p \in A$, so that $p \in A \subseteq S$. Because A is an open set, it is a neighborhood of p, and, therefore, its superset S is also a neighborhood of p, meaning that $p \in Int(S)$. Consequently, $p \in A \Rightarrow p \in Int(S) \Rightarrow A \subseteq Int(S)$, and, therefore, $Int(S)$ contains every open subset of S. This means that $Int(S)$ is the largest open set contained in S.■

Theorem[378]*:* The union of any collection of open sets is an open set.

Proof: Suppose that $\{U_a | a \in \mathcal{A}\}$ is an arbitrary collection of open sets U_a, and that $V = \cup_a U_a$. If $V = \emptyset$, then obviously V is an open set. If $V \neq \emptyset$, then let x be an arbitrary element of V, so that $x \in U_a$ for some $a \in \mathcal{A}$. Because U_a is an open set, it follows that U_a is a neighborhood of x. Consequently, for some $\varepsilon > 0$, and because $U_a \subseteq V$,

$x \in (x-\varepsilon, x+\varepsilon) \subseteq U_a \Rightarrow x \in (x-\varepsilon, x+\varepsilon) \subseteq V$
$\Rightarrow V\ is\ a\ neighborhood\ of\ x$
$\Rightarrow V\ is\ a\ neighborhood\ of\ each\ of\ its\ elements.$

This means that V is an open set.■

Theorem[379]*:* The intersection of a finite collection of open sets is an open set.

Proof: Consider two open sets U_1 and U_2. If $U_1 \cap U_2 = \emptyset$, then $U_1 \cap U_2$ is an open set. If $U_1 \cap U_2 \neq \emptyset$, then let x be an arbitrary element of $U_1 \cap U_2$, that is, $x \in U_1$ and $x \in U_2$. Because U_1 and U_2 are open sets, U_1 is a neighborhood of x, and U_2 is also a neighborhood of x. Because x is an arbitrary element of $U_1 \cap U_2$, it follows that $U_1 \cap U_2$ is a neighborhood of each of its points. This means that $U_1 \cap U_2$ is an open set. This proof can be extended to any finite number of open sets.■

Remark: The intersection of an infinite collection of open sets need not be open. For instance, the intersection of the open intervals $\left(-\frac{1}{n}, \frac{1}{n}\right)$ in $\mathbb{R}$, where n is any positive integer, is the set $\{0\}$, which is not open, since $[a, a] = \{a\}$. Moreover, if $U_n = \left(2-\frac{1}{n}, 3+\frac{1}{n}\right)$, where $n \in \mathbb{N}$, then $\cap_{n\in\mathbb{N}} U_n = [2,3]$, which is not an open set.

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[378] Ibid.
[379] Ibid.

2.3.3. Nested Intervals and Cantor's Intersection Theorem

Let $[a_1, b_1] \supseteq [a_2, b_2] \supseteq \cdots$, that is, $[a_n, b_n] \supseteq [a_{n+1}, b_{n+1}]\ \forall n \in \mathbb{N}$. Then these intervals are called "nested intervals," since each interval contains its successor.

Cantor's Intersection Theorem[380]*:* If $\{I_n = [a_n, b_n] | n \in \mathbb{N}\}$ is a family of nested closed intervals such that the length of the nth subinterval, namely, $b_n - a_n$, tends to zero as n tends to infinity, then there exists an ε such that:

i. $\forall \delta > 0, \exists m | (\varepsilon - \delta, \varepsilon + \delta) \supseteq I_n\ \forall n \geq m$ and
ii. $\bigcap_{n \in \mathbb{N}} I_n = \{\varepsilon\}$.

Proof: The fact that the intervals of the given family are nested implies that

$a_1 \leq a_2 \leq a_3 \leq \cdots \leq b_3 \leq b_2 \leq b_1$, so that
$a_n \leq a_{n+1} \leq b_{n+1} \leq b_n\ \forall n \in \mathbb{N}$.

Let $A = \{a_1, a_2, a_3, \ldots\}$ and $B = \{b_1, b_2, b_3, \ldots\}$. Then A and B are non-empty bounded subsets of $\mathbb{R}$. Therefore, due to the Completeness Axiom of $\mathbb{R}$ (mentioned in section 2.2.4), A has a supremum in $\mathbb{R}$, and B has an infimum in $\mathbb{R}$. Let $\sup(A) = a$ and $\inf(B) = b$, so that $a_n \leq a$ and $b_n \geq b$ for any $n \in \mathbb{N}$. Hence,

$0 \leq b - a \leq b_n - a_n$.

Because, by hypothesis, $b_n - a_n \to 0$ as $n \to \infty$, $b - a = 0$, and, then, let $\varepsilon = b = a$. Consequently, $(\varepsilon - \delta, \varepsilon)$ contains $a_n\ \forall n \geq m_1$, and $(\varepsilon, \varepsilon + \delta)$ contains $b_n\ \forall n \geq m_2$, so that $(\varepsilon - \delta, \varepsilon + \delta)$ contains $a_n, b_n\ \forall n \geq m$ where $m = max\{m_1, m_2\}$. Therefore, $(\varepsilon - \delta, \varepsilon + \delta) \supseteq I_n\ \forall n \geq m$. Moreover, the fact that, $\forall n \in \mathbb{N}$, $a_n \leq \varepsilon$ and $b_n \geq \varepsilon$ implies that $a_n \leq \varepsilon \leq b_n$, so that $\varepsilon \in [a_n, b_n]\ \forall n \in \mathbb{N}$, and, since the intervals are nested, $\varepsilon \in \bigcap_{n \in \mathbb{N}} [a_n, b_n]$, that is, $\varepsilon \in \bigcap_{n \in \mathbb{N}} I_n$.

We can prove the uniqueness of this ε by *reductio ad absurdum* as follows: Suppose that there exists another element $\varepsilon' \neq \varepsilon$ in the intersection $\bigcap_{n \in \mathbb{N}} I_n$. If $\varepsilon' > \varepsilon$, then $(\varepsilon' - \varepsilon) > 0$. Given that $a_n \leq \varepsilon < \varepsilon' \leq b_n$, it follows that $(b_n - a_n) \geq (\varepsilon' - \varepsilon) > 0$, which contradicts the hypothesis that $b_n - a_n \to 0$ as $n \to \infty$. Hence, $\varepsilon' \ngtr \varepsilon$. Similarly, we can show that $\varepsilon' \nless \varepsilon$. Hence, $\varepsilon' = \varepsilon$, which implies that $\bigcap_{n \in \mathbb{N}} I_n = \{\varepsilon\}$.■

Remark: If the intervals are not closed, then Cantor's Intersection Theorem may not be valid. For instance, consider the following family of open intervals:

$$\left\{I_n = \left(a, a + \frac{1}{n}\right) | n \in \mathbb{N}\right\}.$$

Then $I_n \supseteq I_{n+1}\ \forall n \in \mathbb{N}$, but $\bigcap_{n \in \mathbb{N}} I_n = \emptyset$, which is not a singleton.

[380] Ibid.

2.3.4. Closure Points and Accumulation Points

A real number p is called a "closure point" of a set $S \subseteq \mathbb{R}$ if every neighborhood of p contains a point of S. The set of all closure points of S is called the "closure" of S, and it is denoted by $Cls(S)$. Therefore, every point of $S \subseteq \mathbb{R}$ is a closure point of S.

A real number p is called an "accumulation point," or a "limit point," or a "cluster point" of S if every deleted neighborhood of p contains at least one point of S, symbolically, $S \cap N'_\varepsilon(p) \neq \emptyset \; \forall \varepsilon > 0$ (i.e., every neighborhood of p contains at least one point of S other than p). For instance, if $A = [a, b]$ and $B = (a, b)$, then every member of A is an accumulation point of A and of B, since, for instance, $\forall \varepsilon > 0$, the neighborhood $(a - \varepsilon, a + \varepsilon)$ of a contains infinitely many elements of A and of B. Moreover, every real number is an accumulation point of the set $\mathbb{Q}$ of all rational numbers as well as of the set $\mathbb{R}$ of all real numbers, since, for instance, given an arbitrary real number $p, \forall \varepsilon > 0$, the neighborhood $(p - \varepsilon, p + \varepsilon)$ contains infinitely many real numbers as well as infinitely many rational numbers. On the other hand, the set $\mathbb{N}$ of all natural numbers, the set $\mathbb{Z}$ of all integral numbers, and the empty set have no accumulation point. Moreover, no finite set has any accumulation point, because, if, for instance, $A = \{a_1, a_2, a_3, \dots, a_n\}$, and if p is an arbitrary real number, then, if we define $d_1 = |a_1 - p|, d_2 = |a_2 - p|, \dots, d_n = |a_n - p|$, and if $k = min\{d_1, d_2, \dots, d_n\}$, we realize that the neighborhood N with center p and radius $\frac{k}{2}$ contains no point of A, and, therefore, p, which is an arbitrary real number, is not an accumulation point of A.

Notice that, if A is a non-empty subset of $\mathbb{R}$, then:

i. if A is bounded from above and $\sup(A) \notin A$, then $\sup(A)$ is an accumulation point of A;
ii. if A is bounded from below and $\inf(A) \notin A$, then $\inf(A)$ is an accumulation point of A.

Every accumulation point of a set is also a closure point of that set, but not conversely. For instance, given the set $A = \left\{\frac{1}{n} | n \in \mathbb{N}\right\}$, $0 = \inf(\mathrm{A})$ and $0 \notin A$, and, therefore, 0 is an accumulation point of A, but 1 is a closure point of A without being an accumulation point of A, since the neighborhood $(1 - \varepsilon, 1 + \varepsilon)$, where $\varepsilon > 0$, does not contain a member of A other than 1.

If all the elements of a set S are accumulation points of S, then S is said to be "dense-in-itself." For instance, the open interval (a, b) is dense-in-itself, because all its members are accumulation points of (a, b). Moreover, the closed interval $[a, b]$ and $\mathbb{Q}$ are dense-in-themselves.

The set of all accumulation points of a set S is called the "derived set" of S, and it is denoted by $D(S)$. For instance, $D(\mathbb{N}) = \emptyset$, $D(\mathbb{Z}) = \emptyset$, $D(\mathbb{R}) = \mathbb{R}$, $D\big((a, b)\big) = [a, b]$, and $D([a, b]) = [a, b]$. If, for a set S, $D(S) = S$, then S is said to be a "perfect set." In other words, a perfect set is a set that is dense-in-itself, and it contains all its accumulation points. For instance, $\mathbb{R}$ and the closed interval $[a, b]$ are perfect sets.

By the definition of a derived set, the following can be easily verified:

Theorem[381]*:* If A and B are non-empty subsets of $\mathbb{R}$, then

$$A \subseteq B \Rightarrow D(A) \subseteq D(B),$$
$$D(A \cup B) = D(A) \cup D(B),$$
$$D(A \cap B) = D(A) \cap D(B).$$

The following theorem is a very important result regarding accumulation points, and it was originally proved by Bernard Bolzano (1781–1848), who was a Roman Catholic priest, a professor of theology at the Philosophical Faculty of the University of Prague, and a prominent mathematician, and, subsequently, its meaning and ramifications were investigated and highlighted by Karl Weierstrass (1815–97), who was a prominent German mathematician, and he taught at the University of Berlin.

Bolzano–Weierstrass Theorem[382]*:* Every infinite and bounded subset of $\mathbb{R}$ has at least one accumulation point.

Proof: Suppose that$S$ is a bounded and infinite set in $\mathbb{R}$, so that $\exists a, b \in \mathbb{R} | x \in [a, b]\ \forall x \in S$, and $S \subseteq [a, b]$. Let us bisect the interval $[a, b]$ at point c, so that we obtain two subintervals $[a, c]$ and $[c, b]$. Then at least one of these subintervals contains infinitely many elements of S, and we rename this subinterval as $[a_1, b_1]$. The length of $[a_1, b_1]$ is $b_1 - a_1 = \frac{b-a}{2}$. Subsequently, we bisect $[a_1, b_1]$ at point c_1, so that we obtain two new subintervals $[a_1, c_1]$ and $[c_1, b_1]$. Then at least one of these new subintervals contains infinitely many elements of S, and we rename this subinterval as $[a_2, b_2]$. The length of $[a_2, b_2]$ is $b_2 - a_2 = \frac{b-a}{2^2}$. Repeating the same process of bisection and selection, we obtain a set of intervals

$[a_1, b_1], [a_2, b_2], \ldots$ such that:

the intervals are nested (i.e., each is contained in the preceding), and the length of the nth subinterval, namely, of $[a_n, b_n]$, is $b_n - a_n = \frac{b-a}{2^n}$, and it tends to 0 as n tends to ∞. Therefore, by Cantor's Intersection Theorem, it holds that, for some ε,

$$\cap_{n \in \mathbb{N}} [a_n, b_n] = \{\varepsilon\}.$$

Having shown that $\cap_{n \in \mathbb{N}} [a_n, b_n]$ is the singleton of ε, we shall show that this ε is an accumulation point of S. Assume that $b_n - a_n < k$, where $k > 0$, so that $[a_n, b_n] \subseteq (\varepsilon - k, \varepsilon + k)$. Given that each $[a_n, b_n]$ contains infinitely many elements of S, it follows that $(\varepsilon - k, \varepsilon + k)$, which is a neighborhood of ε, contains infinitely many elements of S, and, therefore, ε is an accumulation point of S.■

[381] Ibid.
[382] Ibid.

Remark: The converse of the Bolzano–Weierstrass Theorem is not true, since, for instance, the set $\mathbb{Q}$ of all rational numbers has every real number as its accumulation point, but $\mathbb{Q}$ is not bounded.

2.3.5. Closed Sets

The concept of a closed set is dual to the concept of an open set, in the sense that closed sets are the complements of open sets. A set $S \subseteq \mathbb{R}$ is called “closed” if it contains all its accumulation points. Hence, the derived set $D(S)$ of a closed set S is a subset of S, and all perfect sets are closed. As I have already mentioned, the “closure” of S is denoted by $Cls(S)$, where $Cls(S) = S \cup D(S)$, that is, $Cls(S)$ is the set of all closure points of S. Thus, S is closed if and only if $Cls(S) = S$. If A and B are non-empty subsets of $\mathbb{R}$, it can be easily verified that

i. $A \subseteq B \Rightarrow Cls(A) \subseteq Cls(B)$.
ii. $Cls(A \cup B) = Cls(A) \cup Cls(B)$.
iii. $Cls(A \cap B) \subseteq Cls(A) \cap Cls(B)$.
iv. $Cls\big(Cls(A)\big) = Cls(A)$.

For instance, the empty set has no accumulation point, and, therefore, $D(\emptyset) = \emptyset$; and, from this perspective, namely, because $D(\emptyset) = \emptyset$, the empty set is a perfect and, thus, closed set. Moreover, the sets $\mathbb{N}$ and $\mathbb{Z}$ are closed sets, since $D(\mathbb{N}) = \emptyset = D(\mathbb{Z})$. Because $D([a,b]) = [a,b]$, all closed intervals are perfect and, thus, closed sets, whereas open intervals (a,b) are not closed sets, because two of their accumulation points, namely, a and b, do not belong to (a,b). Notice that, because $D(\mathbb{Q}) = \mathbb{R}$ and, thus, $D(\mathbb{Q}) \notin \mathbb{Q}$, the set $\mathbb{Q}$ is not closed (hence, we realize that the set $\mathbb{Q}$ is neither open nor closed). Moreover, given that finite sets have no accumulation point, every finite set is closed.

Notice that the set $\mathbb{R}$ of all real numbers and the empty set $\emptyset$ are both open and closed.

In general, a subset S of $\mathbb{R}$ is said to be “dense” in a subset B of $\mathbb{R}$ if B is a subset of the closure of S. In particular, S is dense in $\mathbb{R}$ if and only if $Cls(S) = \mathbb{R}$. Because $D(\mathbb{Q}) = \mathbb{R}$ and, thus, $Cls(\mathbb{Q}) = \mathbb{Q} \cup D(\mathbb{Q}) = \mathbb{R}$, it follows that $\mathbb{Q}$ is a dense subset of $\mathbb{R}$. On the other hand, a subset S of $\mathbb{R}$ is said to be “nowhere dense” in $\mathbb{R}$ if $Int\big(Cls(S)\big) = \emptyset$. For instance, if $S = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$, then $D(S) = \{0\}$, and $Cls(S) = S \cup D(S) = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$, which implies that $Int\big(Cls(S)\big) = \emptyset$, since $Cls(S)$ is not a neighborhood of any of its elements, and, thus, the set S is nowhere dense in $\mathbb{R}$.

The supremum (least upper bound) and the infimum (greatest lower bound) of the derived set $D(S)$ of a subset S of $\mathbb{R}$ are respectively called the “limit superior” and the “limit inferior” of the set S. Every bounded infinite set has limit superior and limit inferior. Obviously, the endpoints a and b of the intervals (a,b) and $[a,b]$ are, respectively, the limit inferior and the limit superior of these intervals.

As I have already explained, it can be easily verified that a set S is closed if and only if its complement, $S^{\sim}$, is open.

Theorem[383]*:* The union of a finite collection of closed sets is a closed set.

Proof: Let $S = U_1 \cup U_2 \cup \dots \cup U_n$ where U_α (with $\alpha = 1,2,..,n$) is a closed set. Then $S^\sim = (U_1 \cup U_2 \cup \dots \cup U_n)^\sim$, which, by De Morgan's Laws (mentioned in section 2.1.2), is equal to $U_1^\sim \cap U_2^\sim \cap \dots \cap U_n^\sim$. Because U_α (with $\alpha = 1,2,..,n$) is a closed set, each $U_\alpha^\sim$ is an open set, and, because the intersection of a finite collection of open sets is an open set (as proved in section 2.3.2), $S^\sim$ is an open set, and, thus, S is a closed set.■

Remark: The union of an infinite collection of closed sets need not be closed. For instance, the union of the closed intervals $U_n = \left[\frac{1}{n}, 1\right]$ in $\mathbb{R}$, where $n = 1,2,...$, is $U_1 \cup U_2 \cup \dots = (0,1]$, which is not closed.

Theorem[384]*:* The intersection of any collection of closed sets is a closed set.

Proof: Let $\{U_\alpha | \alpha \in \mathcal{A}\}$ be an arbitrary collection of closed sets, and $S = \cap_{\alpha\in\mathcal{A}} U_\alpha$. Then $S^\sim = (\cap_{\alpha\in\mathcal{A}} U_\alpha)^\sim = \cup_{\alpha\in\mathcal{A}} U_\alpha^\sim$, by De Morgan's Laws (mentioned in section 2.1.2). Because $\cup_{\alpha\in\mathcal{A}} U_\alpha^\sim$ is the union of an arbitrary collection of open sets, which is an open set (as proved in section 2.3.2), it follows that $S^\sim$ is an open set, and, thus, S is a closed set.■

2.3.6. Compactness

Let $\mathcal{C} = \{U_\alpha | \alpha \in \mathcal{A}\}$ be a family of sets, and let S be a subset of $\mathbb{R}$. Then $\mathcal{C}$ is said to be a "cover" of S if S is contained in the union of the members of $\mathcal{C}$, that is, if every element of S belongs to some member of $\mathcal{C}$; symbolically: $S \subseteq \cup_{\alpha\in\mathcal{A}} U_\alpha$. For instance, if U_1 is the set of all odd numbers $(1,3,5,...)$, if U_2 is the set of all even numbers $(0,2,4,6,...)$, and if $\mathcal{C} = \{U_1, U_2\}$, then every element of the set $U = \{0,1,2,3,4,5,6\}$ belongs either to U_1 or to U_2, that is, $U \subset U_1 \cup U_2$, and, therefore, $\mathcal{C}$ is a cover of the set U.

If $\mathcal{C}_1$ and $\mathcal{C}_2$ are covers of a set S, and if $\mathcal{C}_1 \subseteq \mathcal{C}_2$, then $\mathcal{C}_1$ is said to be a "subcover" of S. In other words, a subcollection of members of $\mathcal{C}_2$ that also covers the set S is called a subcover of S. A cover is called "finite" if it contains only a finite number of sets. A cover $\mathcal{C}$ of a set S is said to be an "open cover" of S if each member of $\mathcal{C}$ is an open set.

For instance, let us consider $\mathbb{R}$. Then the family $\mathcal{C} = \{(-n, n) | n \in \mathbb{N}\}$ is an open cover of $\mathbb{R}$, since $\mathbb{R} \subseteq \cup_{n\in\mathbb{N}} (-n, n)$. Suppose that there exists a subcover $\mathcal{C}' = \{(-n_1, n_1), (-n_2, n_2), \dots, (-n_k, n_k)\}$, and let $M = max\{n_1, n_2, \dots, n_k\}$. Then $M \in \mathbb{R}$, but $M \notin \mathcal{C}'$, that is, M remains uncovered by $\mathcal{C}'$, which is a contradiction, and, therefore, $\mathcal{C}'$ is not a subcover of $\mathbb{R}$. Notice that the set $\mathbb{R}$ is closed but not bounded. Hence, an open cover of a closed but unbounded set (e.g, $\mathbb{R}$) may not provide a finite subcover.

Let us consider the closed interval $U = \{x \in \mathbb{R} | 0 \leq x \leq 1\}$. If $\varepsilon > 0$ is fixed, then the family $\mathcal{C} = \{(\alpha - \varepsilon, \alpha + \varepsilon) | \alpha \in U\}$ is an open cover of U. This open cover provides many subcovers. For instance, we may choose the family $\mathcal{C}' = \{(\beta - \varepsilon, \beta + \varepsilon) | \beta \in \{x \in \mathbb{Q} | 0 \leq x \leq 1\}\}$, which is an open cover of U, since every irrational number $x \in [0,1]$ can be

[383] Ibid.
[384] Ibid.

approximated to within ε by some rational number, and $\mathcal{C}'$ is a subset of $\mathcal{C}$, meaning that $\mathcal{C}'$ is a subcover of U. Similarly, the family $\mathcal{C}'' = \{(\gamma - \varepsilon, \gamma + \varepsilon) | \gamma \in \{x \in \mathbb{Q}^{\sim} | 0 \leq x \leq 1\}\}$ is an open cover of U, and it consists of uncountably many sets.

However, consider $\mathcal{C} = \left\{\left[\frac{1}{n}, \frac{2}{n}\right], \{0\} | n \in \mathbb{N} - \{0\}\right\}$, which is a closed cover of the closed and bounded set $U = \{x \in \mathbb{R} | 0 \leq x \leq 1\}$. Suppose that there exists a subcover

$$\mathcal{C}' = \left\{\left[\frac{1}{n_1}, \frac{2}{n_1}\right], \left[\frac{1}{n_2}, \frac{2}{n_2}\right], \dots, \left[\frac{1}{n_k}, \frac{2}{n_k}\right], \{0\}\right\},$$

and let $M = max\{n_1, n_2, \dots, n_k\}$. Then $\frac{1}{M+1} \in U$, but $\frac{1}{M+1} \notin \mathcal{C}'$, which is a contradiction, and, therefore, $\mathcal{C}'$ is not a subcover of U. Hence, a closed cover of a closed and bounded set (e.g., U) may not provide a finite subcover.

Let us consider the open interval $I = (0,1)$, which is a bounded but not closed set. Because it is not closed, it does not contain all its accumulation points (specifically, the two accumulation points that it does not contain are 0 and 1). Then we can construct an open cover $\mathcal{C}$ of $(0,1)$ such that its open sets get infinitely close to at least one (or maybe both) of the endpoints of I, but $\mathcal{C}$ never quite reaches any of the endpoints of I. The family

$$\mathcal{C} = \left\{\left(\frac{1}{n}, 1\right) | n \in \mathbb{N} - \{0\}\right\}$$

is an open cover of $I = (0,1)$. We can prove that $\mathcal{C}$ does not provide a finite subcover by *reductio ad absurdum* as follows: Suppose that there exists a finite subcover

$$\mathcal{C}' = \left\{\left(\frac{1}{n_1}, 1\right), \left(\frac{1}{n_2}, 1\right), \dots, \left(\frac{1}{n_k}, 1\right)\right\},$$

and let$M = max\{n_1, n_2, \dots, n_k\} < \infty$. Then $\frac{1}{M} \in (0,1)$, but $\frac{1}{M} \notin \mathcal{C}'$, which is a contradiction, and, therefore, $\mathcal{C}'$ is not a subcover. Hence, an open cover of a bounded but not closed set (e.g., I) may not provide a finite subcover.

A set S is said to be "compact" if each open cover of S admits a finite subcover. It is widely acknowledged that "compactness' is a topological generalization of the concept of finiteness. Topology is preoccupied with the behavior of an object on an open set, and compactness implies that there are only finitely many possible behaviors. In particular, "finiteness" is a very important concept, because finiteness implies that something is constructible "by hand" (thus giving rise to constructive results), and finite objects (being "pseudo-finite" in their nature) are well-behaved ones. In other words, even though "compactness" is not exactly "finiteness," a compact object behaves like a finite set with regard to important topological properties.

Heine–Borel Theorem[385]*:* A set is compact if and only if it is closed and bounded.

[385] Ibid. This theorem is named after the German mathematician Heinrich Eduard Heine (1821–81) and the French mathematician (and politician) Félix Édouard Justin Émile Borel (1871–1956).

Proof: First, we shall prove that every open cover of a closed and bounded set admits a finite subcover by *reductio ad absurdum* as follows: In particular, let S be a closed and bounded set, and let $\mathcal{C} = \{U_\alpha | \alpha \in \mathcal{A}\}$ be an open cover of S, so that $S \subseteq \cup_{\alpha \in \mathcal{A}} U_\alpha$. Moreover, because S is bounded (by hypothesis), there exist two real numbers a and b such that $S \subseteq [a, b]$. For the sake of contradiction, assume that S does not have a finite subcover. Let us bisect $[a, b]$ at c, so that we obtain two subintervals $[a, c]$ and $[c, b]$. Then at least one of these subintervals contains a subset of S that does not have a finite subcover, and we rename this subinterval as $[a_1, b_1]$.The length of $[a_1, b_1]$ is $b_1 - a_1 = \frac{b-a}{2}$. Subsequently, we bisect $[a_1, b_1]$ at point c_1, so that we obtain two new subintervals $[a_1, c_1]$ and $[c_1, b_1]$. Then at least one of these new subintervals contains a subset of S that does not have a finite subcover, and we rename this subinterval as $[a_2, b_2]$. Repeating this process of bisection and selection, we obtain nested closed intervals $[a_n, b_n]$, where $n = 1,2, ...$, such that:

i. the length of $[a_n, b_n]$, which is equal to $\frac{b-a}{2^n}$, tends to 0 as $n \to \infty$, and
ii. each $[a_n, b_n]$ contains a subset of S that does not have a finite subcover.

Hence, applying Cantor's Intersection Theorem, we obtain $[a_n, b_n] \subset (\varepsilon - \delta, \varepsilon + \delta)$ for $\delta > 0$, and $\cap_{n \in \mathbb{N}} [a_n, b_n] = \{\varepsilon\}$, so that ε is an accumulation point of the set S. Because S is a closed set (by hypothesis), $\varepsilon \in S$. Moreover, $\mathcal{C}$ is an open cover of S, so that, for some n, $\varepsilon \in U_n$, and, since U_n is an open set, $\varepsilon \in (\varepsilon - \delta, \varepsilon + \delta) \subset U_n$. Hence, because of (i), $[a_n, b_n] \subset U_n$ for some n, so that $[a_n, b_n]$ is covered by a single member U_n of $\mathcal{C}$, which contradicts (ii). Therefore, S has a finite subcover. In other words, a closed and bounded set is compact.

Now, we shall prove that every compact set is closed and bounded.

We can prove that every compact set is bounded as follows: If S is a compact set, then every open cover of S has a finite subcover. Let $\mathcal{C} = \{(-n, n) | n \in \mathbb{N}\}$, so that $\mathcal{C}$ is an open cover of S that provides a finite subcover $\mathcal{C}_1 = \{(-n_1, n_1), (-n_2, n_2), \dots, (-n_k, n_k)\}$. Then, by the definition of a cover, $S \subseteq \cup_{i=1}^{k} (-n_i, n_i) = (-M, M)$, where $M = max\{n_1, n_2, \dots, n_k\}$. Hence, $S \subseteq (-M, M)$, which proves that S is bounded.

We can prove that every compact set is closed as follows: If S is a compact set, and if $a \notin S$, then let

$$U_n = \left(-\infty, a - \frac{1}{n}\right) \cup \left(a + \frac{1}{n}, \infty\right), \text{ where } n \in \mathbb{N} - \{0\},$$

so that each U_n is an open set (since it is the union of two open sets). Let $\mathcal{C} = \{U_n | n \in \mathbb{N}\}$, so that $\mathcal{C}$ is an open cover of S. Because S is compact, it admits a finite subcover, say $\mathcal{C}_1 = \{U_{n_1}, U_{n_2}, \dots, U_{n_k}\}$. If $M = max\{n_1, n_2, \dots, n_k\}$, then

$$S \subseteq \cup_{i=1}^{k} U_{n_i} = U_M \Rightarrow S \subset \left(-\infty, a - \frac{1}{M}\right) \cup \left(a + \frac{1}{M}, \infty\right),$$

so that $\left(a - \frac{1}{M}, a + \frac{1}{M}\right)$ does not contain any point of S. This fact implies that a neighborhood of a does not contain a point of S, and, therefore, a is not an accumulation point of S.

Because a is an arbitrary point that does not belong to S, no point outside S can be an accumulation point of S, and, therefore, S is a closed set.■

For instance, the sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, $\mathbb{R}$ and any open interval (a, b) are not compact sets, whereas any finite subset of $\mathbb{R}$ and any closed interval $[a, b]$ are compact sets (i.e., every open cover of a finite subset of $\mathbb{R}$ and every open cover of a closed interval $[a, b]$ provide a finite subcover).

2.3.7. Relative Topology and Connectedness

If S is a set of real numbers, and if $A \subseteq S$, then the set A is said to be "open relative to the set S" if there exists an open set X such that $A = S \cap X$. A set $B \subseteq S$ is said to be "closed relative to the set S" if there exists a closed set Y such that $B = S \cap Y$. Notice that $\emptyset$ is both open and closed relative to any subset S of $\mathbb{R}$, and that every subset of $\mathbb{R}$ is both open and closed relative to itself. If, for instance, $S = \{1,2,3\}$, then: (i) S is open relative to $\mathbb{Z}^{+}$, since $S = \mathbb{Z}^{+} \cap (0,4)$; (ii) S is not open relative to $\mathbb{R}$, since $S \neq \mathbb{R} \cap (0,4)$; (iii) S is closed relative to $\mathbb{Z}$, since $S = \mathbb{Z} \cap [1,3]$, and, simultaneously, S is open relative to $\mathbb{Z}$, since $S = \mathbb{Z} \cap (0,4)$.

"Connectedness" is a topological term that means that a set (or a metric space, or a topological space) is "one piece," or, in other words, that it is not made up of two or more pieces.[386] As I mentioned in section 2.1.2, two sets A and B are said to be disjoint if their intersection is the empty set. However, there is a stronger condition on A and B than disjointness, and this condition is known as "separation." Two sets A and B are said to be "separated" if $Cls(A) \cap B = \emptyset$ and $A \cap Cls(B) = \emptyset$, that is, if each is disjoint from the other's closure. For instance, in $\mathbb{R}$, the sets $[0,1]$ and $(1,2]$ are disjoint but not separated, whereas the sets $[0,1)$ and $(1,2]$ are separated between them (the number 1 belongs to both of their closures). Obviously, any two separated sets are automatically disjoint. However, the condition of separation is not as strong as requiring that the distance between separated sets should be positive. For instance, the distance between the separated sets $[0,1)$ and $(1,2]$ is zero. Therefore, a set (or a metric space, or a topological space) is "connected" if it is not possible to be represented as the union of two separated sets A and B. For instance, a region U in $\mathbb{R}^2$ is connected if and only if any point in U can be joined to any other point in U by a polygonal path lying within U.

A set or space X is said to be disconnected if there exist two non-empty sets A and B such that $X = A \cup B$, and $Cls(A) \cap B = \emptyset$, and $A \cap Cls(B) = \emptyset$. In this case, $Cls(A) = B^{\sim} = A$, and, therefore, A is closed. Similarly, B must be a closed set. If both A and B are closed sets, then the statements $Cls(A) \cap B = \emptyset$ and $A \cap Cls(B) = \emptyset$ are both equivalent to the single statement $A \cap B = \emptyset$. Therefore, a set or space X is said to be connected if there exists no pair of closed sets A and B such that $X = A \cup B$ and $A \cap B = \emptyset$, and, since, in such a case, the sets A and B are complementary, we can equally say that they are both open. Consequently, we obtain the following general definition of connectedness[387]: A set or space X (X being a subset in $\mathbb{R}^n$) is "connected" if it cannot be expressed as $X = A \cup B$ where A and B are non-empty disjoint sets, and both A and B are open relative to X (or, equivalently, both A and

[386] Ibid.
[387] Ibid.

$B$are closed relative to X); otherwise, X is "disconnected." In view of the foregoing, a set or space X is connected if and only if $\emptyset$ and X are the only sets in X that are both open and closed.

For instance, the set $S = (0,1) \cap \mathbb{Q}$ is a disconnected set, since, if $A = \left(0, \frac{1}{\sqrt{2}}\right) \cap \mathbb{Q}$ and $B = \left(\frac{1}{\sqrt{2}}, 1\right) \cap \mathbb{Q}$, then $S = A \cup B$ and $A \cap B = \emptyset$. On the other hand, the empty set and the singleton of any real number are connected sets.

Theorem[388]*:* A non-empty, non-singleton set S in $\mathbb{R}$ is connected if and only if it is an interval. Notice that the assumption that S is an interval means that, if $x < z < y$, and if $x \in S$ and $y \in S$, then $z \in S$.

Proof: Let S be connected, and, for the sake of contradiction, suppose that S is not an interval. Then, for some points $x, y \in S$ with $x < y$, there exists a point $z \in (x, y)$ such that $z \notin S$ (if S were an interval, then $z \in S$). Hence, $U_1 = S \cap (-\infty, z)$ and $U_2 = S \cap (z, \infty)$ are non-empty open sets in S such that $S = U_1 \cup U_2$ and $U_1 \cap U_2 = \emptyset$. Consequently, if S is not an interval, then it is not connected. The aforementioned contradiction implies that S is an interval.

Now, let us assume that S is an interval. We shall prove that S is connected by *reductio ad absurdum*. Suppose that S is not connected. Then $S = A \cup B$, $A \neq \emptyset$, $B \neq \emptyset$, $Cls(A) \cap B = \emptyset$, and $A \cap Cls(B) = \emptyset$. Let $a \in A$ and $b \in B$ with $a < b$. Suppose that $I = [a, b]$, $A_1 = A \cap I$, and $B_1 = B \cap I$. Hence, $I = A_1 \cup B_1$ is a disconnection of I. If $c = sup(A_1)$, then $c \in Cls(A_1)$, and $c \notin B_1$. But, if $c < x \leq b$, then $x \notin A_1$, since we have assumed that $c = sup(A_1)$. Thus, $(x, b] \subset B_1$, which implies that $c \in Cls(B_1)$, so that $c \notin A_1$. But, given that S is an interval, and since $c \in I = A_1 \cup B_1$, c must belong to either A_1 or B_1. The aforementioned contradiction implies that S is connected.■

Generalization: From the aforementioned theorem, $\mathbb{R}^n$ is connected ($n = 1,2,3, \ldots$).

2.4. Sequences of Real Numbers

By the term "sequence of real numbers," we mean a function whose domain is $\mathbb{N}$ and whose range is any subset of $\mathbb{R}$. If, $\forall n \in \mathbb{N}$, there exists a unique real number u_n, then $(u_n)_{n \in \mathbb{N}}$, or simply (u_n), is said to be a sequence of real numbers, and, in essence, it is a set of real numbers $u_1, u_2, u_3, \ldots$ put in a definite order and formed according to a definite rule. The range of a sequence $(u_n)_{n \in \mathbb{N}}$ is often denoted by $\{u_n\}_{n \in \mathbb{N}}$. If a sequence has a finite number of terms, then it is called a finite sequence; otherwise, it is called an infinite sequence.

Two sequences (u_n) and (v_n) are said to be "equal" if $u_n = v_n \; \forall n \in \mathbb{N}$. A sequence (u_n) is said to be a "constant sequence" if $u_n = c \; \forall n \in \mathbb{N}$. If a sequence (u_n) is such that, $\forall \varepsilon > 0, \exists m \in \mathbb{N} | |u_n| < \varepsilon \; \forall n \geq m$, then the sequence is said to be a "null sequence."

A sequence (u_n) is said to be "bounded from above" if there exists some real number k such that $u_n \leq k \; \forall n \in \mathbb{N}$, and (u_n) is said to be "bounded from below" if there exists some

[388] Ibid.

real number l such that $u_n \geq l \; \forall n \in \mathbb{N}$. A sequence (u_n) is said to be "bounded" if there exist real numbers k and l such that $l \leq u_n \leq k \; \forall n \in \mathbb{N}$. Obviously, the boundedness of a sequence can be defined in terms of its range also, in the sense that a sequence is bounded if its range is bounded. For instance, the sequence (u_n) where $u_n = n^2 \; \forall n \in \mathbb{N}$ is bounded from below, but it is not bounded from above; the sequence (u_n) where $u_n = -n^2 \; \forall n \in \mathbb{N}$ is bounded from above, but it is not bounded from below; the sequence $(u_n) = (2, -2, 1, -1, 1 - 1, 1, -1, \dots)$, whose range is the set $\{u_n\} = \{-2, -1, 1\}$, is bounded with infimum equal to -2 and supremum equal to 1. The sequence $(u_n)_{n \in \mathbb{N}}$ where $u_n = 1 - \frac{1}{n}$is bounded with $\inf\,(u_n) = 0$ and $\sup\,(u_n) = 1$

A real number p is said to be an "accumulation point," or a "limit point," of a sequence (u_n) if the (open) neighborhood $(p - \varepsilon, p + \varepsilon)$ contains infinitely many terms of u_n for every $\varepsilon > 0$. An accumulation point of (u_n) need to be an accumulation point of its range $\{u_n\}$. For instance, the sequence $(u_n) = (-1)^n = (1, -1, 1 - 1, 1, -1, \dots) \; \forall n \in \mathbb{N}$ has two accumulation points, namely, -1 and 1, but the range of this sequence, namely, the set $\{u_n\} = \{-1, 1\}$, has not accumulation point, since it is a finite set.

Limit and Convergence of a Sequence[389]

A real number l is said to be the "limit" of a sequence $(u_n)_{n \in \mathbb{N}}$ if,

$$\forall \varepsilon > 0, \exists m \in \mathbb{N} | |u_n - l| < \varepsilon \; \forall n \geq m.$$

If this is the case, then we write $lim_{n \to \infty} u_n = l$, and we say that the sequence $(u_n)_{n \in \mathbb{N}}$ "converges" to l.

On the other hand, if,

$$\forall L > 0, \exists m \in \mathbb{N} | u_n > L \; \forall n \geq m,$$

then $u_n \to \infty$ as $n \to \infty$, and then we say that the sequence $(u_n)_{n \in \mathbb{N}}$ "diverges" to ∞. If,

$$\forall L > 0, \exists m \in \mathbb{N} | -u_n > L \; \forall n \geq m,$$

then $u_n \to -\infty$ as $n \to \infty$, and then we say that the sequence $(u_n)_{n \in \mathbb{N}}$ "diverges" to $-\infty$. For instance, the sequence $(u_n)_{n \in \mathbb{N}} = (n)_{n \in \mathbb{N}}$ diverges to ∞, and the sequence $(u_n)_{n \in \mathbb{N}} = (-n)_{n \in \mathbb{N}}$ diverges to $-\infty$.

If a sequence $(u_n)_{n \in \mathbb{N}}$ neither converges nor diverges to ∞ or $-\infty$, then it is said to "oscillate." If a bounded sequence oscillates, then it is said to "oscillate finitely." If an unbounded sequence oscillates, then it is said to "oscillate infinitely." For instance, the sequence$(u_n)_{n \in \mathbb{N}} = ((-1)^n)_{n \in \mathbb{N}}$ oscillates finitely, and the sequence $(u_n)_{n \in \mathbb{N}} = ((-1)^n n)_{n \in \mathbb{N}}$ oscillates infinitely.

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[389] See: Apostol, *Mathematical Analysis*; Fraleigh, *Calculus with Analytic Geometry*; Landau, *Foundations of Analysis*; Nikolski, *A Course of Mathematical Analysis*; Rudin, *Principles of Mathematical Analysis*; Spivak, *Calculus*; in conjunction with Cauchy, *Cours d'Analyse.*

Properties of the Limit of a Sequence[390]*:* Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be two arbitrary infinite sequences such that $lim_{n\to\infty}u_n = l_1$ and $lim_{n\to\infty}v_n = l_2$. Then:

i. If the limit of a sequence exists, then it is unique (i.e., a sequence can converge to only one limit).
ii. $lim_{n\to\infty}(u_n + v_n) = lim_{n\to\infty}u_n + lim_{n\to\infty}v_n = l_1 + l_2$.
iii. $lim_{n\to\infty}(u_n - v_n) = lim_{n\to\infty}u_n - lim_{n\to\infty}v_n = l_1 - l_2$.
iv. $lim_{n\to\infty}(u_n \cdot v_n) = lim_{n\to\infty}u_n \cdot lim_{n\to\infty}v_n = l_1 \cdot l_2$.
v. $lim_{n\to\infty}\frac{u_n}{v_n} = \frac{lim_{n\to\infty}u_n}{lim_{n\to\infty}v_n} = \frac{l_1}{l_2}$, provided that $lim_{n\to\infty}v_n = l_2 \neq 0$. If $l_2 = 0$ and $l_1 \neq 0$, then $\frac{lim_{n\to\infty}u_n}{lim_{n\to\infty}v_n}$ does not exist, and, if $l_2 = 0 = l_1$, then $\frac{lim_{n\to\infty}u_n}{lim_{n\to\infty}v_n}$ may or may not exist.
vi. $lim_{n\to\infty}u_n^r = (lim_{n\to\infty}u_n)^r = l_1^r$, provided that l_1^r exists, where r is a real number.
vii. Every convergent sequence is bounded.
viii. The limit of a convergent sequence is an accumulation point of the given sequence.

Proof:

i. First of all, the uniqueness of the limit of a sequence can be proved by thinking geometrically as follows: in the neighborhood of the limit of a sequence, there are infinitely many terms of the given sequence, whereas, in any other area, there only a few or no terms of the given sequence. Moreover, this theorem can be proved by *reductio ad absurdum* as follows: For the sake of contradiction, suppose that the limit of a convergent sequence $(u_n)_{n\in\mathbb{N}}$ is not unique, so that $lim_{n\to\infty}u_n = \alpha$ and $lim_{n\to\infty}u_n = \beta$. Then, by the definition of a limit,
$\forall\varepsilon > 0, \exists m | |u_n - \alpha| < \frac{\varepsilon}{2} \& |u_n - \beta| < \frac{\varepsilon}{2}\ \forall n \geq m$. Hence,
$|\alpha - \beta| = |\alpha - u_n + u_n - \beta| \leq |a - u_n| + |u_n - \beta|$
$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \Rightarrow \alpha = \beta$.
ii. By hypothesis,
$\forall\varepsilon > 0, \exists m_1, m_2 | |u_n - l_1| < \frac{\varepsilon}{2} \& |v_n - l_2| < \frac{\varepsilon}{2}\ \forall n \geq m_1 \&\ \forall n \geq m_2$.
Then $|u_n + v_n - (l_1 + l_2)| \leq |u_n - l_1| + |v_n - l_2|$
$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\ \forall n \geq m = max\{m_1, m_2\}$. Therefore,
$lim_{n\to\infty}(u_n + v_n) = lim_{n\to\infty}u_n + lim_{n\to\infty}v_n = l_1 + l_2$.
iii. The proof is similar to that of (ii).
iv. The proof is similar to that of (ii).
v. The proof is similar to that of (iv).
vi. The proof is similar to that of (iv).
vii. Assume that the sequence $(u_n)_{n\in\mathbb{N}}$ is convergent, and $lim_{n\to\infty}u_n = l_1$. Then, $\forall\varepsilon > 0, \exists m | |u_n - l_1| < \varepsilon\ \forall n \geq m$. If $\varepsilon = 1$, then
$|u_n - l_1| < 1 \Leftrightarrow l_1 - 1 < u_n < l_1 + 1\ \forall n \geq m$. Let
$a = min\{l_1 - 1, u_1, u_2, \dots, u_{m-1}\}$ and
$b = max\{l_1 + 1, u_1, u_2, \dots, u_{m-1}\}$, so that

[390] Ibid.

$a \leq u_n \leq b$ for every $n \in \mathbb{N}$, meaning that $(u_n)_{n\in\mathbb{N}}$ is bounded. However, notice that a bounded sequence may not converge. For instance, the sequence $(u_n)_{n\in\mathbb{N}} = (-1,1,-1,1,\dots)$ is bounded, but it is not convergent.

viii. Assume that the sequence $(u_n)_{n\in\mathbb{N}}$ is convergent, and $lim_{n\to\infty}u_n = l_1$. Then, $\forall\varepsilon > 0, \exists m||u_n - l_1| < \varepsilon\ \forall n \geq m$, so that, $\forall n \geq m$,
$l_1 - \varepsilon < u_n < l_1 + \varepsilon \Leftrightarrow u_n \in (l_1 - \varepsilon, l_1 + \varepsilon)$ for infinitely many n, and, therefore, l_1 is an accumulation point of $(u_n)_{n\in\mathbb{N}}$. However, notice that, if l_1 is an accumulation point of a sequence $(u_n)_{n\in\mathbb{N}}$, then this sequence need not be convergent. For instance, the sequence $(u_n)_{n\in\mathbb{N}} = (1,2,1,4,1,6,\dots)$ has an accumulation point equal to 1, but it is not convergent.■

Remark: A sequence can have several accumulation points, but it can have at most one limit. For instance, the sequence $((-1)^{n+1})_{n\in\mathbb{N}}$ does not converge, but it has two accumulation points, namely, -1 and $+1$. Notice that the limit of a sequence is formally defined by the aforementioned ε-m definition, according to which the fact that a sequence $(u_n)_{n\in\mathbb{N}}$ converges to a number l means that, as long as the subscript n of $(u_n)_{n\in\mathbb{N}}$ is large enough, all terms u_n of this sequence fall within a small neighborhood of the number l, whereas an accumulation point has lots of but not all the terms of a sequence near it (the limit of a sequence is an accumulation point of the given sequence).

The Squeeze Theorem for Convergent Sequences[391]*:* Given the convergent sequences $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$, and $(w_n)_{n\in\mathbb{N}}$ with $u_n \leq v_n \leq w_n$ after some nth term, it holds that, if $lim_{n\to\infty}u_n = lim_{n\to\infty}w_n = l$, then $lim_{n\to\infty}v_n = l$.

Proof: First of all, this theorem can be proved by thinking geometrically as follows: v_n lies between u_n and w_n, whose distances from l can become as small as we want, and, therefore, the distance of v_n from l can also become as small as we want. Moreover, this theorem can be proved in a more rigorous way as follows: Let $u_n \leq v_n \leq w_n$ after the M*th* term. Then we have to show that

$$\forall\varepsilon > 0, \exists m \in \mathbb{N}||v_n - l| < \varepsilon\ \forall n \geq m.$$

Notice that $lim_{n\to\infty}u_n = l \Leftrightarrow \forall\varepsilon > 0, \exists m_1 \in \mathbb{N}||u_n - l| < \varepsilon\ \forall n \geq m_1$, and

$$lim_{n\to\infty}w_n = l \Leftrightarrow \forall\varepsilon > 0, \exists m_2 \in \mathbb{N}||w_n - l| < \varepsilon\ \forall n \geq m_2.$$

Letting $M = max\{m, m_1, m_2\}$, we ensure that $-\varepsilon < u_n - l < \varepsilon$, $-\varepsilon < w_n - l < \varepsilon$, and $u_n \leq v_n \leq w_n$. Subtracting l from all parts of this inequality, we obtain $u_n - l \leq v_n - l \leq w_n - l$, so that $-\varepsilon < v_n - l < \varepsilon \Leftrightarrow |v_n - l| < \varepsilon$, and, therefore, $lim_{n\to\infty}v_n = l$.■

Cauchy Sequences and the Completeness of the Real Field[392]

A sequence $(u_n)_{n\in\mathbb{N}}$ is said to be a “Cauchy sequence” if,

[391] Ibid.
[392] Ibid.

$\forall \varepsilon > 0, \exists m \in \mathbb{N} | |u_{n+k} - u_n| < \varepsilon \; \forall n \geq m \; \& \; k \in \mathbb{N}$.

Notice that the difference between the definition of a "Cauchy sequence" and the definition of a "convergent sequence" is that the terms of a Cauchy sequence get close to each other, whereas the terms of a convergent sequence get close only to some fixed element in the ordered field. A convergent sequence is always a Cauchy sequence, but, in some ordered fields, a Cauchy sequence may not converge. For instance, consider the sequence of rational numbers

$1, 1.4, 1.41, 1.414, 1.4142, \ldots$

obtained by computing the square root of 2. Then this is a Cauchy sequence, but it is not convergent in the rational field.

Theorem[393]*:* A convergent sequence in an arbitrary ordered field F is a Cauchy sequence in F.

Proof: Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of elements of F such that $\lim_{n\to\infty} u_n = u$. Then

$\forall \varepsilon > 0 \; in \; F, \exists n_0 > 0 \; in \; \mathbb{N} | |u_n - u| < \frac{\varepsilon}{2} \; \forall n \geq n_0$. Hence,
$|u_n - u_m| \leq |u_n - u| + |u - u_m| < \varepsilon \; \forall m, n \geq n_0$.■

An ordered field F is "complete" if and only if every Cauchy sequence of elements of F converges to an element in F (the concept of completeness was studied in section 2.2.4).

Remark: The rational field is not complete; for instance, as I have already mentioned, the Cauchy sequence of rational numbers

$1, 1.4, 1.41, 1.414, 1.4142, \ldots$,

obtained by computing the $\sqrt{2}$, is not convergent in the rational field. However, the construction of the real field, which was explained in section 2.2.4, and the concept of a neighborhood, which was studied in sections 2.2.6 and 2.3.1, imply that the real field is complete, and, in fact, the extension of the rational field to the real field is the extension of an incomplete ordered field to a complete ordered field.

Baire's Category Theorem[394]*:* Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of closed subsets of $\mathbb{R}$ such that $\cup_{n=1}^{\infty} A_n$ contains an interval. Then at least one of the sets A_n contains an interval.

Proof: Assume that I_0 is a closed and bounded interval in $\cup_{n=1}^{\infty} A_n$. For the sake of contradiction, we shall assume that the theorem is not true, and we shall try to inductively construct a decreasing sequence of closed intervals $I_0, I_1, I_2, \ldots$ such that, $\forall n > 1$, $I_n \cap A_n =$

[393] Ibid.
[394] Ibid.

$\emptyset$. Since I_0 is not contained in A_1, there is a point $x_1 \in I_0 - A_1$. Since A_1 is closed, there is an interval about x_1 that does not meet A_1, and, in this interval, we can choose a closed interval $I_1 \subseteq I_0$. Repeating the same process, inductively, if we have chosen I_{n-1}, then there is a point $x_n \in I_{n-1} - A_n$, and we can find a closed interval $I_n \subseteq I_{n-1}$ such that $I_n \cap A_n = \emptyset$. The intersection of a nested sequence of closed and bounded intervals cannot be the empty set, and, therefore, there exists a point x such that $x \in \bigcap_{n=0}^{\infty} I_n$. Hence, $x \in I_0$, but $x \notin \bigcup_{n=1}^{\infty} A_n$, which contradicts the fact that $\bigcup_{n=1}^{\infty} A_n \supseteq I_0$. This contradiction implies that the theorem is true.■

Remark: This theorem was proved by the French mathematician René-Louis Baire in his 1899 doctoral thesis, and it is usually stated in the more general context of complete metric spaces as follows: Let X be a complete metric space (that is, one in which every Cauchy sequence converges). Then (i) the countable intersection of open, dense sets is dense, and (ii) X is not a countable union of nowhere dense sets. Hence, the Baire Category Theorem provides important information about the size of certain topological spaces.

Subsequences[395]

Consider a sequence $(u_n)_{n\in\mathbb{N}}$ and a set of positive integers $n_1, n_2, \ldots, n_k, \ldots$ with $n_{k+1} > n_k$, where $k = 1,2,3, \ldots$ Then $\left(u_{n_k}\right)_{k\in\mathbb{N}}$ is called a "subsequence" of $(u_n)_{n\in\mathbb{N}}$. For instance, $(u_{2n})_{n\in\mathbb{N}}$, $(u_n^2)_{n\in\mathbb{N}}$, and $(u_{3n-2})_{n\in\mathbb{N}}$ are subsequences of $(u_n)_{n\in\mathbb{N}}$. In other words, $\left(u_{n_k}\right)_{k\in\mathbb{N}}$ is a subsequence of $(u_n)_{n\in\mathbb{N}}$ if $\left(u_{n_k}\right) \subseteq (u_n)$ and n_k is a strictly increasing sequence of natural numbers, where $k \in \mathbb{N}$. A subsequence is an infinite subset that preserves the order of the original sequence. If $lim_{k\to\infty} u_{n_k} = l'$, then l' is called a "subsequential limit" of $(u_n)_{n\in\mathbb{N}}$.

Theorem[396]*:* Every accumulation point of a subsequence of a sequence of real numbers is also an accumulation point of the given sequence. The converse is not necessarily true.

Proof: Let $\left(u_{n_k}\right)_{k\in\mathbb{N}}$ be a subsequence of the sequence $(u_n)_{n\in\mathbb{N}}$, and p be an accumulation point of $\left(u_{n_k}\right)_{k\in\mathbb{N}}$. Hence, by definition, for every $\varepsilon > 0$, $u_{n_k} \in (p - \varepsilon, p + \varepsilon) \Rightarrow u_n \in (p - \varepsilon, p + \varepsilon)$, meaning that p is also an accumulation point of $(u_n)_{n\in\mathbb{N}}$.

However, the converse is not necessarily true. For instance, consider the infinite sequence $(u_n)_{n\in\mathbb{N}} = (1,1,2,1,2,3,1,2,3,4,1,2,3,4,5, \ldots)$, which is built as follows: at first the number 1 is written out, then the numbers from 1 to 2 are written out, then the numbers from 1 to 3, then the numbers from 1 to 4, etc. This sequence has infinitely many accumulation points, namely, $1,2,3, \ldots$ (even though it is not convergent). On the other hand, the subsequence $\left(u_{n_k}\right)_{k\in\mathbb{N}} = (1,2,3, \ldots)$ of the given sequence has no accumulation point.■

Theorem[397]*:* Any bounded sequence contains a convergent subsequence, that is, it has at least one accumulation point (in essence, this theorem is a reformulation of the Bolzano–Weierstrass Theorem, proven in section 2.3.4, in terms of sequences).

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[395] Ibid.
[396] Ibid.
[397] Ibid.

Proof: Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence. Then there exists an interval $[a_1, b_1]$ such that $a_1 \leq u_n \leq b_1$. Either $\left[a_1, \frac{a_1+b_1}{2}\right]$ or $\left[\frac{a_1+b_1}{2}, b_1\right]$ contains infinitely many terms of $(u_n)_{n\in\mathbb{N}}$, meaning that there exist infinitely many n such that u_n is in $\left[a_1, \frac{a_1+b_1}{2}\right]$, or there exist infinitely many n such that u_n is in $\left[\frac{a_1+b_1}{2}, b_1\right]$. If $\left[a_1, \frac{a_1+b_1}{2}\right]$ contains infinitely many terms of $(u_n)_{n\in\mathbb{N}}$, then let $[a_2, b_2] = \left[a_1, \frac{a_1+b_1}{2}\right]$; otherwise, let $[a_2, b_2] = \left[\frac{a_1+b_1}{2}, b_1\right]$.

Moreover, either $\left[a_2, \frac{a_2+b_2}{2}\right]$ or $\left[\frac{a_2+b_2}{2}, b_2\right]$ contains infinitely many terms of $(u_n)_{n\in\mathbb{N}}$. If $\left[a_2, \frac{a_2+b_2}{2}\right]$ contains infinitely many terms of $(u_n)_{n\in\mathbb{N}}$, then let $[a_3, b_3] = \left[a_2, \frac{a_2+b_2}{2}\right]$; otherwise, let $[a_3, b_3] = \left[\frac{a_2+b_2}{2}, b_2\right]$. By mathematical induction, we can repeat this process and, thus, construct a sequence of intervals $([a_n, b_n])_{n\in\mathbb{N}}$ such that, $\forall n \in \mathbb{N}$, $[a_n, b_n]$ contains infinitely many terms of $(u_n)_{n\in\mathbb{N}}$, $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$, and $b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n)$, that is, $b_n - a_n = \frac{b_1 - a_1}{2^{n-1}}$.

Cantor's Intersection Theorem (proven in section 2.3.3) implies that the intersection of all the intervals $[a_n, b_n]$ is a single point u. Now, we have to show that u is an accumulation point of $(u_n)_{n\in\mathbb{N}}$. Hence, we shall construct a subsequence of $(u_n)_{n\in\mathbb{N}}$ that converges to u.

Given that each of the intervals $[a_n, b_n]$ contains infinitely many terms of $(u_n)_{n\in\mathbb{N}}$, we choose one term x_{n_1} from $[a_1, b_1]$, then we choose one term x_{n_2} from $[a_2, b_2]$, then we choose one term x_{n_3} from $[a_3, b_3]$, etc. Then $\left(u_{n_k}\right)_{k\in\mathbb{N}}$ is a subsequence of $(u_n)_{n\in\mathbb{N}}$, and $a_n \leq u_{n_k} \leq b_n$ for every $k \in \mathbb{N}$. Because $lim_{n\to\infty} a_n = lim_{n\to\infty} b_n = u$, the Squeeze Theorem for Convergent Sequences implies that $u_{n_k} \to u$.■

Theorem[398]*:* If a Cauchy sequence $(u_n)_{n\in\mathbb{N}}$ of elements in an ordered field F has a subsequence $\left(u_{n_k}\right)_{k\in\mathbb{N}}$ that converges to u, then $(u_n)_{n\in\mathbb{N}}$ converges to u.

Proof: Let $\varepsilon > 0$ in F. Then there exists a positive integer n_0 such that $|u_n - u_m| < \frac{\varepsilon}{2}$ $\forall m, n \geq n_0$. Moreover, there exists a positive integer k_0 such that $\left|u_{n_k} - u\right| < \frac{\varepsilon}{2}$ $\forall k \geq k_0$. If $k \geq k_0$ such that $n_k \geq n_0$, then $|u_n - u| \leq \left|u_n - u_{n_k}\right| + \left|u_{n_k} - u\right| < \varepsilon$ whenever $n \geq n_0$.■

Let $\mathcal{S}$ be the set of all Cauchy sequences of rational numbers with the binary operations

$(u_n) + (v_n) = (u_n + v_n)$ and $(u_n)(v_n) = (u_n v_n)$.

Moreover, let $\mathcal{N}$ be the subset of $\mathcal{S}$ that consists of the "null sequences," namely, of those sequences which converge to 0. Then we define a relation R in $\mathcal{S}$ as follows:

$(u_n)R(v_n)$ if $(u_n) - (v_n) = (u_n - v_n) \in \mathcal{N}$.

[398] Ibid.

It can be easily verified that this relation is an equivalence relation, and it determines a partition of $\mathcal{S}$ into equivalence classes which can be denoted by $\overline{(u_n)}$. Given the aforementioned notation and definitions, the real number system $\mathbb{R}$ is the quotient set$\mathcal{S}/R$ with the operations of addition and multiplication defined as follows:

$\overline{(u_n)} + \overline{(v_n)} = \overline{(u_n) + (v_n)}$ and $\overline{(u_n)} \cdot \overline{(v_n)} = \overline{(u_n) \cdot (v_n)}$.

Monotonic Sequences[399]

A sequence $(u_n)_{n\in\mathbb{N}}$ is said to be "increasing" if $u_{n+1} \geq u_n\ \forall n \in \mathbb{N}$, and it is said to be "strictly increasing" if $u_{n+1} > u_n\ \forall n \in \mathbb{N}$. For instance, the sequence $u_n = n$, including the terms 1,2,3,4,5, ..., is strictly increasing, and the sequence

$$v_n = \begin{cases} \frac{n+1}{2}\ if\ n\ is\ odd \\ \frac{n}{2}\ if\ n\ is\ even \end{cases},$$

including the terms 1,1,2,2,3,3,4,4, ..., is increasing but not strictly increasing.

A sequence $(u_n)_{n\in\mathbb{N}}$ is said to be "decreasing" if $u_{n+1} \leq u_n\ \forall n \in \mathbb{N}$, and it is said to be "strictly decreasing" if $u_{n+1} < u_n\ \forall n \in \mathbb{N}$. For instance, the sequence $u_n = \frac{1}{n}$ is strictly decreasing.

A sequence $(u_n)_{n\in\mathbb{N}}$ is said to be "monotonic" if it is either increasing or decreasing. A sequence $(u_n)_{n\in\mathbb{N}}$ is said to be "non-monotonic" if it is neither increasing nor decreasing. For instance, the sequence $u_n = (-1)^n$ is non-monotonic.

Theorem[400]*:* (i) Let $(u_n)_{n\in\mathbb{N}}$ be an increasing real sequence. Then it is convergent if and only if it is bounded from above. (ii) Let $(u_n)_{n\in\mathbb{N}}$ be a decreasing real sequence. Then it is convergent if and only if it is bounded from below. (iii) Given (i) and (ii), a monotonic sequence is convergent if and only if it is bounded.

Proof: (i) Let $(u_n)_{n\in\mathbb{N}}$ be an increasing sequence, and $lim_{n\to\infty} u_n = u$, so that,

$$\forall \varepsilon > 0, \exists m \in \mathbb{N} | |u_n - u| < \varepsilon\ \forall n \geq m.$$

Let $\varepsilon = 1$, so that, whenever $n \geq k$ for some $k \in \mathbb{N}$,

$$|u_n - u| < 1 \Rightarrow u - 1 < u_n < u + 1, \forall n \geq k,$$

which implies that $u_n \leq n_0$ for $n_0 = max\{u + 1, u_1, u_2, \dots, u_{k-1}\}$, and, therefore, $(u_n)_{n\in\mathbb{N}}$ is bounded from above.

Now, assuming that $(u_n)_{n\in\mathbb{N}}$ is bounded from above and increasing, we shall prove that it converges to a real number p. Let $\sup(u_n) = p$, where $p \in \mathbb{R}$. We shall prove $lim_{n\to\infty} u_n = p$. Let $\varepsilon > 0$ be such that $p - \varepsilon$ is not an upper bound of $(u_n)_{n\in\mathbb{N}}$, so that

399 Ibid.
400 Ibid.

$p - \varepsilon < u_m$ for some $m \in \mathbb{N}$. $(*)$

Because $(u_n)_{n\in\mathbb{N}}$ is increasing, $u_m \leq u_n \ \forall n \geq m$. $(**)$

Because of $(*)$ and $(**)$,$\forall n \geq m$,

$p - \varepsilon < u_m \leq u_n \Rightarrow p - \varepsilon < u_n$. $(***)$

Moreover, $u_n < p + \varepsilon$, given thatsup $(u_n) = p$. $(****)$

Because of $(***)$ and $(****)$,

$$\forall \varepsilon > 0, \exists m \in \mathbb{N} | p - \varepsilon < u_n < p + \varepsilon \Rightarrow |u_n - p| < \varepsilon \ \forall n \geq m.$$

Therefore, $lim_{n\to\infty} u_n = p$.

(ii) The proof is similar to that of (i), and the point of convergence is the infimum. (iii) It is a straightforward combination of (i) and (ii).■

Hilbert Space

The set $H = \{u = (u_1, u_2, \ldots) | u_i \in \mathbb{R} \ \& \sum_{i=1}^{\infty} u_i^2 < \infty\}$ is called a "Hilbert space, and it is often denoted by $\mathbb{R}^\infty$. The pair (H, d), where $d(u, v) = [\sum_{i=1}^{\infty}(u_i - v_i)^2]^{\frac{1}{2}}$ with $u, v \in H$ is a metric space (it can be easily verified that it satisfies the requirements of the definition of a metric space). A Hilbert space precludes the possibility of containing a sequence that may converge to something not in the space, and, therefore, it is complete. It is named after the great German mathematician David Hilbert (1862–1943), who has made foundational contributions to functional analysis and geometry, and he was one of Albert Einstein's mathematical mentors. The set $I^\infty = \left\{u = (u_1, u_2, \ldots) | |u_n| \leq \frac{1}{n}, n \in \mathbb{N}\right\}$ is called the "Hilbert cube."

Alphabets and Languages[401]

If f is a sequence whose domain X is finite and consists of n consecutive natural numbers, and if Y is the codomain of f, then f defines a "string of length n in Y," or a "word of length n in Y." Obviously, any such sequence is an n-tuple. For instance, if $X = \{1,2,3,4,5\}$, $Y = \{A, B, C, D\}$, and the sequence $f: X \to Y$ is defined by $f(1) = A, f(2) = B, f(3) = D, f(4) = A, and\ f(5) = D$, then the sequence is the string $ABDAD$ of length 5 in Y (that is, the 5-tuple $ABDAD$). In other words, if A is an "alphabet," namely, a set whose elements are called "letters," then a "word," or "string," from A (or over A, or on A) is a finite sequence of letters.

If S is any non-empty set, then we denote by $S_n$the set of all strings of length n in S, and by S^* the set of all strings, including the null string with no elements. Any subset of S^* is called a "language over the alphabet S." The union and the intersection of two languages over an alphabet are also languages over the same alphabet. If $u = (u_1 u_2 u_3 \ldots u_m)$ and $v =$

[401] See: Balakrishnan, *Introductory Discrete Mathematics*, pp. 207–18; Yablonsky, *Introduction to Discrete Mathematics*, parts I and IV.

$(v_1 v_2 v_3 \dots v_n)$ are two strings of lengths m and n, respectively, in S^*, then the "concatenation" of u and v is the string uv in S^* of length $m+n$ defined as $uv = (u_1 u_2 u_3 \dots u_m v_1 v_2 \dots v_n)$. In other words, concatenation means appending one string to the end of another string.

Any function from $A \times A$ into A is called a "binary operator" on A. The function $c: S^* \times S^* \to S^*$ defined by $c(u, v) = uv$, where uv is the concatenation of the strings u and v, is a binary operator on S^*.

Let K and L be languages over an alphabet A. Then the language KL over A can be defined as follows: KL consists of all words over A formed by concatenating words in K with words in L. Therefore,

$$KL = \{w | w = uv, where\ u \in K\ and\ v \in L\}.$$

String distance functions (known also as string metrics) are used in several areas, such as DNA analysis, RNA analysis, ontology merging (i.e., the act of bringing together two conceptually divergent formal systems or the instance data associated with two formal systems), image analysis, fraud detection, fingerprint analysis, evidence-based machine learning, data mining, incremental search, data integration, and semantic knowledge integration.[402] In bioinformatics, in particular, a sequence alignment is a way of arranging the sequences of DNA and RNA, or protein in order to identify regions of similarity that may derive from functional, evolutionary, or structural relationships between sequences.[403]

2.5. INFINITE SERIES AND INFINITE PRODUCTS

As I have already mentioned, by a (real) sequence, we mean a function $f: \mathbb{N} \to \mathbb{R}$ whose images are $a_1, a_2, a_3, \dots, a_n, \dots$ By adding $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$, we obtain an "infinite summation." Such an infinite sum is called an "infinite series."[404] If, however, we add finitely many terms, then we get a sum that is a number, namely, every finite summation converges. The founders of the modern theory of infinite series are Isaac Newton and James Gregory in the seventeenth century, and the Bernoulli family mathematicians (Jacob, John, Nicolaus, and Daniel), Leonhard Euler, and Joseph Lagrange in the eighteenth century. In fact, the eighteenth-century mathematicians were thinking of infinite series as infinite polynomials (mathematical expressions consisting of variables, coefficients, and the operations of addition, subtraction, multiplication, and non-negative integral exponents), and they tried to develop an arithmetic system of infinite polynomials (see section 2.6).

The basic idea in the study of infinite series is that an infinite summation of numbers can have a finite sum. Some of the early work on series was motivated by paradoxes related to the concept of infinity, with which many ancient Greek mathematicians were preoccupied. In the fifth century B.C., the Greek mathematician and philosopher Zeno posed the following paradox: Consider a race between the legendary Greek hero Achilles and a tortoise over 100

[402] See: Navarro, "A Guided Tour to Approximate String Matching."
[403] See: Mount, *Bioinformatics.*
[404] See: Fraleigh, *Calculus with Analytic Geometry;* Hyslop, *Infinite Series;* Knopp, *Theory and Application of Infinite Series.*

meters. Suppose that the tortoise starts 80 meters ahead, and Achilles can run 10 times as fast as the tortoise. Then, after 10 sec., when Achilles will have run 80 meters, reaching the point where the tortoise started, the tortoise will have run only 8 meters farther. Then it will take Achilles 1 sec. more to cover that distance, but, during the same time, the tortoise will have run 0.8 meters farther. Then it will take Achilles 0.1 sec. to reach this third point, while the tortoise moves ahead by 0.08 meters, etc. Thus, whenever Achilles reaches somewhere the tortoise has been, the tortoise is still ahead, and it seems that the tortoise will stay ahead. In fact, Zeno's paradox can be resolved as follows: the total time that it would take Achilles to catch up, in seconds, is $10 + 1 + 0.1 + 0.01 + 0.001 + \cdots$, which is an infinite series. But this infinite series is equal to 11.111…, which is a finite number. In particular, let $x = 0.1 + 0.01 + 0.001 + \cdots$. In fact, $0.1 + \frac{x}{10} = 0.1 + 0.01 + 0.001 + \cdots$, and, therefore, $x = 0.1 + \frac{x}{10} \Rightarrow 10x = 1 + x \Rightarrow 9x = 1x \Rightarrow x = \frac{1}{9}$. Hence, the time for Achilles to catch up is $11\frac{1}{9}$ sec.

The infinite series

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} + \cdots$$

is an infinite decreasing geometric progression, namely, a particular case of

$$a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots,$$

which is decreasing if $|r| < 1$. We form the sequences

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_n &= a_1 + a_2 + a_3 + \cdots + a_n = \textstyle\sum_{k=1}^{n} a_k, \end{aligned}$$

in order to compute $\sum_{n=1}^{\infty} a_n$, because, if the limit $\lim\limits_{n\to\infty} \sum_{k=1}^{n} a_k$ exists, then its value is equal to $\sum_{n=1}^{\infty} a_n$. Therefore, we can compute $a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots$ by computing the limit of the summation $a + ar + ar^2 + ar^3 + \cdots + ar^n$ as $n \to \infty$. Set $S_n = a + ar + ar^2 + ar^3 + \cdots + ar^n$. Then

$$rS_n = ar + ar^2 + ar^3 + ar^3 + \cdots + ar^n + ar^{n+1}, \text{ and}$$
$$S_n - rS_n = a - ar^{n+1} \Rightarrow S_n = \frac{a - ar^{n+1}}{1-r}.$$

Hence, $S_n = \frac{a}{1-r} - \frac{ar^{n+1}}{1-r} \Rightarrow \lim\limits_{n\to\infty} S_n = \frac{a}{1-r} - \frac{a}{1-r}\left(\lim\limits_{n\to\infty} r^{n+1}\right)$. Because $|r| < 1$, it follows that $\lim\limits_{n\to\infty} r^{n+1} = 0$, so that the limit of S_n as $n \to \infty$ is equal to $\frac{a}{1-r}$, that is, $a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots = \frac{a}{1-r}$ if $|r| < 1$. Going back to the initial series of our example, it follows that $1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^n}+\cdots=\frac{1}{1-\frac{1}{2}}=2$ if we apply the formula $\lim_{n\to\infty} S_n=\frac{a}{1-r}$ for $a=1$ and $r=\frac{1}{2}$.

Now, let us study the behavior of the series $1+2+3+4+\cdots$, which is an increasing arithmetic progression. It is obvious that the given series tends to infinity. Let $S_n=a_1+(a_1+d)+(a_1+2d)+\cdots+(a_n-2d)+(a_n-d)+a_n$ be the sum of the first n terms of an arithmetic series. Moreover, we can write $S_n=a_n+(a_n-d)+(a_n-2d)+\cdots(a_1+2d)+(a_1+d)+a_1$. If we add these equivalent expressions of S_n term by term, then we obtain $2S_n=n(a_1+a_n)\Rightarrow S_n=\frac{n}{2}(a_1+a_n)=\frac{n}{2}\{a_1+[a_1+(n-1)d]\}=\frac{n}{2}[2a_1+(n-1)d]$.

Notice that the series $\sum_{n=1}^{\infty}\frac{1}{n}$ is called harmonic, because it satisfies the property

$$\frac{2}{a_n}=\frac{1}{a_{n-1}}+\frac{1}{a_{n+1}}\ \forall n\geq 2.$$

In general, we say that a series $\sum_{n=1}^{\infty}a_n$ converges to a real number L, and we write $\sum_{n=1}^{\infty}a_n=L$ if and only if $\lim_{n\to\infty}S_n=L$ (S_n stands for the nth partial sum, namely, the general form of the summation of n terms of an infinite series). A series $\sum_{n=1}^{\infty}a_n$ tends to $+\infty$ or $-\infty$ if and only if $\lim_{n\to\infty}S_n=+\infty\ or-\infty$, respectively. If the limit of S_n as $n\to\infty$ does not exist, then we say that the series$\sum_{n=1}^{\infty}a_n$ diverges.

Remarks: (i) If $\sum_{n=1}^{\infty}a_n=L_1$ and $\sum_{n=1}^{\infty}b_n=L_2$, then $\sum_{n=1}^{\infty}(ma_n+nb_n)=mL_1+nL_2$, where $L_1,L_2\in\mathbb{R}$. (ii) If infinitely many terms are added to or subtracted from a converging series (resp. a diverging one), then the new series will still converge (resp. diverge). (iii) If $\sum_{n=1}^{\infty}a_n$ converges, then $lim_{n\to\infty}a_n=0$, since, if S_n is the nth partial sum of the given series, converging to some L, then $a_n=S_{n+1}-S_n\Rightarrow lim_{n\to\infty}a_n=lim_{n\to\infty}(S_{n+1}-S_n)=L-L=0$.

Comparison Test[405]*:* If $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ are two series such that $0\leq a_n\leq cb_n\ \forall n\geq n_0$, then $\sum_{n=1}^{\infty}b_n<\infty\Rightarrow\sum_{n=1}^{\infty}a_n<\infty$, and $\sum_{n=1}^{\infty}a_n=\infty\Rightarrow\sum_{n=1}^{\infty}b_n=\infty$. Notice that the same comparison test applies if $\lim_{n\to\infty}\frac{a_n}{b_n}=L$, where $0<L<\infty$.

Proof: It logically derives from the following facts: if the larger series converges, then the smaller series must also converge; and, if the smaller series is unbounded, then the larger series must also be unbounded.■

Special Ratio Test[406]*:* If $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ are two series with positive terms and there exists an $n_0\in\mathbb{N}$ such that $\frac{a_{n+1}}{a_n}\leq\frac{b_{n+1}}{b_n}\ \forall n\geq n_0$, then $\sum_{n=1}^{\infty}b_n<\infty\Rightarrow\sum_{n=1}^{\infty}a_n<\infty$, and $\sum_{n=1}^{\infty}a_n=\infty\Rightarrow\sum_{n=1}^{\infty}b_n$.

[405] Ibid.
[406] Ibid.

Proof: It follows the logic of the aforementioned Comparison Test.■

D'Alembert's Ratio Test[407]*:* Let $\sum_{n=1}^{\infty} a_n$ be a series with $a_n \geq 0 \ \forall n \in \mathbb{N}$. This series converges if there exists an $n_0 \in \mathbb{N}$ such that $\frac{a_{n+1}}{a_n} \leq c < 1 \ \forall n \geq n_0$, and it tends to $+\infty$ if $\frac{a_{n+1}}{a_n} \geq c > 1 \ \forall n \geq n_0$.

Proof: $\frac{a_{n+1}}{a_n} \leq c < 1 \ \forall n \geq n_0 \Rightarrow \begin{cases} a_{n_0+1} \leq c a_{n_0} \\ \vdots \\ a_{n_0+k} \leq c a_{n_0+k-1} \end{cases}$.

If we multiply the aforementioned inequalities by parts, then $a_{n_0+k} \leq c^k a_{n_0}$. Hence,

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{n_0-1} a_n + \sum_{n=n_0}^{\infty} a_n = \sum_{n=1}^{n_0-1} a_n + \sum_{k=0}^{\infty} a_{n_0+k} \leq \sum_{n=1}^{n_0-1} a_n + a_{n_0} \sum_{k=0}^{\infty} c^k < +\infty$, since $\sum_{n=1}^{n_0-1} a_n$ is a fixed number, and $\sum_{k=0}^{\infty} c^k$ is a geometric series with common ratio $c < 1$ that converges. If $c > 1$, then $a_{n+1} \geq c a_n > a_n \ \forall n \geq n_0$, that is, $(a_n)_{n \in \mathbb{N}}$ is an increasing sequence, so that $\lim_{n \to \infty} a_n \neq 0$. Therefore, the series $\sum_{n=1}^{\infty} a_n$ is not convergent, and, because it is a series of positive terms, it tends to $+\infty$.■

Corollary[408]*:* If the terms of a series $\sum_{n=1}^{\infty} a_n$ with $a_n > 0 \ \forall n \in \mathbb{N}$ satisfy the condition $lim_{n \to \infty} \frac{a_{n+1}}{a_n} = L \in \mathbb{R}$, then we distinguish the following cases:

i. if $0 \leq L < 1$, then this series converges;
ii. if $L > 1$, then this series tends to $+\infty$; and
iii. if $L = 1$, then the criterion does not apply.

Cauchy's nth *Root Test*[409]*:* Let $\sum_{n=1}^{\infty} a_n$ be a series with $a_n \geq 0 \ \forall n \in \mathbb{N}$. This series converges if there exists an $n_0 \in \mathbb{N}$ such that $\sqrt[n]{a_n} \leq c < 1 \ \forall n \geq n_0$, and it tends to $+\infty$ if $\sqrt[n]{a_n} \geq c > 1 \ \forall n \geq n_0$.

Proof: Notice that
$\sqrt[n]{a_n} \leq c < 1 \ \forall n \geq n_0 \Rightarrow \begin{cases} a_{n_0} \leq c^{n_0} \\ \vdots \\ a_{n_0+k} \leq c^{n_0+k} \end{cases}$. Hence,

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{n_0-1} a_n + \sum_{n=n_0}^{\infty} a_n = \sum_{n=1}^{n_0-1} a_n + \sum_{k=0}^{\infty} a_{n_0+k} \leq \sum_{n=1}^{n_0-1} a_n + c^{n_0} \sum_{k=0}^{\infty} c^k < +\infty$. If $\sqrt[n]{a_n} \geq c > 1$, then $a_n > 1 \ \forall n \geq n_0$, and, therefore, $\lim_{n \to \infty} a_n \neq 0$, which implies that, in this case, the series is not convergent, and, because it is a series of positive terms, it tends to $+\infty$.■

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[407] Ibid.
[408] Ibid.
[409] Ibid.

Corollary[410]*:* If a series $\sum_{n=1}^{\infty} a_n$ with $a_n \geq 0\ \forall n \in \mathbb{N}$ satisfies the condition that $lim_{n\to\infty} \sqrt[n]{a_n} = L \in \mathbb{R}$, then we distinguish the following cases:

i. if $0 \leq L < 1$, then this series converges;
ii. if $L > 1$, then this series tends to $+\infty$;
iii. if $L = 1$, then the criterion does not apply.

Remark:$\sum_{n=1}^{\infty} a_n < \infty \Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N} | \forall m, n \in \mathbb{N}\ with\ m, n > n_0, |S_m - S_n| < \varepsilon$, that is, an infinite series is convergent if and only if the sequence of its partial sums is a Cauchy sequence.

Leibniz's Alternating Series Test[411]*:* A series in which successive terms have opposite signs is called "alternating series." If $\sum_{n=1}^{\infty} a_n$ is an alternating series such that $|a_{n+1}| \leq |a_n|$ for all n and $\lim_{n\to\infty} a_n = 0$, then the series converges.

Proof: Given that $\sum_{n=1}^{\infty} a_n$ is an alternating series, a_n has either the sign $(-1)^n\ \forall n \in \mathbb{N}$, or the sign $(-1)^{n+1}\ \forall n \in \mathbb{N}$. Let us consider the odd-numbered partial sums of the given series. We realize that

$$S_{2n+1} = (a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) + \cdots + (a_{2n-1} - a_{2n}) + a_{2n+1}.$$

Because $|a_{n+1}| \leq |a_n|$, all the terms in the parentheses are non-negative, and, therefore, $S_{2n+1} \geq 0\ \forall n \in \mathbb{N}$. Moreover,

$$S_{2n+3} = S_{2n+1} - a_{2n+2} + a_{2n+3} = S_{2n+1} - (a_{2n+2} - a_{2n+3}),$$

and, because $a_{2n+2} - a_{2n+3} \geq 0$, we obtain

$$S_{2n+3} \leq S_{2n+1}.$$

Consequently, the sequence of odd-numbered partial sums is bounded from below by 0, and it is decreasing, meaning that it is convergent. For this reason, S_{2n+1} converges to some limit L. Now, let us consider the even-numbered partial sums of the given series. We find that $S_{2n+2} = S_{2n+1} - a_{2n+2}$, and, because $a_{2n+2} \to 0$,

$$lim_{n\to\infty} S_{2n+2} = lim_{n\to\infty} S_{2n+1} - lim_{n\to\infty} a_{2n+2} = L - 0 = L,$$

meaning that the even partial sums also converge to L. Because both the odd and the even sums converge to L, we realize that the partial sums converge to L, which proves the theorem.■

[410] Ibid.
[411] Ibid.

A series $\sum_{n=1}^{\infty} a_n$ is "absolutely convergent" if and only if $\sum_{n=1}^{\infty}|a_n|$ is convergent. If, however, $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty}|a_n|$ does not converge, then $\sum_{n=1}^{\infty} a_n$ is said to be a "conditionally convergent" series.

Remarks: $\sum_{n=1}^{\infty}|a_n| < \infty \Rightarrow \sum_{n=1}^{\infty} a_n < \infty$. This can be proved as follows: If we assume that the series $\sum_{n=1}^{\infty}|a_n|$ is convergent and $b_n = |a_n| - a_n$, then $a_n \leq |a_n|\ \forall n \in \mathbb{N} \Rightarrow 0 \leq b_n = |a_n| - a_n \leq |a_n| + |a_n| = 2|a_n|$, and, therefore, applying the Comparison Test, the series $\sum_{n=1}^{\infty} b_n$ converges. Then $\sum_{n=1}^{\infty} a_n$ converges, too, since $a_n = |a_n| - b_n\ \forall n \in \mathbb{N}$, and the series $\sum_{n=1}^{\infty}|a_n|$ and $\sum_{n=1}^{\infty} b_n$ converge.

Kummer's Test (the Prussian mathematician Eduard Kummer stated and proved this test in 1835)[412]*:* (i) A series $\sum_{n=1}^{\infty} a_n$, where $a_n \neq 0\ \forall n \in \mathbb{N}$, converges absolutely if there exist a sequence $(b_n)_{n\in\mathbb{N}}$ with positive terms and a constant $k > 0$ such that $0 < k \leq b_n - b_{n+1}\left|\frac{a_{n+1}}{a_n}\right|\ \forall n \in \mathbb{N}$. (ii) A series $\sum_{n=1}^{\infty} a_n$, where $a_n > 0\ \forall n \in \mathbb{N}$, tends to $+\infty$ if there is a sequence $(b_n)_{n\in\mathbb{N}}$ such that the series $\sum_{n=1}^{\infty}\frac{1}{b^n}$ tends to $+\infty$ and $b_n - b_{n+1}\left(\frac{a_{n+1}}{a_n}\right) \leq 0\ \forall n \in \mathbb{N}$.

Proof:

(i) Consider the following inequalities:

$k|a_1| \leq b_1|a_1| - b_2|a_2|$
$k|a_2| \leq b_2|a_2| - b_3|a_3|$
$\vdots$
$k|a_n| \leq b_n|a_n| - b_{n+1}|a_{n+1}|$.

Adding these inequalities by parts, we obtain

$k\sum_{i=1}^{n}|a_i| \leq b_1|a_1| - b_{n+1}|a_{n+1}| \leq b_1|a_1|$. Hence, $\sum_{i=1}^{n}|a_i| \leq \frac{b_1|a_1|}{k}$, which implies that the partial sums converge; $\{S_n\}_{n\in\mathbb{N}}$ is an increasing and bounded sequence, namely, convergent. Therefore, the series $\sum_{n=1}^{\infty}|a_n|$ converges.

(ii) We form the following inequalities:

$b_1a_1 \leq b_2a_2$
$b_2a_2 \leq b_3a_3$
$\vdots$
$b_{n-1}a_{n-1} \leq b_na_n$.

Multiplying these inequalities by parts, we obtain $\frac{b_1a_1}{b_n} \leq a_n\ \forall n \in \mathbb{N}$. Because $\sum_{n=1}^{\infty}\frac{1}{b_n} = +\infty$, the Comparison Test implies that $\sum_{n=1}^{\infty} a_n = +\infty$.■

[412] Ibid.

The Cauchy Product of Two Series of Real Numbers: The (Cauchy) product of two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ is a series $\sum_{n=1}^{\infty} c_n$ such that

$$c_n = a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1 = \sum_{k=0}^{\infty} a_{k+1}\, b_{n-k}.$$

A "power series in x"is a series of the form

$$\sum_{k=0}^{\infty} a_k\, x^k = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \cdots,$$

and a "power series in $(x - x_0)$" is a series of the form

$$\sum_{k=0}^{\infty} a_k\, (x - x_0)^k = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_k(x - x_0)^k + \cdots,$$

where x_0 is a real number.

By the term "radius of convergence" of the power series $\sum_{k=0}^{\infty} a_k\, x^k$, we mean a real number r such that $\sum_{k=0}^{\infty} a_k\, x^k$ converges if $|x| < r$ and does not converge if $|x| > r$. In other words, r is the radius of the largest disk (open 2-dimensional ball) in which the power series converges. At $x = r$ and at $x = -r$, we cannot decide if the series converges or diverges. If $\sum_{k=0}^{\infty} a_k\, x^k$ converges only at 0, then we say that its radius of convergence is 0. If $\sum_{k=0}^{\infty} a_k\, x^k$ converges for all real numbers, then we say that its radius of convergence is ∞.

Power Series Test[413]*:* In case of a power series $\sum_{n=0}^{\infty} a_n\, x^n$, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = l \Rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| |x| = l|x|$.

Proof: If $l|x| < 1 \Leftrightarrow |x| < \frac{1}{l}$, which implies that the series converges in the open interval $\left(-\frac{1}{l}, \frac{1}{l}\right)$, and $\frac{1}{l}$ is the radius of convergence. If $l|x| = 0 \Leftrightarrow l|x| < 1\ \forall x \in \mathbb{R}$, and, therefore, the series converges $\forall x \in \mathbb{R}$. If $l|x| = 1 \Leftrightarrow |x| = \frac{1}{l}$, and we cannot reach any conclusion. If $l = \infty$, then the series tends to infinity $\forall x \in \mathbb{R} - \{0\}$, but it converges at $x = 0$.■

Remark: When a power series is convergent in $(-a, a)$, then it is also convergent in every closed interval $[-k, k] \subset (-a, a)$. At the endpoints of the open interval $(-a, a)$, we must test the series for convergence or divergence separately.

Binomial Series[414]*:* The binomial coefficient is defined by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

where $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot n$, and $n \in \mathbb{N}$. Notice that the binomial coefficient gives the number of combinations of n elements taken k at the time (for this reason, $n = 0 \Rightarrow 0! = 1$). Thus, the binomial coefficient is the answer to the following question that was posed by the

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[413] Ibid.
[414] Ibid.

French-Jewish philosopher and mathematician Levi ben Gershon (known also by his Graecized name as Gersonides): how many are the ways in which one can choose k objects from n objects?

Remarks: (i) $\binom{n}{k} = \binom{n}{n-k}$ for $0 \leq k \leq n$. (ii) *Pascal's Identity:* for any positive integers k and n, $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$, meaning that the number of ways of choosing k things from n things is equal to the number of ways of choosing $k-1$ things from $n-1$ things added to the number of ways of choosing k things from $n-1$ things. (iii) $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.

Binomial Theorem (it was originally stated by Isaac Newton in 1676, and it was originally proved by John Colson in 1736)[415]*:* $(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{k}a^{n-k}b^k + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n = \sum_{k=0}^{n}\binom{n}{k}\, a^{n-k}b^k$.

Proof: The binomial theorem can be proved by mathematical induction as follows:

For $n = 1$, $(a+b)^1 = a + b = \binom{1}{0}a^1 + \binom{1}{1}a^0 b^1$.

Let the binomial theorem be true for n, namely:

$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$.

Claim that $(a+b)^{n+1} = (a+b)(a+b)^n$

$= (a+b)\left[\binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n\right] = a^{n+1} + \left[\binom{n}{0} + \binom{n}{1}\right]a^n b + \left[\binom{n}{1} + \binom{n}{2}\right]a^{n-1}b^2 + \cdots \left[\binom{n}{k-1} + \binom{n}{k}\right]a^{n-k+1}b^k + \cdots + \left[\binom{n}{n-1} + \binom{n}{n}\right]ab^n + b^{n+1}$. From Pascal's Identity, it follows that

$(a+b)^{n+1} = a^{n+1} + \binom{n+1}{1}a^n b + \cdots + \binom{n+1}{k}a^{n-k+1}b^k + \cdots + \binom{n+1}{n}ab^n + b^{n+1}$. Moreover, as I have already mentioned, $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.

Thus, the binomial theorem is true for $n+1$.■

Notice that, for any real number m, and for $|x| < 1$, the binomial series for $(1+x)^m$ is

$\sum_{k=0}^{\infty}\binom{m}{k}x^k = 1 + mx + \frac{m(m-1)}{2!}x^2 + \cdots + \frac{m(m-1)\dots(m-k+1)}{k!}x^k + \cdots$.

By the Ratio Test, the radius of convergence of $\sum_{k=0}^{\infty}\binom{m}{k}x^k$ is $r = 1$, so that the given series converges if $-1 < x < 1$, for which reason we have assumed that $|x| < 1$.

Infinite Products[416]*:* Notice that $\prod_{k=1}^{n} a_k = a_1 \cdot a_2 \cdot \ldots \cdot a_n$. An infinite product $\prod_{k=1}^{\infty} a_k$ is said to "converge" if $a_k \neq 0\ \forall k \in \mathbb{N}$, and, for P_n denoting the nth partial product, it holds

[415] Ibid.
[416] Ibid.

that $lim_{n\to\infty}P_n = lim_{n\to\infty}\prod_{k=1}^{n} a_k = p$ with$p \neq 0$. If $lim_{n\to\infty}P_n$ does not exist or is equal to 0, then we say that the corresponding infinite product "diverges."

Cauchy's Criterion of Convergence for Infinite Products[417]*:* An infinite product $\prod_{n=1}^{\infty} a_n$ converges if and only if:

$\forall\varepsilon > 0, \exists n_0 \in \mathbb{N} | \forall n \in \mathbb{N}\ with\ n > n_0 \&\ \forall k \in \mathbb{N}$, it holds that

$$|a_{n+1} \cdot a_{n+2} \cdot \ldots \cdot a_{n+k} - 1| < \varepsilon. \qquad (*)$$

Proof: Suppose that the infinite product $\prod_{n=1}^{\infty} a_n$ converges, so that $a_n \neq 0\ \forall n \in \mathbb{N}$ and $lim_{n\to\infty}P_n = p$, where $P_n = a_1 \cdot a_2 \cdot \ldots \cdot a_n$. Since $p \neq 0$, $\exists M > 0 | |P_n| > M$. Because $(P_n)_{n\in\mathbb{N}}$ converges, it is a Cauchy sequence, and, therefore,

$$\forall\varepsilon > 0, \exists n_0 \in \mathbb{N} | \forall n \in \mathbb{N}\ with\ n > n_0 \&\ \forall k \in \mathbb{N}, |P_{n+k} - P_n| < \varepsilon M.$$

Dividing by $|P_n|$, we obtain $(*)$.

Conversely, suppose that $(*)$ holds. If $\varepsilon = \frac{1}{2}$, then there exists an $n_1 \in \mathbb{N}$ such that

$$\frac{1}{2} < |Q_n| < \frac{3}{2}, \qquad (**)$$

where $Q_n = a_{n+1} \cdot a_{n+2} \cdot \ldots \cdot a_{n+k}$. Hence, if $(Q_n)_{n\in\mathbb{N}}$ is convergent, then it converges to a non-zero number. We can prove that $(Q_n)_{n\in\mathbb{N}}$ is convergent as follows: Because of $(*)$, there exists an $n_0 \in \mathbb{N}$ such that

$$\left|\frac{Q_{n+k}}{Q_n} - 1\right| < \frac{\varepsilon}{3},$$

and, therefore, by $(**)$, we realize that, if $m = max\{n_0, n_1\}$, then

$$|Q_{n+k} - Q_n| < \frac{\varepsilon}{3}|Q_n| < \frac{3}{2} \cdot \frac{\varepsilon}{3} < \varepsilon\ \forall n > m,$$

meaning that $(Q_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, and, hence, it converges. Consequently, the given infinite product converges.■

Remark: If, in $(*)$, we set $k = 1$, we realize that, if $\prod_{n=1}^{\infty} a_n$ converges, then $lim_{n\to\infty}P_n = 1$. For this reason, we often write $a_n = 1 + u_n$, $n \in \mathbb{N}$. Then $\prod_{n=1}^{\infty} a_n = \prod_{n=1}^{\infty}(1 + u_n)$, and, if $\prod_{n=1}^{\infty}(1 + u_n)$ converges, then $lim_{n\to\infty}u_n = 0$. This is a necessary condition for the infinite product $\prod_{n=1}^{\infty}(1 + u_n)$ to be convergent, but it is not a sufficient condition, since, for instance, the infinite product $\prod_{n=1}^{\infty}\left(1 + \frac{1}{n}\right)$ diverges, whereas $lim_{n\to\infty}\frac{1}{n} = 0$.

[417] Ibid.

Theorem[418]*:* Let $u_n > 0 \; \forall \, n \in \mathbb{N}$. Then the infinite product $\prod_{n=1}^{\infty}(1+u_n)$ converges if and only if the infinite series $\sum_{n=1}^{\infty} u_n$ converges.

Proof: Suppose that $\sum_{n=1}^{\infty} u_n$ converges. Let $S_n = u_1 + u_2 + \cdots + u_n$ and $P_n = (1+u_1)(1+u_2) \ldots (1+u_n)$ the sequences of the nth partial sums and the partial products, respectively. Notice that $e^x \geq 1 + x \; \forall x > 0$, because $e^x = lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \geq lim_{n\to\infty}\left(1+n\frac{x}{n}\right)^n = 1 + x$. Hence, it holds that

$$P_n = (1+u_1)(1+u_2) \ldots (1+u_n) \leq e^{u_1} e^{u_2} \ldots e^{u_n} = e^{S_n} \; \forall \, n \in \mathbb{N},$$

and, therefore, $(P_n)_{n\in\mathbb{N}}$ is bounded. Because $(P_n)_{n\in\mathbb{N}}$ is also increasing, that is, $\frac{P_{n+1}}{P_n} > 1$, it converges to some number p, where $p \geq 1$. Consequently, the infinite product $\prod_{n=1}^{\infty}(1+u_n)$ converges.

Conversely, suppose that $\prod_{n=1}^{\infty}(1+u_n)$ converges. Then $lim_{n\to\infty} P_n = p$, and $P_n \leq p \; \forall \, n \in \mathbb{N}$. Since

$$(1+u_1)(1+u_2) \ldots (1+u_n) \geq 1 + u_1 + u_2 + \cdots + u_n,$$

it holds that $1 + S_n \leq P_n \leq p \; \forall \, n \in \mathbb{N}$. Consequently, $(S_n)_{n\in\mathbb{N}}$ is bounded, and, because $u_n > 0 \; \forall \, n \in \mathbb{N}$, the infinite series $\sum_{n=1}^{\infty} u_n$ converges.■

2.6. The Limit of a Function

Preliminary Concepts[419]

Let f be a single-valued function defined on a subset S of $\mathbb{R}$. Then f is said to be "bounded" if its range $R_f = f(S) = \{f(x) | x \in S\}$ is bounded. In other words, a function f with domain D_f is said to be "bounded" on $A \subseteq D_f$ if there exists a number M such that $|f(x)| \leq M \; \forall x \in A$ (in particular, it said to be "bounded from above" if $f(x) \leq M$, and "bounded from below" if $M \leq f(x)$).

The supremum (least upper bound) of the range of a bounded function is called the "supremum of the function," and the infimum (greatest lower bound) of the range of a bounded function is called the "infimum of the function." Hence, $M = \sup(f)$ on S if $f(x) \leq M \; \forall x \in S$ and, for any $\delta > 0$, $f(x) > M - \delta$ for some $x \in S$; $m = \inf(f)$ on S if $f(x) \geq m \; \forall x \in S$ and, for any $\delta > 0$, $f(x) < m + \delta$ for some $x \in S$. The supremum (resp. the infimum) of f may or may not belong to its range R_f: if $\sup(f) \in R_f$ (resp. $\inf(f) \in R_f$), then $\exists x \in S | f(x) = M$ (resp. $f(x) = m$), and then we say that f "attains" its supremum (resp. infimum). If $f(x)$ is not bounded from above, then $\sup(f) = +\infty$, and, if $f(x)$ is not bounded from below, then $\inf(f) = -\infty$.

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[418] Ibid.

[419] See: Barbeau, *Polynomials*; Kramer, *The Nature and Growth of Modern Mathematics*; Waerden, *Algebra*.

It can be easily proved from the definitions of the supremum and infimum that, given two functions $f, g: U \to \mathbb{R}$ that are bounded on their common domain U, the following propositions hold[420]:

i. $sup\big(kf(x)\big) = \begin{cases} ksup(f(x)), k > 0 \\ kinf(f(x)), k < 0 \end{cases};$
ii. $inf(kf(x)) = \begin{cases} kinf\big(f(x)\big), k > 0 \\ ksup\big(f(x)\big), k < 0 \end{cases};$
iii. $sup[f(x) + g(x)] \leq sup\big(f(x)\big) + sup\big(g(x)\big);$
iv. $inf[f(x) + g(x)] \geq inf(f(x)) + inf\big(g(x)\big).$

For instance, the function $f(x) = x$ over $[0, \infty)$ is not bounded from above; $f(x) = x$ over $[0,1)$ is bounded, but it does not attain its supremum of 1; $f(x) = x + 5$ over $[-2,2]$ is bounded, and it attains both its supremum of 7 and its infimum of 3; if $f(x) = \begin{cases} 1 \; if \; x \in \mathbb{Q} \\ 0 \; if \; x \in \mathbb{Q}^{\sim} \end{cases}$ over $(-\infty, +\infty)$, then the range of f is $R_f = \{0,1\}$, which is bounded, $\sup(f) = 1$, and $\inf(f) = 0$.

A function f is said to be "increasing" in an interval if, for x_1, x_2 in the interval, $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$, and it is said to be "strictly increasing" if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$. A function f is said to be "decreasing" in an interval if, for x_1, x_2 in the interval, $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$, and it is said to be "strictly increasing" if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

A function $y = f(x)$ is said to be "single-valued" if, for each x-value, there is only one y-value. On the other hand, a "multivalued" function is similar to a function, but it may associate several values with each point. For instance, inverse trigonometric functions are multivalued, because trigonometric functions are periodic. Given a function y of x, namely, $y = f(x)$, the inverse function is denoted by $f^{-1}(y) = x$. Notice that, if $y = f(x)$ is single-valued, $f^{-1}(y)$ may be multivalued. For instance, $f(x) = tanx$ is single-valued, but $arctanx$ is multivalued: because $tanx$ is periodic,

$$tan\left(\frac{\pi}{4}\right) = tan\left(\frac{5\pi}{4}\right) = tan\left(\frac{-3\pi}{4}\right) = tan\left(\frac{(2n+1)\pi}{4}\right) = \cdots = 1,$$

and, therefore, $arctan(1)$ is associated with several values, such as $\frac{\pi}{4}, \frac{5\pi}{4}, \frac{-3\pi}{4}, \ldots$

A function of a single variable x is said to be a "polynomial" on its domain if it can be put in the following form:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \tag{P}$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are constants. Hence, every polynomial can be expressed as a finite sum of monomial terms of the form $a_k x^k$, in which the variable is raised to a non-negative integral power. Notice that $x^0 = 1$, and, therefore, $a_0 x^0 = a_0$. For the polynomial (P) with $a_n \neq 0$:

the numbers a_i (where $0 \leq i \leq n$) are called "coefficients";

[420] Ibid.

a_n is the "leading coefficient";
$a_n x^n$ is the "leading term";
a_0 is the "constant term" or the "constant coefficient";
a_1 is the "linear coefficient";
$a_1 x$ is the "linear term";

when the leading coefficient, a_n, is equal to 1, the polynomial is said to be "monic"; the non-negative integer n is the "degree" of the polynomial, and we write $\deg(p) = n$. A "constant polynomial" has only one term, specifically, a_0. A non-zero constant polynomial has degree 0, and, by convention, the "zero polynomial" (with all coefficients vanishing) has degree $-\infty$.

A "zero" of a polynomial $p(x)$ is any number r for which $p(r)$ takes the value 0. Hence, when $p(r) = 0$, we say that r is a "root," or a "solution" of the equation $p(x) = 0$.

Let

$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ and
$q(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_m x^m$

be two arbitrary polynomials. Then we can operate with them as follows:

Sum: $(p + q)(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots$
Difference: $(p - q)(x) = (a_0 - b_0) + (a_1 - b_1)x + (a_2 - b_2)x^2 + \cdots$
Product of a constant and a polynomial: $(cp)(x) = ca_0 + ca_1 x + ca_2 x^2 + \cdots$
Product of two polynomials: $(p \cdot q)(x) = a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots + (a_0 b_k + a_1 b_{k-1} + \cdots + a_i b_{k-i} + \cdots + a_k b_0)x^k + \cdots + (a_n b_m)x^{m+n}$.

Composition of two polynomials: $(p \circ q)(x) = p\big(q(x)\big)$, so that we replace each occurrence of x in the expression for $p(x)$ with $q(x)$.

Notice that we divide one polynomial by another in a manner similar to the division of two integers. First, we arrange the terms of the dividend and the divisor in descending powers of x. If a term is missing, then we write 0 as its coefficient. Then we divide the first term of the dividend by the first term of the divisor to obtain the first term of the quotient. Next, we multiply the entire divisor by the first term of the quotient, and we subtract this product from the dividend. We use the remainder as the new dividend, and we repeat the same process until the remainder is of lower degree than the divisor. As with the division of numbers, the dividend is equal to the product of the divisor and the quotient plus the remainder.

Remainder Theorem[421]*:* If a polynomial $p(x)$ is divided by $x - b$, then the remainder is $p(b)$.

Proof: Let $q(x)$ and r be, respectively, the quotient and the remainder when $p(x)$ is divided by $x - b$. Then, given that

[421] Ibid.

$Dividend = Quotient \times Divisor + Remainder,$

it holds that, for any x,
$p(x) = (x - b)q(x) + r$.
If $x = b$, then $p(b) = r$.■

Factor Theorem[422]*:* Given an arbitrary polynomial $y = p(x)$, b is a zero of $y = p(x)$ if and only of b is a factor of $p(x)$.

Proof: It can be easily verified using the Remainder Theorem.■

Remark: The real number zeros of $y = p(x)$ are also the x-intercepts in the graph of $y = p(x)$. If b is a real number zero with multiplicity n of $y = p(x)$, then the graph of $y = p(x)$ crosses the x-axis at $x = b$ if n is odd, whereas the graph turns around and stays on the same side of the x-axis at $x = b$ if n is even. Hence, the x-intercepts can be obtained from the Factor Theorem, and the behavior of the graph at an x-intercept, say $(b, 0)$, can be determined from the multiplicity of b, or, equivalently, by the highest power of $(x - b)$ that is a factor of $p(x)$. For instance, if $p(x) = (x + 1)(x - 2)^2$, then: by setting $x = 0$, we realize that the y-intercept is $(0,4)$; because $(x + 1)$ is a factor with an odd exponent, $(-1,0)$ is an x-intercept at which the graph crosses the x-axis; because $(x - 2)^2$ is a factor with an even exponent, $(2,0)$ is an x-intercept at which the graph touches the x-axis and then turns around.

In fact, the fundamental problem in algebra consists in finding ways of solving polynomial equations, and, specifically, we seek formulas for zeros/roots in terms of the coefficients of the corresponding polynomial. A well-known example is the "quadratic formula": If we have the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, then we have the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

where the expression $b^2 - 4ac$ is known as the "discriminant," meaning that, if we have a number r such that $r^2 = b^2 - 4ac$, then

$$x_1 = \frac{-b + r}{2a} \text{ and } x_2 = \frac{-b - r}{2a}$$

are the solutions of $ax^2 + bx + c = 0$.

If a function $y = f(x)$ satisfies an equation of the form

$$p_0(x)y^n + p_1(x)y^{n-1} + \cdots + p_{n-1}(x)y + p_n(x) = 0,$$

[422] Ibid.

where $p_0(x), \dots, p_n(x)$ are polynomials in x, then it is said to be an "algebraic function." In other words, an algebraic function is a function that can be defined as the root of a polynomial equation. If a function can be expressed as the quotient of two polynomials, namely,

$$f(x) = \frac{p(x)}{q(x)},$$

then it is called a "rational algebraic function." Thus, an "algebraic curve" (such as the lines and the conic sections studied in section 2.2.6) is defined as a curve with an equation of the form $p(x, y) = 0$ where p is a polynomial in x and y (and, usually, we take rational coefficients). Notice that any rational function $y = \frac{p(x)}{q(x)}$ is the solution to $q(x)y - p(x) = 0$.

If a function cannot be expressed as the quotient of two polynomials, then it is called an "irrational algebraic function." Thus, an algebraic function involving one or more radicals of polynomials is called an irrational function.

If a function is not algebraic, then it is called a "transcendental function." For instance, exponential functions, logarithmic functions, trigonometric functions, and inverse trigonometric functions are transcendental functions (however, a composition of transcendental functions can give an algebraic function, such as $f(x) = cos\big(arcsin(x)\big) = \sqrt{1 - x^2}$, and $g(x) = sin\big(arcsin(x)\big) = x$).

The Limit of a Function

The concept of a limit, or a limiting process, is central to all mathematical analysis.[423] In fact, one can argue that, from the perspective of mathematical analysis, "analysis" means taking limits. Consider an arbitrary function $f(x)$ defined at all values in an open interval of the number line $\mathbb{R}$ containing a point x_0, with the possible exception of x_0 itself, and let L be a real number. The "limit of a function" $f(x)$ at a point x_0 is L if and only if the values of x (where $x \neq x_0$) approach the number x_0 (notice that $f(x_0)$ may not be defined, since, according to the definition of a limit, x tends to x_0, but x never becomes equal to x_0). In other words as x gets closer to x_0, $f(x)$ gets closer and stays close to L; symbolically:

$$lim_{x \to x_0} f(x) = L.$$

Remark: Let a be a real number and c a constant. Then

$lim_{x \to a} x = a$, and
$lim_{x \to a} c = c.$

Equivalently, we can define the limit of a function as follows: If x_0 is an accumulation point of the domain D_f of a function f, then we say that the limit of f is $L \in \mathbb{R} \cup \{-\infty, +\infty\}$ as x tends to $x_0 \in \mathbb{R} \cup \{-\infty, +\infty\}$, and we write $lim_{x \to x_0} f(x) = L$, if and only if, for all the

[423] See: Apostol, *Mathematical Analysis*; Fraleigh, *Calculus with Analytic Geometry*; Hardy, *A Course of Pure Mathematics*; Landau, *Foundations of Analysis*; Nikolski, *A Course of Mathematical Analysis*; Spivak, *Calculus*; in conjunction with Cauchy, *Cours d'Analyse*.

sequences $(x_n)_{n\in\mathbb{N}}$ of numbers in $D_f - \{x_0\}$ that converge to x_0, the corresponding sequence of the values of the function $f(x_n)$, where $n \in \mathbb{N}$, converges to L. When, in the aforementioned definition, we say that "x tends to x_0," we mean that the distance $|x - x_0|$ becomes infinitely close to zero, without ever becoming equal to zero.

One-sided limits: Assume that a function $f(x)$ is defined at all values in an open interval of the real line, and that L is a real number. If the values of $f(x)$ approach L as the values of x approach the number a and $x < a$, then we say that L is the limit of $f(x)$ as x approaches a from the left, and we write

$$lim_{x\to a^-} f(x) = L.$$

By analogy, if the values of $f(x)$ approach L as the values of x approach the number a and $x > a$, then we say that L is the limit of $f(x)$ as x approaches a from the right, and we write

$$lim_{x\to a^+} f(x) = L.$$

If $f(x)$ is a function defined at all values in an open interval of the real line containing a, with the possible exception of a, and if L is a real number, then

$$lim_{x\to a} f(x) = L \Leftrightarrow lim_{x\to a^-} f(x) = L = lim_{x\to a^+} f(x).$$

Moreover, notice that, if $lim_{x\to a} f(x) = L \in \mathbb{R}$, then $f(x)$ is bounded on a region (specifically, on a deleted neighborhood) of a.

The Cauchy epsilon-delta definition of a limit[424]*:* First of all, recall that, as I explained in section 2.2.6, the distance between any two points a and b on the number line $\mathbb{R}$ is $|a - b|$. Therefore, the statement

$$|f(x) - L| < \varepsilon$$

means that the distance between $f(x)$ and L is less than ε, and, by the definition of an absolute value, the statement

$$0 < |x - a| < \delta$$

is equivalent to the statement

$a - \delta < x < a + \delta$, so that $x \neq a$.

Thus, the Cauchy epsilon-delta definition of a limit is the following: Assume that, for all $x \neq a$, an arbitrary function $f(x)$ is defined over an open interval containing a. Then

[424] Ibid.

$$lim_{x \to a} f(x) = L$$

if and only if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that, if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

The statement (with universal quantifier) "for every $\varepsilon > 0$" means "for every positive distance ε from L"; the statement (with the existential quantifier) "there exists a $\delta > 0$" means that there is a positive distance δ from a; and the conditional statement "if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$" means that, if x is closer than δ to a, and $x \neq a$, then the value of $f(x)$ is closer than ε to L. Therefore, the definition of a limit is based on the definition of a deleted neighborhood (studied in section 2.3.1), since a deleted neighborhood can be defined as a set that includes every point for which

$$0 < |x - a| < \delta.$$

Remark: The name of the aforementioned definition of a limit was given in honor of the French mathematician Augustin-Louis Cauchy (1789–1857), who studied this concept in a systematic way, and he was one of the greatest pioneers of mathematical analysis.

Examples:

(i) The limit of $f: [-1,0) \cup (0,1]$ with $f(x) = \begin{cases} x^2 + 1, x < 0 \\ 1 - x, x > 0 \end{cases}$, as $x \to 0$ is 1. We can prove this result as follows: We want an ε such that $0 < \varepsilon < 1$. Then we must prove that $\exists \delta > 0 | |(x^2 + 1) - 1| < \varepsilon$ or $x^2 < \varepsilon \, \forall x$ with $-\delta < x < 0$, and $|(1 - x) - 1| < \varepsilon$ or $x < \varepsilon \, \forall x$ with $0 < x < \delta$. Setting $\delta = \varepsilon$, the previous inequalities hold, so that $lim_{x \to 0^-} f(x) = lim_{x \to 0^+} f(x)$. Therefore, $lim_{x \to 0} f(x) = 1$.

(ii) Given that $f(x) = \frac{10x-10}{4x^2+2x+1}$ is defined $\forall x \in \mathbb{R}$, we can compute its limit as $x \to 1$ by direct substitution, namely: $lim_{x \to 1} \frac{10x-10}{4x^2+2x+1} = \frac{10-10}{4+2+1} = 0$.

(iii) Given $lim_{x \to 2} \frac{5}{(x-2)^{2020}}$, we observe that, as $x \to 2$, the denominator tends to 0, and the given fraction tends to infinity. Therefore, this limit does not exist.

(iv) If we are asked to compute $lim_{x \to 1} f(x)$ when $f(x) = \begin{cases} \frac{10-x}{2}, x \geq 1 \\ 2 + x, x < 1 \end{cases}$, then we observe the following: if x is slightly larger than 1, then $f(x)$ approaches $\frac{9}{2}$, and, if x is slightly smaller than 1, then $f(x)$ approaches 3. Therefore, the required limit does not exist.

The concept of the limit of a function and the concept of boundedness are related to each other through the following theorem.

Theorem[425]*:* Consider a real function f with domain D_f such that $lim_{x \to a} f(x) = L$. Then

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[425] Ibid.

i. f is bounded on some region (deleted neighborhood) of a;
ii. if $L \neq 0$, then there exists a region (deleted neighborhood) of a where $f(x) \neq 0$.

Proof:

(i) Given that $lim_{x \to a} f(x) = L$,

$(\forall \varepsilon > 0)(\exists \delta > 0)\big(\forall x \in D_f\big)[0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon]$.

If $\varepsilon = 1$, then $|f(x)| \leq L + 1$.

If $a \notin D_f$, then we set $M = |L| + 1$. If $a \in D_f$, then we set $M = sup\{|f(a)|, |L| + 1\}$. Hence, if $x \in N'_\delta(a) \cap D_f$, then $|f(x)| \leq M$, meaning that f is bounded on a region (deleted neighborhood) of a.

(ii) Given that $L \neq 0$, if we set $\varepsilon = \frac{1}{2}|L|$, then we obtain

$$-\frac{|L|}{2} + L < f(x) < L + \frac{|L|}{2}. \quad (*)$$

If $L > 0$, then $(*)$ implies that $\frac{L}{2} < f(x) < \frac{3L}{2}$, so that $\frac{|L|}{2} < |f(x)| < \frac{3|L|}{2}$. If $L < 0$, then $(*)$ implies that $\frac{3L}{2} < f(x) < \frac{L}{2} \Leftrightarrow -\frac{3L}{2} > -f(x) > -\frac{L}{2} \Leftrightarrow \frac{|L|}{2} < |f(x)| < \frac{3|L|}{2}$. Therefore, for $L \neq 0$, there exists a region (deleted neighborhood) of a, denoted by $N'_\delta(a)$, where $f(x) \neq 0$.■

Remark: The aforementioned theorem can be equivalently reformulated as follows: if f has a non-zero limit at a, then there exists a region (deleted neighborhood) of a, denoted by $N'_\delta(a)$, where f is bounded away from zero. This implies that the fraction $\frac{1}{f}$ exists and is meaningful $\forall x \in N'_\delta(a) \cap D_f$.

Theorem[426]*:* The limit of a function near a point is unique. In other words, if $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} f(x) = k$, then $l = k$.

Proof: This theorem can be proved by *reductio ad absurdum* as follows: If f converges to l near a, then

$$\forall \varepsilon > 0, \exists \delta_1 > 0 | \forall x, 0 < |x - a| < \delta_1 \Rightarrow |f(x) - l| < \frac{\varepsilon}{2}.$$

Choosing $\delta = min\{\delta_1, \delta_2\}$,

$$0 < |x - a| < \delta \Rightarrow |f(x) - l| < \frac{\varepsilon}{2} \& |f(x) - k| < \frac{\varepsilon}{2}.$$

Notice that $|l - k| = |f(x) - k - f(x) + l| \leq |f(x) - k| + |f(x) - l| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for $0 < |x - a| < \delta$. Therefore, the fact that $|l - k| < \varepsilon$ for every ε implies that $l = k$.■

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[426] Ibid.

Theorem[427]*:* If $\lim_{x \to a} g(x) = l$ and $\lim_{x \to a} h(x) = m$, where $l \in \mathbb{R}$, $m \in \mathbb{R}$, and a is the accumulation point of the common domain D of the functions g and h, then the following properties hold:

i. $\lim_{x \to a}[g(x) + h(x)] = l + m$.
ii. $\lim_{x \to a} g(x) \cdot h(x) = l \cdot m$.
iii. $lim_{x \to a}|g(x)| = |l|$.
iv. $\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{l}$ if $l \neq 0$.
v. $\lim_{x \to a} \frac{g(x)}{h(x)} = \frac{l}{m}$ if $m \neq 0$.

Proof: Let $\varepsilon > 0$. Without loss of generality, we assume that $\varepsilon < 1$. Then, by hypothesis, there exists a $\delta > 0$ such that, for every x that belongs to the common domain D of the functions g and h, and for $0 < |x - a| < \delta$, the following inequalities hold: $|g(x) - l| < \varepsilon$ and $|h(x) - m| < \varepsilon$. For such x, we have the following:

$$|[g(x) + h(x)] - (l + m)| \leq |g(x) - l| + |h(x) - m| < 2\varepsilon,$$

which proves (i);

$$|g(x) \cdot h(x) - l \cdot m| \leq |g(x)||h(x) - m| + |m||g(x) - l|$$
$$\leq (|l| + \varepsilon)\varepsilon + |m|\varepsilon < (|l| + 1 + |m|)\varepsilon,$$

which proves (ii); and

$$\big||g(x)| - |l|\big| \leq |g(x) - l| < \varepsilon,$$

which proves (iii).

We can prove (iv) as follows: since $l \neq 0$, then (given that, if g has a non-zero limit at a, there exists a region (deleted neighborhood) of a, denoted by $N'_{\delta}(a)$, where g is bounded away from zero), there exist positive numbers k and δ_1 such that $|g(x)| > k$, where $k = \frac{|l|}{2}$, $\forall x \in D \cap N'_{\delta_1}(a)$. Let $\delta^* = min\{\delta, \delta_1\}$, so that, $\forall x \in D \cap N'_{\delta^*}(a)$, we have $\left|\frac{1}{g(x)} - \frac{1}{l}\right| = \left|\frac{l - g(x)}{g(x)l}\right| < \frac{\varepsilon}{k|l|}$, which implies that $\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{l}$.

Given (iv), (v) follows from (ii), because the limit of $g \cdot \frac{1}{h}$ as $x \to a$ is $l \cdot \frac{1}{m} = \frac{l}{m}$.■

Corollary: If $\lim_{x \to a} g(x)$ exists, and if n is a positive integer, then there exists $\lim_{x \to a}[g(x)]^n$, and $\lim_{x \to a}[g(x)]^n = \left[\lim_{x \to a} g(x)\right]^n$. Similarly, $lim_{x \to a} \sqrt[n]{g(x)} = \sqrt[n]{lim_{x \to a} g(x)}$.

[427] Ibid.

Theorem[428]*:* If $p(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$ is a polynomial function with $c_0, c_1, c_2, \ldots, c_n$ being real numbers and n being a fixed positive integer, then, $\forall a \in \mathbb{R}$, $lim_{x \to a} p(x) = p(a)$.

Proof: Due to the previous theorem (specifically, by applying the sum, constant multiple, and power properties), we have:

$$lim_{x\to a} p(x) = lim_{x\to a}(c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0)$$
$$= c_n(lim_{x\to a} x)^n + c_{n-1}(lim_{x\to a} x)^{n-1} + \cdots + c_1(lim_{x\to a} x) + lim_{x\to a} c_0$$
$$= c_n a^n + c_{n-1}a^{n-1} + \cdots + c_1 a + c_0 = p(a).\blacksquare$$

Corollary: Let $p(x)$ and $q(x)$ be two polynomial functions, $a \in \mathbb{R}$, and $q(a) \neq 0$. Then $lim_{x\to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$.

Limits at infinity: A function $f(x)$ is said to have a "limit at infinity" if there exists a real number L such that, $\forall \varepsilon > 0$, $\exists M > 0 | |f(x) - L| < \varepsilon \; \forall x > M$, and, in this case, we write $lim_{x\to\infty} f(x) = L$. Moreover, we can write $lim_{x\to-\infty} f(x)$ and $lim_{x\to+\infty} f(x)$ if x increases without bound on the negative or on the positive direction, respectively.

The Squeeze Theorem for Functions[429]*:* Assume that the functions f, g, and h are defined on a set $U \subseteq D$, where D is the common domain of these three functions, and that a is an accumulation point of U. If

$f(x) \leq g(x) \leq h(x) \; \forall x \in U$ and
$lim_{x\to a} f(x) = lim_{x\to a} h(x) = L$, where $a \in \mathbb{R}$, and it may be $+\infty$ or $-\infty$,

then $lim_{x\to a} g(x) = L$.

Proof: It is a straightforward generalization of the Squeeze Theorem for Convergent Sequences, which was proved in section 2.4.■

Example: Applying the Squeeze Theorem, we can compute $lim_{x\to 0} x cos x$ as follows: Because, for all x, $-1 \leq cos x \leq 1$, we have $-x \leq x cos x \leq x$ for $x \geq 0$, and $-x \geq x cos x \geq x$ for $x \leq 0$. Moreover, $lim_{x\to 0}(-x) = 0 = lim_{x\to 0} x$. Therefore, $lim_{x\to 0} x cos x = 0$.

2.7. Continuous Functions

The concept of the continuity of a function is very important not only because it underpins the study of real functions in the context of pure mathematics but also because it plays a preeminent role in the construction of scientific models for the study of cosmology, mechanics, biology, economics, social dynamics, and other scientific fields. Intuitively, the

[428] Ibid.
[429] Ibid.

concept of continuity is connected with the geometric concept of an uninterruptedly extended line.

Consider a function f whose domain is D_f. Let a be an interior point of D_f. Then f is said to be "continuous at the point" a if

$lim_{x \to a} f(x)$ exists finitely and
$lim_{x \to a} f(x) = f(a)$,

namely, if the limit of $f(x)$ as x tends to a is equal to the value of $f(x)$ at a.[430] If a is a boundary point of D_f (i.e., in this case, an endpoint of a closed interval), then we distinguish the following two cases:

i. if $D_f = (x_1, a]$, then $f(x)$ is said to be "continuous from the left" at a if $lim_{x \to a^-} f(x) = f(a)$;
ii. if $D_f = [a, x_2)$, then $f(x)$ is said to be "continuous from the right" at a if $lim_{x \to a^+} f(x) = f(a)$.

The aforementioned definition of continuity can also be given in the following equivalent forms[431]:

(i) A function f is continuous at $a \in D_f$ if and only if, for every sequence (x_n) with $lim_{n \to \infty} x_n = a$, where $x_n \in D_f$, it holds that $lim_{n \to \infty} f(x_n) = f(a)$. The sequential definition of continuity was originally developed by the German mathematician Heinrich Eduard Heine (1821–81).
(ii) A function f is continuous at $x = a \in D_f$ if and only if, $\forall \varepsilon > 0, \exists \delta > 0 || x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$.
(iii) In section 2.2.6, I defined the continuity of a mapping between metric spaces, and, in section 2.3, I defined the continuity of a mapping between topological spaces. Moreover, in view of 2.2.4, if we use the concept of continuity for a line, then continuity relates the set of all the points of the given line to the set $\mathbb{R}$ of all real numbers.

A function f is said to be "continuous over (or on, or in) an open interval" (x_1, x_2) if f is continuous at every point in that interval (x_1 may be $-\infty$, and/or x_2 may be $+\infty$). A function f is said to "continuous over (or on, or in) the closed interval" $[x_1, x_2]$ if the following conditions hold: (i) f is continuous at every x in the open interval (x_1, x_2); (ii) $f(x_1)$ and $f(x_2)$ both exist; and (iii) $lim_{x \to x_1^+} f(x) = f(x_1)$, and $lim_{x \to x_2^-} f(x) = f(x_2)$.

Remarks: If we compare the definition of the limit of a function (studied in section 2.6) with the definition of the continuity of a function, we realize that they have the same structure, but they also have the following differences:

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[430] Ibid.
[431] Ibid.

i. In the case of the limit of a function (Cauchy epsilon-delta definition), we have $0 < |x-a| < \delta$, namely, $x \neq a$, whereas, in the case of continuity, we have only $|x-a| < \delta$, meaning that the definition of continuity holds also when $x = a$.
ii. Instead of the value L that is used in the definition of the limit of a function, the definition of the continuity of a function uses the value $f(a)$, meaning that, in the case of the continuity of a function, the function must be defined at the point a. Indeed, it is meaningless to talk about the continuity (or the discontinuity) of a function at a point that does not belong to its domain of definition.
iii. In the definition of the limit of a function (Cauchy epsilon-delta definition), the point a must be an accumulation point of the domain of definition D_f of the corresponding function, and, therefore, it may not belong to D_f, whereas, in the definition of the continuity of a function, the point a must belong to the domain of definition D_f of the corresponding function (the definition of an accumulation point was stated in section 2.3.4).

Theorem[432]*:* The constant function $f(x) = c$ and the identity function $f(x) = x$ are continuous $\forall x \in \mathbb{R}$.

Proof: We shall prove that $f(x)$ is continuous at an arbitrary point x_0. If $f(x) = c$, then $f(x_0) = c$, and then, $\forall \varepsilon > 0, |f(x) - f(x_0)| = 0 < \varepsilon \ \forall x, x_0 \in \mathbb{R}$. If δ is an arbitrary positive real number, then, $\forall x \in \mathbb{R}$ with $|x - x_0| < \delta$, it holds that $|f(x) - f(x_0)| = 0 < \varepsilon$, meaning that $f(x) = c$ is continuous.

If $f(x) = x$, then it suffices to show that, $\forall \varepsilon > 0, \exists \delta > 0 | |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon \ \forall x \in \mathbb{R}$. But, since $|f(x) - f(x_0)| = |x - x_0|$, we must merely set $\delta = \varepsilon$ to prove the continuity of $f(x) = x$.∎

The concept of a continuous function underpins the concept of a "curve" in $\mathbb{R}^n$, since a curve $C \subset \mathbb{R}^n$ is defined by a set

$$\{x = (x_1, \dots, x_n) \in \mathbb{R}^n | f_i(x) = c_i, for\ 1 \leq i \leq n-1\},$$

where each $f_i: \mathbb{R}^n \to \mathbb{R}$ is a continuous function, and $c_i \in \mathbb{R}$. For instance, in $\mathbb{R}^2$, consider the circle $C_1 = \{(x,y) \in \mathbb{R}^3 | x^2 + y^2 = 1\}$ and the parabola $C_2 = \{(x,y) \in \mathbb{R}^3 | x^2 - y = 0\}$. A "parametrized curve" in $\mathbb{R}^n$ is a continuous function

$$\gamma: (\alpha, \beta) \to \mathbb{R}^n | t \xrightarrow{\gamma} \big(\gamma_1(t), \dots, \gamma_n(t)\big),$$

where $-\infty \leq \alpha < \beta \leq \infty$, and the $\gamma_i: \mathbb{R} \to \mathbb{R}$ are continuous functions. For instance, a parametrization for the aforementioned circle C_1 is $\gamma_1: (-\infty, \infty) \to \mathbb{R}^2 |\ t \xrightarrow{\gamma_1} \big(\cos(t), sin(t)\big)$, and a parametrization for the aforementioned parabola C_2 is $\gamma_2: (-\infty, \infty) \to \mathbb{R}^2 |\ t \xrightarrow{\gamma_2} (t, t^2)$.

In general, the concept of continuity underpins the concept of a manifold, which is an abstraction of the notion of a curved surface. A mapping $f: X \to Y$ of metric spaces is said to

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[432] Ibid.

be a “homeomorphism” (in which case X and Y are said to be homeomorphic) if the following three conditions are satisfied: f is continuous, f is bijective, and the inverse mapping f^{-1} is continuous. A metric space M is said to be an “n-dimensional manifold” (or simply “manifold”) if any point x of M is contained in a neighborhood $U \subset M$ homeomorphic to a domain V of a Euclidean space $\mathbb{R}^n$. Thus, an n-dimensional manifold M is locally homeomorphic to a domain V of a Euclidean space $\mathbb{R}^n$, and then the dimension of M is equal to n. The concept of a manifold indicates that topology, the most abstract study of the structure of space, precedes geometry, and that, since a manifold is locally Euclidean while its global structure may be non-Euclidean, different geometries can be simultaneously valid.

Furthermore, by the definition of the continuity of a function and by the definition of an isolated point (stated in section 2.3.1), we realize that, given a function $f: D_f \to \mathbb{R}$, where $D_f \subseteq \mathbb{R}$, and a point $a \in D_f$ such that a is an isolated point of D_f, it holds that f is continuous at a. For instance, $f: \mathbb{N}^* \to \mathbb{R}$ with $f(x) = \frac{1}{x}$ is continuous, since every point of $\mathbb{N}$ is an isolated point. In general, notice that any point $x_0 \in D_f$ will be either an isolated point or a point of accumulation of D_f.

The points at which a function f is not continuous are called “points of discontinuity,” and f is called “discontinuous” at these points.

Types of Discontinuity

i. If both $lim_{x \to x_0^+} f(x)$ and $lim_{x \to x_0^-} f(x)$ exist but $lim_{x \to x_0^+} f(x) \neq lim_{x \to x_0^-} f(x)$, then f is said to have a “jump discontinuity” at $x = x_0$, as shown in Figure 2.13. If $lim_{x \to x_0^+} f(x) \neq f(x_0)$ and $lim_{x \to x_0^-} f(x) = f(x_0)$, then f has a “jump discontinuity from the right.” If $lim_{x \to x_0^-} f(x) \neq f(x_0)$ and $lim_{x \to x_0^+} f(x) = f(x_0)$, then f has a “jump discontinuity from the left.”

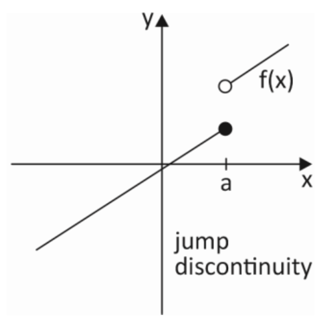

Figure 2.13. Jump Discontinuity.

ii. If $lim_{x \to x_0} f(x)$ and $f(x_0)$ exist but $lim_{x \to x_0} f(x) \neq f(x_0)$, then f is said to have a “removable discontinuity” at $x = x_0$, as shown in Figure 2.14 (i.e., x_0 is a hole, a point on the graph of f at which f is undefined). We say that a discontinuity is “removable” because we can redefine the value of the function at a point of discontinuity in such a way that the new function is continuous at that point (i.e., a removable discontinuity can be “filled in” if we make the function a piecewise function and define a part of the function at the point where the hole is).

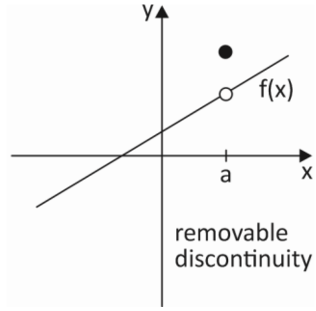

Figure 2.14. Removable Discontinuity.

iii. If $lim_{x\to x_0^+} f(x) = +\infty$ or $lim_{x\to x_0^-} f(x) = -\infty$, namely, if at least one of the one-sided limits of the function tends to infinity, then f is said to have an "infinite discontinuity" at $x = x_0$, as shown in Figure 2.15.

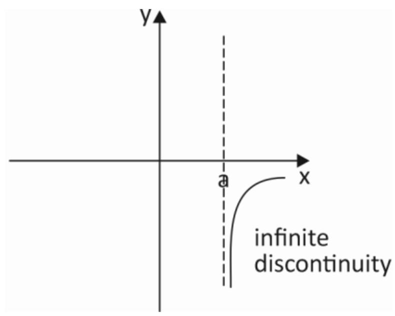

Figure 2.15. Infinite Discontinuity.

iv. If neither $lim_{x\to x_0^+} f(x)$ nor $lim_{x\to x_0^-} f(x)$ exists, then f is said to "discontinuous" at $x = x_0$.

v. If only one of the limits $lim_{x\to x_0^+} f(x)$ and $lim_{x\to x_0^-} f(x)$ exists, then f is said to have a "mixed discontinuity" at $x = x_0$.

Examples: (i) The function $f(x) = [x]$, where $[x]$ denotes the greatest integer $\leq x$, is continuous at every non-integral value of x, and it is discontinuous at every integral value of x. *Proof:* Let $x_0 = a + b$ be a non-integral value of x with $a \in \mathbb{Z}$ and $b \in (0,1)$. Then $f(x_0) = [a + b] = a$. Notice that $lim_{x\to x_0^+} f(x) = lim_{h\to 0} f(x_0 + h) = lim_{h\to 0}[a + b + h] = a$, and $lim_{x\to x_0^-} f(x) = lim_{h\to 0} f(x_0 - h) = lim_{h\to 0}[a + b - h] = a$. Therefore, $lim_{x\to x_0} f(x) = f(x_0)$ for every non-integral value of x. Now, assume that $x_0 = a$, where $a \in \mathbb{Z}$, is an integral value of x. Then $f(x_0) = [x_0] = [a] = a$. Notice that $lim_{x\to x_0^+} f(x) = lim_{h\to 0}[x_0 + h] = lim_{h\to 0}[a + h] = a$, and $lim_{x\to x_0^-} f(x) = lim_{h\to 0}[x_0 - h] = lim_{h\to 0}[a - h] = a - 1$. Hence, in this case, $lim_{x\to x_0^-} f(x) = a - 1 \neq f(x_0) = lim_{x\to x_0^+} f(x)$.■ (ii) Dirichlet's function, named after the German mathematician Johann Peter Gustav Lejeune Dirichlet (1805–59), who formulated it, is defined as follows:

$f(x) = \begin{cases} 1\ if x \in \mathbb{Q} \\ -1\ if\ x \in \mathbb{Q}^{\sim} \end{cases}$.

This function is everywhere discontinuous over the reals. *Proof:* Since both rational numbers (where $f = 1$) and irrational numbers (where $f = -1$) are dense in ℝ, every open interval, however small, contains both rational and irrational numbers. Therefore, it is impossible to draw the function properly, because we should draw two horizontal lines (for $f = 1$ and $f = -1$) such that they will have holes everywhere, that is, there will be no piece of uninterrupted line (given that there are rational and irrational points everywhere): as regards the line corresponding to $f = 1$, whenever we take two rational numbers, there will always be some irrational number between them making a hole in this line; and, similarly, as regards the line corresponding to $f = -1$, whenever we take two irrational numbers, there will always be some rational number between them making a hole in this line. Consequently, $f(x)$ has no accumulation point anywhere in its domain, meaning that $f(x)$ is everywhere discontinuous over the reals. However, notice that, if we take the absolute value $|f(x)|$, then $|f(x)| = 1\ \forall x \in \mathbb{R}$, which is everywhere continuous over ℝ.■

Theorem[433]*:* Let $f(x)$ and $g(x)$ be two functions continuous at $x = x_0$. Then:

(i) $f(x) + g(x)$,
(ii) $f(x) - g(x)$,
(iii) $f(x) \cdot g(x)$, and
(iv) $\frac{f(x)}{g(x)}, g(x_0) \neq 0$,

are continuous at $x = x_0$.

Proof: The proof is a direct application of the definition of continuity.■

Remark: The converse may not be true. For instance, if

$f(x) = \begin{cases} -x\ if\ x < 0 \\ 1\ if\ x \geq 0 \end{cases}$ and $g(x) = \begin{cases} 1\ if\ x < 0 \\ x\ if\ x \geq 0 \end{cases}$'

then $f(x) + g(x)$ and $f(x) \cdot g(x)$ are continuous at 0 even though neither $f(x)$ nor $g(x)$ is continuous at 0.

Theorem[434]*:* If $f(x)$ is continuous at $x = x_0$, then $|f(x)|$ is also continuous at $x = x_0$, but the converse may not be true.

Proof: If $f(x)$ is continuous at $x = x_0$, then, by definition,

$$\forall \varepsilon > 0, \exists \delta > 0 \,|\, |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon. \qquad (*)$$

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[433] Ibid.
[434] Ibid.

Moreover, $\big||f(x)| - |f(x_0)|\big| \leq |f(x) - f(x_0)|$. $(**)$

$(*)$ and $(**)$ imply that

$|x - x_0| < \delta \Rightarrow \big||f(x)| - |f(x_0)|\big| < \varepsilon,$

and, therefore, $|f(x)|$ is continuous at $x = x_0$.
In order to show that the converse is not always true, it is enough to give an example. Indeed, let

$$f(x) = \begin{cases} 1\ if\ x \in \mathbb{Q} \\ -1\ if\ x \in \mathbb{Q}^{\sim} \end{cases}.$$

Then $|f(x)| = 1\ \forall x \in \mathbb{R}$, and, hence, it is continuous $\forall x \in \mathbb{R}$, whereas $f(x)$ is not continuous $\forall x \in \mathbb{R}$.■

Theorem[435]*:* The composition of continuous functions is a continuous function.

Proof: Consider a function f continuous at $x_0 \in D_f$ and a function g defined on the range R_f of f and continuous at $f(x_0) \in R_f$. Then we shall prove that the function $h = g \circ f$ is continuous at x_0. Because f is continuous at x_0, we have:

$\forall \varepsilon^* > 0, \exists \delta > 0 \big| |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon^*\ \forall x \in D_f.$

Because g is continuous at $f(x_0) \in R_f$, we have:

$\forall \varepsilon > 0, \exists \delta_1 > 0 \big| |f(x) - f(x_0)| < \delta_1 \Rightarrow |h(x) - h(x_0)| = \big|g\big(f(x)\big) - g\big(f(x_0)\big)\big| < \varepsilon\ \forall f(x) \in R_f.$

Therefore, we must merely set $\varepsilon^* = \delta_1$ to prove the continuity of $g \circ f$.■

Theorem[436]*:* If a function f is continuous on a compact set S, then the image $f(S)$ of S under f is also compact, namely, the continuous image of a compact set is compact.

Proof: Let $f: S \to \mathbb{R}$ be a continuous function on the compact set S. Then

$f(S) = \{f(x) | x \in S\}$

is the image set, or range, of f. By the definition of continuity on a set, for every neighborhood $N(f(x))$ of $f(x)$, there exists a neighborhood $N(x)$ of $x \in S$ such that

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[435] Ibid.
[436] Ibid.

$f(N(x) \cap S) \subset N(f(x))$.

In order to prove that $f(S)$ is compact, we must prove that every open cover of $f(S)$ provides a finite subcover. Let $\mathcal{C} = \{U_\alpha | \alpha \in \mathcal{A}\}$ be an open cover of $f(S)$, symbolically:

$f(S) \subset \cup_{\alpha \in \mathcal{A}} U_\alpha$,

where each $U_\alpha \in \mathcal{C}$ is an open set (see section 2.3.6). Hence, for any $x \in S$, $f(x)$ belongs to some open set $U_x \in \mathcal{C}$, so that $f(x)$ is an interior point of U_x, and, therefore, there exists some neighborhood $N(f(x))$ of $f(x)$ such that $N(f(x)) \subset U_x$. Thus, we have: $f(N(x) \cap S) \subset N(f(x)) \subset U_x$. Moreover, by hypothesis, S is compact, meaning that every open cover $\mathcal{B} = \{N(x) | x \in S\}$ of S will have a finite subcover, say $\mathcal{B}_1 = \{N(x_\alpha) | a = 1,2, \dots, m\}$, so that

$$S \subset \cup_{\alpha=1}^{m} N(x_\alpha) \Rightarrow S = \cup_{\alpha=1}^{m} N(x_\alpha) \cap S$$
$$= (N(x_1) \cap S) \cup (N(x_2) \cap S) \cup \dots \cup (N(x_m) \cap S).$$

Hence, $S = \cup_{\alpha=1}^{m} N(x_\alpha) \cap S \Rightarrow f(S) = \cup f(N(x_\alpha) \cap S)$
$\Rightarrow f(S) \subset \cup N(f(x_\alpha)) \subset \cup U_{x_\alpha} \Rightarrow f(S) \subset \cup U_{x_\alpha}$, where $a = 1,2, \dots, m$. Therefore,

$\{U_{x_1}, U_{x_2}, \dots, U_{x_m}\}$ is an open cover of the set $f(S)$.

Consequently, the cover $\mathcal{C}$ of $f(S)$ has a finite subcover, namely, $\{U_{x_\alpha} | a = 1,2, \dots, m\}$, and, therefore, $f(S)$ is compact.■

Corollary[437]*:* If a function f is continuous on a closed interval, then it is bounded on that interval.

Proof: Let f be continuous on $[a, b]$. Because $[a, b]$ is compact, the aforementioned theorem implies that $f([a, b])$ is also compact, and, therefore, $f([a, b])$ is closed and bounded.

However, the converse may not be true: for instance, if $f(x) = [x]\ \forall x \in [-1,1]$, where $[x]$ denotes the greatest integer not greater than x, then $f(x)$ is a bounded function, but it is discontinuous at $x = 0$.

Moreover, notice that a function that is continuous on an open interval may not be bounded on that interval: for instance, the function

$$f(x) = \begin{cases} \frac{1}{x}\ if\ x \neq 0 \\ 0\ if\ x = 0 \end{cases}$$

is continuous on the open interval $(0,1)$, but it is not bounded on this interval.■

Theorem[438]*:* If f is a continuous function on $[a, b]$, then f attains its supremum (least upper bound) and its infimum (greatest lower bound) in $[a, b]$. In other words, if $f: [a, b] \rightarrow$

[437] Ibid.
[438] Ibid.

$\mathbb{R}$ is continuous over $[a, b]$, then there exist $p, q \in [a, b]$ such that q is a maximum for f, and p is a minimum for f.

Proof: The fact that f is continuous on $[a, b]$ implies that f is bounded on $[a, b]$. Let $\sup(f) = M$ and $\inf(f) = m$. We must prove that there exist p and q in $[a, b]$ such that $f(p) = m$ and $f(q) = M$.

First, we shall prove that f attains its supremum in $[a, b]$. For the sake of contradiction, assume that $\sup(f)$ is not attained in $[a, b]$, namely, that $\nexists q \in [a, b] | f(q) = M$. Then, $\forall x \in [a, b]$,

$f(x) \neq M \Rightarrow M - f(x) > 0$, given that $\sup(f) = M$.

Because M can be considered to be a constant function, and because, by hypothesis, $f(x)$ is continuous on $[a, b]$, it holds that $M - f(x)$ is continuous on $[a, b]$, and $\frac{1}{M-f(x)}$ is also continuous on $[a, b]$. Consequently, $\frac{1}{M-f(x)}$ is bounded on $[a, b]$. Let u be an upper bound of $\frac{1}{M-f(x)}$, so that, $\forall x \in [a, b]$,

$$\frac{1}{M-f(x)} \leq u \Rightarrow M - f(x) \geq \frac{1}{u} \Rightarrow f(x) \leq M - \frac{1}{u},$$

and, therefore, $M - \frac{1}{u}$ is also an upper bound of $f(x)$, which contradicts the assumption that $\sup(f) = M$. This contradiction implies that f attains its supremum in $[a, b]$, namely, that $\exists q \in [a, b] | f(q) = M$.

Similarly, we can prove that f attains its infimum in $[a, b]$.■

Remark: As I have already explained, in set theory, the "maximum" is the largest element of a set, while the "supremum" is the least upper bound of a set. However, if the maximum exists, then there is no difference between the maximum and the supremum. For instance, given the set $A = \{1,2,3,4\}$ in the reals, the maximum is 4, and the supremum is 4 as well. But, given the set $B = \{x | x < 2\}$, the maximum of B is not 2, because 2 does not belong to B, and, in fact, in this case, the maximum is not well defined, whereas the supremum is clearly 2. For this reason, in real analysis, it is often more convenient and more useful to consider the supremum of a given set rather than the maximum. By analogy, we can think about the minimum and the infimum (greatest lower bound).

Theorem[439]*:* Assume that a function $f(x)$ is continuous at a point $x = c$, and that $f(c) \neq 0$. Then there exists a neighborhood of c, say $(c - \delta, c + \delta)$, where $\delta > 0$, such that $f(x)$ and $f(c)$ have the same sign $\forall x \in (c - \delta, c + \delta)$.

Proof: Because f is continuous at $x = c$, it holds that there exist $\delta, \varepsilon > 0$ for which

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$$

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[439] Ibid.

$\Leftrightarrow f(c) - \varepsilon < f(x) < f(c) + \varepsilon \; \forall x \in (c - \delta, c + \delta)$. $(*)$

Let $\varepsilon > 0$ be such that $\varepsilon < |f(c)|$. $(**)$

First, suppose that $f(c) > 0$. Then $(*)$ and $(**)$ imply that $f(c) - \varepsilon$ and $f(c) + \varepsilon$ are positive, and that $f(x)$ lies between them $\forall x \in (c - \delta, c + \delta)$, so that $f(x) > 0 \; \forall x \in (c - \delta, c + \delta)$. Hence, $f(c) > 0 \Rightarrow f(x) > 0 \; \forall x \in (c - \delta, c + \delta)$.

Second, suppose that $f(c) < 0$. Then $(*)$ and $(**)$ imply that $f(c) - \varepsilon$ and $f(c) + \varepsilon$ are negative, and that $f(x)$ lies between them $\forall x \in (c - \delta, c + \delta)$, so that $f(x) < 0 \; \forall x \in (c - \delta, c + \delta)$. Hence, $f(c) < 0 \Rightarrow f(x) < 0 \; \forall x \in (c - \delta, c + \delta)$.

As a conclusion, $f(x)$ and $f(c)$ have the same sign $\forall x \in (c - \delta, c + \delta)$.■

Bolzano's Theorem[440]*:* Assume that a function $f(x)$ is continuous on $[a, b]$, and that $f(a)$ and $f(b)$ are of opposite signs, that is, $f(a) \cdot f(b) < 0$. Then there exists a point $c \in (a, b)$ such that $f(c) = 0$.

Proof: (i) We can prove this theorem using geometry as follows: Every continuous line of simple curvature (i.e., one that does not cross itself) of which the ordinates are first negative and then positive (or conversely) must necessarily intersect the x-axis. (ii) We can prove this theorem using pure mathematical analysis as follows: Given that $f(a)$ and $f(b)$ are of opposite signs, suppose that $f(a) < 0$ and $f(b) > 0$. Let us bisect the interval $[a, b]$ at the point c, so that $[a, b]$ has two subintervals $[a, c]$ and $[c, b]$. Then there are three possibilities: $f(c) = 0$, or $f(c) < 0$, or $f(c) > 0$.

If $f(c) = 0$, then the theorem is obviously verified.

If $f(c) > 0$, then let us rename the first subinterval $[a, c]$ of $[a, b]$ as the interval $[a_1, b_1]$, so that $f(a_1) < 0$ and $f(b_1) > 0$.

If $f(c) < 0$, then let us rename the second subinterval $[c, b]$ of $[a, b]$ as the interval $[a_1, b_1]$, so that $f(a_1) < 0$ and $f(b_1) > 0$.

The length of $[a_1, b_1]$ is equal to $\frac{b-a}{2}$.

Repeating this process of bisection and selection, we obtain the nested closed intervals $[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n], \ldots$ such that:

in each $[a_n, b_n]$, $f(a_n) < 0$ and $f(b_n) > 0$;

the length of $[a_n, b_n]$ is equal to $\frac{b-a}{2^n}$, and it tends to 0 as $n \to \infty$, that is,

$lim_{n\to\infty}(b_n - a_n) = 0$; and, by Cantor's Intersection Theorem,
$\cap_{n\in\mathbb{N}} [a_n, b_n] = \{k\}$ (see section 2.3.3).

Given that f is continuous on $[a, b]$, f is continuous at k, so that

$lim_{n\to\infty} f(a_n) = f(k) \& lim_{n\to\infty} f(b_n) = f(k)$. $(*)$

[440] Ibid. This theorem was proved by Bernard Bolzano in 1817, and Augustin-Louis Cauchy published another proof in 1821.

Moreover,

$$f(a_n) < 0 \Rightarrow lim_{n\to\infty} f(a_n) \leq 0 \;\&\; f(b_n) > 0 \Rightarrow lim_{n\to\infty} f(b_n) \geq 0. \qquad (**)$$

Because of $(*)$ and $(**)$,

$$lim_{n\to\infty} f(a_n) = lim_{n\to\infty} f(b_n) = 0,$$

and, therefore, $f(k) = 0$; $k \neq a$ since $f(a) < 0$, and $k \neq b$ since $f(b) > 0$. Hence, $k \in (a, b)$.■

Remark: If $f: [a, b] \to [a, b]$ is continuous on $[a, b]$, then there exists a point $x_0 \in [a, b]$ such that $f(x_0) = x_0$; such a point is said to be a "fixed point" of f.

Intermediate Value Theorem[441]*:* If f is continuous on $[a, b]$, and $f(a) \neq f(b)$, then f assumes every value between $f(a)$ and $f(b)$. This Intermediate Value Theorem can be geometrically interpreted as follows: if a continuous function f assumes the values $f(a)$ and $f(b)$, then it assumes every value in between, so that every horizontal line between $f(a)$ and $f(b)$ intersects the graph of f in at least one point. It is a generalization of the aforementioned Bolzano's Theorem.

Proof: Assume that k is a number between $f(a)$ and $f(b)$. Let us define a function

$$g(x) = f(x) - k$$

on $[a, b]$, so that $g(a) = f(a) - k$ and $g(b) = f(b) - k$. Because $f(a) \neq f(b)$, it follows that $g(a)$ and $g(b)$ are of opposite signs. Moreover, f is given to be continuous on $[a, b]$. Therefore, according to the aforementioned Bolzano's Theorem, there exists a number $c \in (a, b)$ such that $g(c) = 0 \Rightarrow g(c) = k$. Because k is an arbitrary value between $f(a)$ and $f(b)$, it holds that f takes every value between $f(a)$ and $f(b)$.■

Corollary: If f is continuous on $[a, b]$, if $\inf(f) = m$, and if $\sup(f) = M$, then f assumes every value between m and M.

Theorem[442]*:* Assume that a function f is continuous on $[a, b]$. Then $[a, b]$ can always be divided into a finite number of subintervals such that, $\forall \varepsilon > 0$,

$$|f(x_1) - f(x_2)| < \varepsilon,$$

where x_1 and x_2 belong to the same subinterval.

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[441] Ibid.
[442] Ibid.

Proof: For the sake of contradiction, suppose that $[a, b]$ cannot be divided into a finite number of subintervals such that, $\forall \varepsilon > 0$,

$$|f(x_1) - f(x_2)| < \varepsilon,$$

where x_1 and x_2 belong to the same subinterval.

Let us bisect $[a, b]$ at point c, obtaining the subintervals $[a, c]$ and $[c, b]$. We rename that part in which the result is false as $[a_1, b_1]$. Then we bisect $[a_1, b_1]$, and we rename that part in which the result is false as $[a_2, b_2]$. Repeating the same process, we obtain the nested closed intervals $[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n], \ldots$ such that the length of $[a_n, b_n]$ is equal to $\frac{b-a}{2^n}$, and it tends to 0 as $n \to \infty$. Hence, by Cantor's Intersection Theorem, $\cap_{n \in \mathbb{N}} [a_n, b_n] = \{k\}$ (see section 2.3.3).

Given that f is continuous on $[a, b]$, it holds that f is continuous at $x = k$, so that

$$|x - k| < \delta \Rightarrow |f(x) - f(k)| < \frac{\varepsilon}{2}.$$

Let $(b_n - a_n) < \delta$, so that $(k - \delta, k + \delta) \supset [a_n, b_n]$. If x_1 and x_2 are two arbitrary points in $[a_n, b_n]$, then

$$|f(x_1) - f(k)| < \frac{\varepsilon}{2} \& |f(x_2) - f(k)| < \frac{\varepsilon}{2},$$

so that $|f(x_1) - f(x_2)| = |f(x_1) - f(k) + f(k) - f(x_2)|$
$$\leq |f(x_1) - f(k)| + |f(k) - f(x_2)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, $|f(x_1) - f(x_2)| < \varepsilon$, where $x_1, x_2 \in [a_n, b_n]$, which implies that the theorem is true in the subinterval $[a_n, b_n]$, thus contradicting our assumption that the theorem is not true. This contradiction proves that the theorem is true.■

Notice that a function $f: I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, is invertible if and only if it is one-to-one, namely, $\forall x, y \in I, x \neq y \Rightarrow f(x) \neq f(y)$. Moreover, notice that a strictly monotonic function is bijective (i.e., one-to-one and onto) and, therefore, invertible. In the following theorem, we shall show that a continuous strictly monotonic function $f: I = [a, b] \to \mathbb{R}$ is invertible, and that f^{-1} is continuous and monotonic on $f(I)$, and, in particular, f^{-1} has the same kind of monotonicity as f.

Continuous Inverse Function Theorem[443]*:* If $f: I = [a, b] \to \mathbb{R}$ is a continuous strictly monotonic function defined on an interval I, then its inverse is also continuous and strictly monotonic on $f(I)$; f^{-1} has the same kind of monotonicity as f.

Proof: First, we shall consider the case in which f is strictly increasing on I; we work similarly in case f is strictly decreasing on I. Then $f(I) = [f(a), f(b)]$, and, since f is strictly increasing, f is bijective and, therefore, invertible. We can show that f^{-1} is strictly increasing as follows: Let $y_1, y_2 \in f(I)$ with $y_1 < y_2$. Then, for some $x_1, x_2 \in I$, $y_1 =$

[443] Ibid.

$f(x_1)$and $y_2 = f(x_2)$. It must hold that $x_1 < x_2$, because, otherwise, namely, if $x_1 \geq x_2$, it would hold that $y_1 = f(x_1) \geq y_2 = f(x_2)$, contradicting the assumption that $y_1 < y_2$. Therefore,

$$f^{-1}(y_1) = x_1 < f^{-1}(y_2) = x_2,$$

and, since y_1 and y_2 are arbitrary elements of $f(I)$ with $y_1 < y_2$, it follows that f^{-1} is strictly increasing on $f(I)$.

Now, using the definition of continuity, we shall show that f^{-1} is continuous on $J = f(I) = [f(a), f(b)]$ as follows: If $y_0 \in [f(a), f(b)]$, then there exists a unique $x_0 \in [a, b]$ such that $f(x_0) = y_0$. For $\varepsilon > 0$ and without loss of generality, let $\varepsilon \leq min\{x_0 - a, b - x_0\}$. Then the points $x_0 - \varepsilon$ and $x_0 + \varepsilon$ belong to $[a, b]$, and they are associated with the points $y_0 - \delta_1$ and $y_0 + \delta_2$ of $[f(a), f(b)]$, where $\delta_1 > 0$, $\delta_2 > 0$, and $f(x_0 - \varepsilon) = y_0 - \delta_1$, and $f(x_0 + \varepsilon) = y_0 + \delta_2$. If $\delta = min\{\delta_1, \delta_2\} > 0$, then, $\forall y \in (y_0 - \delta, y_0 + \delta)$, it holds that $y_0 - \delta_1 < y < y_0 + \delta_2$. In this case, since f^{-1} is strictly increasing on $f(I)$, we have that

$$x_0 - \varepsilon = f^{-1}(y_0 - \delta_1) < f^{-1}(y) < f^{-1}(y_0 + \delta_2) = x_0 + \varepsilon,$$

so that

$$|f^{-1}(y) - f^{-1}(y_0)| = |f^{-1}(y) - x_0| < \varepsilon,$$

namely, f^{-1} is continuous at y_0.

In order to show that f^{-1} is continuous at the endpoint $f(a)$ of J, we choose $\varepsilon \leq b - a$, and we set $f(a) + \delta_2 = f(a + \varepsilon)$. Then, $\forall y \in [f(a), f(a) + \delta_2)$, it holds that

$$f(a) \leq y < f(a) + \delta_2 \leq f(b).$$

Hence, $a = f^{-1}\big(f(a)\big) \leq f^{-1}(y) < a + \varepsilon$,

so that $\left|f^{-1}(y) - f^{-1}\big(f(a)\big)\right| < \varepsilon$,

namely, f^{-1} is continuous from the right of $f(a)$. Similarly, we can show that f^{-1} is continuous from the left of $f(b)$.■

Piecewise Continuity: A function is called “piecewise continuous” (or “sectionally continuous”) over $[a, b]$ if $[a, b]$ can be subdivided into finitely many intervals over each of which the function is continuous and has finite right-hand and left-hand limits. A piecewise continuous function has only finitely many points of discontinuity. Thus, a function $f: [a, b] \to \mathbb{R}$ is called piecewise continuous if there exist $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ so that

i. f is continuous on (x_k, x_{k+1}) for all $k = 0,1,2, \dots, n - 1$, and
ii. the limits $lim_{x \to x_{k+1}^-} f(x)$ and $lim_{x \to x_k^+} f(x)$ exist and are finite for all $k = 0,1,2, \dots, n - 1$.

Uniform Continuity: A function f is called "uniformly continuous" over (or on) its domain $D_f \subseteq \mathbb{R}$ if the following condition is satisfied:

$$\forall\varepsilon > 0, \exists\delta > 0 \,||x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon, for\ any\ x_1, x_2 \in D_f.$$

Example: For instance, let $f(x) = x^2\ \forall x \in [0,1]$. Then $f(x)$ is uniformly continuous over $[0,1]$. In order to prove the uniform continuity of this function, we must prove the following:

$$\forall\varepsilon > 0, \exists\delta > 0 |\forall x_1, x_2 \in [0,1], |x_1 - x_2| < \delta \Rightarrow |x_1^2 - x_2^2| < \varepsilon.$$

We have: $|x_1^2 - x_2^2| = |x_1 - x_2||x_1 + x_2| \leq 2|x_1 - x_2| < 2\delta$,

which is less than ε if we choose $\delta = \frac{\varepsilon}{2}$.

Remark: In case of continuity, δ depends on both ε and $x = a$ (as mentioned earlier, a function f is continuous at $x = a \in D_f$ if and only if, $\forall\varepsilon > 0, \exists\delta > 0 ||x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$), but, in case of uniform continuity, δ depends only on ε, and it remains uniform for every point belonging to D_f. Continuity pertains to the behavior of a function at a point, namely, it characterizes the local behavior of a function. On the other hand, uniform continuity is established on a set, and, therefore, it is a global property of a function. Hence, in order to be more easily and more emphatically distinguished from uniform continuity, continuity is often referred to as "pointwise continuity" (and a continuous function is called "pointwise continuous" to be more easily and more emphatically distinguished from a uniformly continuous function).

Theorem[444]*:* If a function $f: X \to \mathbb{R}$ is uniformly continuous on X, then it is also pointwise continuous on X; but the converse is not necessarily true.

Proof: Let $f: X \to \mathbb{R}$ be uniformly continuous on X, so that,

$$\forall\varepsilon > 0, \exists\delta > 0 ||f(x_1) - f(x_2)| < \varepsilon\ whenever\ |x_1 - x_2| < \delta, \forall x_1, x_2 \in X.$$

Let $x_2 = a$, so that

$$|f(x_1) - f(a)| < \varepsilon\ whenever\ |x_1 - a| < \delta.$$

Hence, $f: X \to \mathbb{R}$ is pointwise continuous at $x = a$. Because a is an arbitrary point, f is pointwise continuous at every point of X.

In order to prove that pointwise continuity does not necessarily imply uniform continuity, it suffices to give an example. For instance, if $f: (0,1) \to \mathbb{R}$ is defined by $f(x) = \frac{1}{x}$, then

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[444] Ibid.

$f(x)$is continuous on $(0,1)$, but is not uniformly continuous on $(0,1)$. In order to show that $f(x)=\frac{1}{x}$ is not uniformly continuous on $(0,1)$, we have to show that

$$\exists\varepsilon>0|\forall\delta>0,\exists x_1,x_2\in(0,1)||x_1-x_2|<\delta\Rightarrow|f(x_1)-f(x_2)|\geq\varepsilon.$$

For $\varepsilon=1$, and, for any $\delta\in(0,1)$, if we choose $x_1=\delta$ and $x_2=\frac{\delta}{2}$, we obtain

$$|x_1-x_2|=\frac{\delta}{2}<\delta,|f(x_1)-f(x_2)|=\left|\frac{1}{x_1}-\frac{1}{x_2}\right|=\frac{1}{\delta}>1.$$

Hence, f is not uniformly continuous.■

Theorem[445]*:* If a function $f:[a,b]\to\mathbb{R}$ is continuous on the closed interval $[a,b]$, then it is uniformly continuous on $[a,b]$.

Proof: Because f is continuous on $[a,b]$, then $[a,b]$ can be divided into finitely many subintervals such that

$$|f(x_1)-f(x_2)|<\frac{\varepsilon}{2},\varepsilon>0,$$

where x_1 and x_2 belong to the same subinterval. Let $\delta_1,\delta_2,\dots,\delta_m$ be the lengths of the subintervals into which $[a,b]$ has been divided, and let $\delta=min\{\delta_1,\delta_2,\dots,\delta_m\}$. Then $\delta>0$. We choose two elements x_1 and x_2 of $[a,b]$ such that $|x_1-x_2|<\delta$. Then there are two possibilities: either x_1 and x_2 belong to the same interval, or they belong to two consecutive subintervals, namely, two subintervals with a common endpoint. If x_1 and x_2 belong to the same interval, then

$$|f(x_1)-f(x_2)|<\frac{\varepsilon}{2}<\varepsilon,$$

and, therefore, f is uniformly continuous on $[a,b]$. If x_1 and x_2 belong to two consecutive subintervals, then let the common endpoint of the two subintervals to which x_1 and x_2 belong be k, so that

$$|f(x_1)-f(k)|<\frac{\varepsilon}{2}\&|f(x_2)-f(k)|<\frac{\varepsilon}{2}.$$

Then $|f(x_1)-f(k)|\leq|f(x_1)-f(k)|+|f(k)-f(x_2)|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}<\varepsilon,$

so that f is uniformly continuous on $[a,b]$.■

Theorem[446]*:* Let $X\subseteq\mathbb{R}$ and $f:X\to\mathbb{R}$. If there is a constant $k>0$ such that $|f(x_1)-f(x_2)|\leq k|x_1-x_2|\ \forall x_1,x_2\in X$, then we say that f satisfies a “Lipschitz condition.” If $f:X\to\mathbb{R}$ satisfies a Lipschitz condition, then it is uniformly continuous on X.

[445] Ibid.

Proof: We must prove that

$\forall \varepsilon > 0, \exists \delta > 0 | \forall x_1, x_2 \in X, |x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon.$

Because $f: X \to \mathbb{R}$ satisfies a Lipschitz condition, if we choose $\delta = \frac{\varepsilon}{k}$, then, when $|x_1 - x_2| < \delta$, we obtain $|f(x_1) - f(x_2)| < k \cdot \frac{\varepsilon}{k} = \varepsilon$.■

Remark: The converse need not hold: For instance, the function $f(x) = \sqrt{x}$ with $x \in [0,2]$ is uniformly continuous (since it is continuous on a closed interval), but it does not satisfy a Lipschitz condition, since there exists no $k > 0$ such that $|\sqrt{x}| \leq k|x|$ $\forall x \in [0,2]$.

2.8. COMPLEX NUMBERS[447]

If we adjoin to the real field $\mathbb{R}$ a root i of the polynomial $x^2 + 1 = 0$, which is irreducible to $\mathbb{R}$, we obtain the "field of complex numbers" $\mathbb{C} \equiv \mathbb{R}(i)$. In other words, a (2-dimensional) number of the form $a + bi$, where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$, is called a "complex number"; the number a is called the "real part" of $a + bi$, b is called the "imaginary part" of $a + bi$, and $i = \sqrt{-1}$ is called the "imaginary unit."

In his book *Algebra*, published in 1673, the English mathematician and clergyman John Wallis (Savilian Professor of Geometry at Oxford) argued that "imaginary quantities" arise from the supposed square root of a negative number. Moreover, because the product of two square roots each of which has a negative radicand gives a negative number, Wallis became increasingly conscious of the need for a scientific inquiry into the system of the negative numbers. In particular, Wallis observed that there is a relation between the negative sign of a number and the notion of direction, in the sense of advancing (moving forward) and retreating (moving backward). In his *Algebra*, Wallis mentions that the square root signifies "a Mean Proportional between a Positive and a Negative Quantity," and he exemplifies the algebraic significance of "imaginary quantities" as follows:

> For like as $\sqrt{bc}$ signifies a Mean Proportional between $+b$ and $+c$; or between $-b$ and $-c$ (either of which, by Multiplication, makes $+bc$): So doth $\sqrt{-bc}$ signify a Mean Proportional between $+b$ and $-c$, or between $-b$ and $+c$; either of which being Multiplied, makes $-bc$.[448]

Furthermore, in his *Algebra*, Wallis exemplifies the geometric significance of "imaginary quantities" as shown in Figure 2.16, arguing as follows: if, for instance, we move forward from point A, then we take $AB = +b$, and forward from thence, $BC = +c$, making $AC = +AB + BC = +b + c$, the diameter of a circle, so that $BP = \sqrt{+bc}$ is "the Sine, or Mean Proportional"; whereas, if we move backward from point A, then we take $AB = -b$, and then

[446] Ibid.
[447] See: Andreescu and Andrica, *Complex Numbers from A to . . . Z.*
[448] Wallis, "On Imaginary Numbers," p. 48.

forward from that B, $BC = +c$, making $AC = -AB + BC = -b + c$, the diameter of a circle, so that $BP = \sqrt{-bc}$ is "the Tangent, or Mean Proportional."[449]

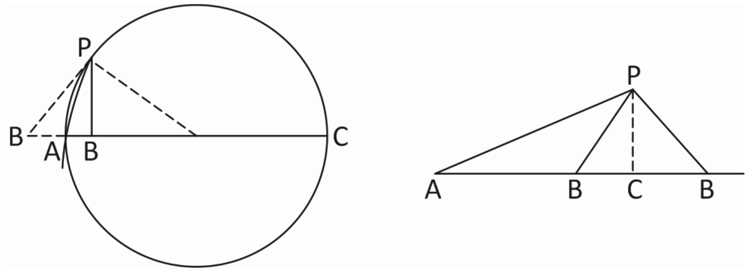

Figure 2.16. The Geometric Significance of Imaginary Numbers.

In 1833, at the Royal Irish Academy, the Irish mathematician and astronomer Sir William Rowan Hamilton[450] presented the complex numbers as ordered pairs of real numbers, thus denoting a complex number by an ordered pair (a, b), and denoting the imaginary unit by $i = \sqrt{-1}$, so that $i^2 = (0,1) \cdot (0,1) = (-1,0) = -1$. In fact, as Hamilton has originally shown, the complex number system $\mathbb{C}$ is the set $\mathbb{R} \times \mathbb{R}$ with operations of addition and multiplication defined as follows:

i. Addition of complex numbers: $(a, b) + (c, d) = (a + c, b + d)$,
ii. Multiplication of complex numbers: $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$.

Remarks: The zero of $\mathbb{C}$ is $(0,0)$, and the unit of $\mathbb{C}$ is $(1,0)$. If $(a, b) \in \mathbb{C}$, then, if we set $i = (0,1)$, and if we identify the complex numbers whose second element is zero with the corresponding real numbers, then we obtain $(a, b) = (a, 0) + (0, b) = (a, 0) + (b, 0) \cdot (0,1) = a + bi$, thus obtaining the familiar notation for complex numbers. The proof of the fact that $\mathbb{C}$ is a field follows directly from the definition of a field (see section 2.2.4).

In simpler notation, for any complex numbers:

$(a + bi) + (c + di) = (a + c) + (b + d)i$,
$(a + bi) - (c + di) = (a - c) + (b - d)i$,
$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$, and
$\frac{(a+bi)}{(c+di)} = \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$.

The (complex) "conjugate" of $a + bi$ is $a - bi$, and the conjugate of a complex number z is denoted by $\bar{z}$. The conjugate has the following properties[451]:

i. $z \cdot \bar{z} \in \mathbb{R}$.
ii. $z = \bar{z} \Leftrightarrow z \in \mathbb{R}$.
iii. $\overline{z + w} = \bar{z} + \bar{w}$.

[449] Ibid, p. 49.
[450] Hamilton, "On Quaternions."
[451] See: Andreescu and Andrica, *Complex Numbers from A to . . . Z*.

iv. $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$.
v. $\overline{z^n} = (\bar{z})^n \ \forall n \in \mathbb{N}$.

Let A be the subfield of $\mathbb{C}$ that consists of those complex numbers whose second element is zero; symbolically

$$A = \{(x, 0) | x \in \mathbb{R}\}.$$

Then the function f from $\mathbb{R}$ onto A defined by

$$f(x) = (x, 0)$$

is an isomorphism of $\mathbb{R}$ onto A. Hence, the real field can be considered as a subfield of the complex field (see section 2.2.4).

Given that $i^2 = (0,1)^2 = (-1,0) = -1$, it follows that $\mathbb{C}$ cannot be an ordered field, because, in an ordered field, the square of any non-zero element is positive. Even though $\mathbb{C}$ is not an ordered field, we can define an absolute value in the complex field that has the same basic properties as the absolute value in the real field. The (complex) "modulus," that is, the "absolute value," of the complex number $z = (x, y)$ is defined as follows:

$$|z| = \sqrt{x^2 + y^2}.$$

The absolute value $|z|$ has the following properties[452]:

i. $|z| \geq 0$; $|z| = 0 \Leftrightarrow z = 0$.
ii. $|z_1 z_2| = |z_1||z_2|$.
iii. $|z_1 + z_2| \leq |z_1| + |z_2|$.

If we consider the complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ to be the points in the Euclidean plane whose coordinates are (x_1, y_1) and (x_2, y_2), respectively, then $|z_1 - z_2|$ is the distance between the points z_1 and z_2.

The trigonometric form of a complex number[453]*:* A complex number $z = a + bi$ is represented on the complex plane by a point M with coordinates a and b, as shown in Figure 2.17. The straight line segment OM, joining the point M to the origin $O(0,0)$, is the "radius vector" of the point M, and its length, namely, the "modulus," or "absolute value," of the complex number $z = a + bi$ is $|z| = \sqrt{a^2 + b^2}$. The angle φ that the radius vector of the point $M(a, b)$ makes with the positive horizontal axis (Figure 2.17) is the "argument," or "amplitude," of the complex number $z = a + bi$. If φ is the argument of the complex number $z = a + bi$, then $\varphi + 2\pi k$ is also the argument of $z = a + bi$ for any integral value of k, but, in order for the angle φ to be defined uniquely, we set $-\pi < \varphi \leq \pi$. As shown in Figure 2.17, the radius vector of the point $M(a, b)$, which represents a complex number $z = a + bi$,

[452] Ibid.
[453] Ibid.

intersects the unit circle at the point P. For instance, if $z = -5i$, then this complex number is represented by the point $M(0, -5)$, as shown in Figure 2.17, and the radius vector of $M(0, -5)$ makes an angle $-\frac{\pi}{2}$ with the positive direction of the x-axis, which implies that the argument of $z = -5i$ is $-\frac{\pi}{2}$.

As I explained in section 2.2.6, the point P illustrated in Figure 2.17 has coordinates $cos\varphi$ and $sin\varphi$. Moreover, because $OP = 1$, and $OM = |z|$, the coordinates of $M(a, b)$ are $a = |z|cos\varphi$ and $b = |z|sin\varphi$. Therefore, we obtain the trigonometric form of a complex number:

$$z = a + bi = |z|cos\varphi + |z|sin\varphi \cdot i = |z|(cos\varphi + isin\varphi).$$

De Moivre's Formula[454]: $(cos\varphi + isin\varphi)^n = cos(n\varphi) + isin(n\varphi)$,

which can be easily proved by mathematical induction.

Corollary: If $z = |z|(cos\varphi + isin\varphi)$, then

$$z^n = |z|^n\big(cos(n\varphi) + isin(n\varphi)\big).$$

Euler's Formula[455]:$e^{i\varphi} = cos\varphi + isin\varphi$,

where e is the base of the natural logarithm, $i = \sqrt{-1}$, as shown in Figure 2.17, so that, given a real number φ, we can plot the complex number $e^{i\varphi}$ on the unit circle, since $e^{i\varphi} = cos\varphi + isin\varphi$. When $\varphi = \pi$, then Euler's formula reduces to

$$e^{i\pi} + 1 = 0,$$

which is known as Euler's identity.

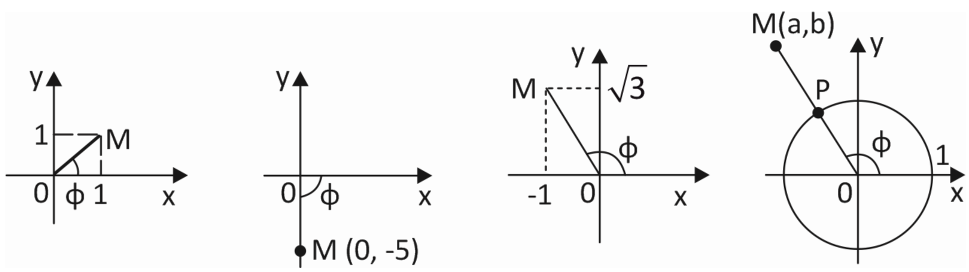

Figure 2.17. Euler's Formula.

Let f be an one-to-one mapping of the extended real line into ℝ defined as follows:

$$f(-\infty) = -1,$$

[454] Ibid.
[455] Ibid.

$f(x) = \frac{x}{1+|x|}, x \in \mathbb{R}$,
$f(+\infty) = 1$.

Then the function

$$d(x, y) = |f(x) - f(y)|\ \forall x, y \in \mathbb{R} \cup \{-\infty, +\infty\}$$

is a metric on the extended real line, and the metric space of the extended real line is denoted by $\overline{\mathbb{R}}$. Notice that $\overline{\mathbb{R}}$ is isometric to the metric space that consists of the closed interval $[-1,1]$ with the Euclidean metric d_E (for the concept of an "isometry," see section 2.2.6). This metric space, which can be simply denoted by $[-1,1]$, is called a subspace of $\mathbb{R}$.

By analogy, the complex plane $\mathbb{C}$ can be extended by adjoining a point called infinity and denoted by ∞. Let $\overline{\mathbb{C}} \equiv \mathbb{C} \cup \{\infty\}$. A familiar model of the $\overline{\mathbb{C}}$ is known as the "Riemann sphere," named after the great nineteenth-century German mathematician Bernhard Riemann. Arguably, the n-dimensional sphere is the simplest non-Euclidean geometry. However, notice that Euclidean geometry is a local geometry on the sphere (in regions where the curvature of the sphere tends to zero), and the geometry on the sphere (Riemannian geometry) is a generalization of Euclidean geometry.

By the term "stereographic projection," we mean a mapping from the sphere to the plane.[456] It is obtained by looking at the north pole of the sphere and drawing straight lines from the north pole down to the plane. Each such line hits the sphere once, and it also hits the plane once, and, therefore, the mapping from the sphere to the plane maps the point at which the line hits the sphere to the point at which the line hits the plane, as shown in Figure 2.18.

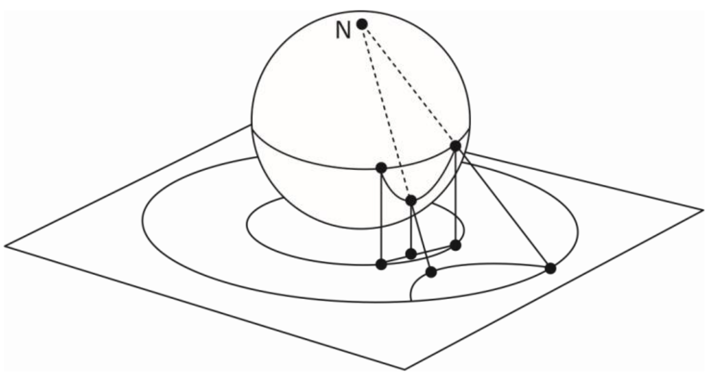

Figure 2.18. Stereographic Projection.

Let S^n denote the unit n-sphere in the Euclidean space $\mathbb{R}^{n+1}$. A "great circle" of S^n is the shortest distance between two points on the surface S^n. One way to represent S^n is the stereographic projection of S^n minus the north pole $(0,0,\ldots,0,1)$ onto $\mathbb{R}^n \subset \mathbb{R}^{n+1}$. Such a projection is bijective and preserves angles. Moreover, we can represent S^n by parametrization.

The Riemann sphere S^2 can be visualized as the unit 2-sphere defined by $S^2 = \{(x_1, x_2, x_3) | x_1^2 + x_2^2 + x_3^2 = 1\}$ in the 3-dimensional Euclidean space $\mathbb{R}^3$. Let us identify the

[456] See: Pedoe, *Geometry*.

plane $x_3 = 0$ with the complex plane $\mathbb{C}$ via $(x_1, x_2, 0) \leftrightarrow x_1 + x_2 i$. Let $N = (0,0,1)$ denote the north pole of the unit 2-sphere S^2. If P is any point belonging to S^2 with $N \neq P$, then we obtain a complex number $\zeta_N(P)$ by intersecting line NP with the complex plane $\mathbb{C} \subset \mathbb{R}^3$. The mapping $P \to \zeta_N(P)$ from the unit 2-sphere S^2 minus the north pole $(0,0,1)$ to the complex plane $\mathbb{C}$ (corresponding to the plane $x_3 = 0$) is the "stereographic projection" of the unit 2-sphere S^2 through the north pole. This mapping is invertible, and its inverse is obtained by considering a complex number z and the line through N and z, and then taking the intersection point of this line with S^2 that is not N. The inverse mapping is also called stereographic projection. Let $P = (x_1, x_2, x_3) \in S^2$ with $P \neq N$, so that the parametric form of the line NP is $(kx_1, kx_2, kx_3 + (1-k))$, and it intersects $\mathbb{C}$ when $kx_3 + (1-k) = 0$, that is, when $k = 1/(1-x_3)$, which is defined, because the fact that $P \neq N$ implies that $x_3 \neq 1$. Therefore,

$$\zeta_N(P) = \frac{x_1}{1-x_3} + \frac{x_2}{1-x_3} i.$$

For the inverse mapping, consider $z = x + iy \in \mathbb{C}$. If $\zeta_N^{-1}(z) = (x_1, x_2, x_3)$, then $x_1/(1-x_3) = x$ and $x_2/(1-x_3) = y$, so that $x_1 = x(1-x_3)$ and $x_2 = y(1-x_3)$. Given that $\zeta_N^{-1}(z)$ lies on the unit 2-sphere, it holds that $x_1^2 + x_2^2 + x_3^2 = 1$, and, therefore,

$$x^2(1-x_3)^2 + y^2(1-x_3)^2 + x_3^2 = 1$$
$$\Rightarrow (x^2 + y^2 + 1)x_3^2 - 2(x^2 + y^2)x_3 + (x^2 + y^2 - 1) = 0.$$

One solution is $x_3 = 1$, and it corresponds to the fact that the line through N and z intersects the unit 2-sphere at N, and, for the other point of intersection, it holds that

$$x_3 = \frac{x^2+y^2-1}{x^2+y^2+1} = \frac{|z|^2-1}{|z|^2+1}.$$

Thus,

$$x_1 = x(1-x_3) = \frac{2x}{|z|^2+1} = \frac{z+\bar{z}}{|z|^2+1},$$
$$x_2 = y(1-x_3) = \frac{2y}{|z|^2+1} = \frac{z-\bar{z}}{i(|z|^2+1)},$$

and, given these definitions of x_1, x_2, and x_3, we obtain the following formula:

$$\zeta_N^{-1}(z) = \left(\frac{z+\bar{z}}{|z|^2+1}, \frac{z-\bar{z}}{i(|z|^2+1)}, \frac{|z|^2-1}{|z|^2+1}\right).$$

Notice that the stereographic projection $\zeta_N(P)$ will cover the whole sphere except the north pole $(0,0,1)$, and the stereographic projection $\zeta_N^{-1}(z)$ will cover the whole sphere except the south pole $(0,0,-1)$. Hence, one needs two complex planes, one for each projection, which are "glued" back-to-back at $x_3 = 0$. If $P \to N$, namely, if P approaches N, then

$$|\zeta_N(P)|^2 = \frac{x_1^2+x_2^2}{(1-x_3)^2} = \frac{1-x_3^2}{(1-x_3)^2} = \frac{1+x_3}{1-x_3} \to \infty.$$

Consequently, we can extend ζ_N continuously to all of the unit 2-sphere S^2 by introducing a point at infinity, denoted by ∞, and setting $\zeta_N(N) = \infty$. In this way, we obtain the extended complex plane $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, and the stereographic projection $\zeta_N: S^2 \to \bar{\mathbb{C}}$ is defined as above. When we consider S^2 as being identified with $\bar{\mathbb{C}}$, we call S^2 the "Riemann sphere."

Consequently

$$f(z) = \frac{1}{1+|z|^2}(x, y, |z|^2), z \in \mathbb{C} \text{ and } f(\infty) = (0,0,1),$$

then f is an one-to-one mapping of the extended complex plane $\bar{\mathbb{C}}$ onto the sphere $S^2 = \{(x_1, x_2, x_3) | x_1^2 + x_2^2 + x_3^2 = x_3\}$ in the 3-dimensional Euclidean space $\mathbb{R}^3$. The metric defined by $d(z_1, z_2) = |f(z_1) - f(z_2)|$ is said to be the "chordal metric," determining the metric space $\bar{\mathbb{C}}$. If this is the case, then

$$d(z_1, z_2) = \frac{|z_1-z_2|}{(1+|z_1|^2)^{\frac{1}{2}}+(1+|z_2|^2)^{\frac{1}{2}}},\ z_1, z_2 \in \mathbb{C}, \text{ and}$$
$$d(z, \infty) = \frac{1}{(1+|z|^2)^{\frac{1}{2}}}, z \in \mathbb{C}.$$

A thorough study of the chordal metric was made by the renowned Greek-German mathematician Constantine Carathéodory (1873–1950).[457]

The usefulness of the complex numbers derives from the fact that every non-constant polynomial over $\mathbb{C}$ can be factored into a product of linear factors. This fact is known as the Fundamental Theorem of Algebra. Because the complex space $\mathbb{C}$ can be identified with $\mathbb{R}^2$ as a metric space, the Heine–Borel Theorem (proven in section 2.3.6) applies to $\mathbb{C}$ also. Thus, we can prove that the complex field is algebraically closed, meaning that any non-constant polynomial over $\mathbb{C}$ can be factored into a product of linear factors; symbolically: If

$$p(z) = \sum_{k=0}^{n} a_k z^k, \text{ where } a_k \in \mathbb{C}, n \geq 1, \text{ and } a_n \neq 0,$$

then, according to the Fundamental Theorem of Algebra,

$$p(z_0) = 0 \text{ for some } z_0 \in \mathbb{C},$$

and, therefore, $p(z) = (z - z_0)q(z)$, where q is a polynomial of degree $n - 1$. If $n - 1 \geq 1$, then $q(z)$ has a zero (complex root), and, hence, it has a linear factor. In this way, we can show that any non-constant polynomial over $\mathbb{C}$ can be written as a product of linear factors.

The Fundamental Theorem of Algebra (Euler–Gauss)[458]*:* If $p(z)$ is a non-constant polynomial over $\mathbb{C}$, then $p(z_0) = 0$ for some $z_0 \in \mathbb{C}$.

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[457] Carathéodory, *Funktionentheorie*.

[458] See: Barbeau, *Polynomials*; Ebbinghaus, Hermes, Hirzebruch, Koecher, Mainzer, Neukirch, Prestel, and Remmert, *Numbers*; Haaser and Sullivan, *Real Analysis*. In 1608, in his *Arithmetica Philosophica*, Peter Roth

Proof: Let $p(z) = \sum_{k=0}^{n} a_k z^k$, $n \geq 1$, and $a_n \neq 0$, and let

$m = inf\{|p(z)||z \in \mathbb{C}\}$.

Given that $\left|p\left(re^{ix}\right)\right| \geq r^n(|a_n| - r^{-1}|a_{n-1}| - \cdots - r^{-n}|a_0|)$,$\left|p\left(re^{ix}\right)\right|$ becomes infinitely large as r tends to infinity. Therefore, there exists a real number s such that $\left|p\left(re^{ix}\right)\right| \geq m + 1$ whenever $r > s$. If $A = \left\{re^{ix}|r \leq s\right\}$, then A is compact in the metric space $\mathbb{C}$, and $m = inf\{|p(z)||z \in A\}$. Because $|p|$ is a continuous real-valued function on A, then $|p|$ has a minimum value in A (see section 2.7). Hence, $\exists z_0 \in A||p(z_0)| = m$.

If $m = 0$, then the theorem is true.
If $m \neq 0$, then let $q(z) = \frac{p(z+z_0)}{p(z_0)}, z \in \mathbb{C}$,

so that q is a polynomial of degree n, and $|q(z)| \geq 1\ \forall z \in \mathbb{C}$. (∗)

Because $q(0) = 1$, the polynomial $q(z)$ can be written as follows:

$$q(z) = 1 + b_k z^k + \cdots + b_n z^n,$$

where k is the smallest positive integer $\leq n$ such that $b_k \neq 0$. Because the absolute value of $-\frac{|b_k|}{b_k}$ is equal to 1, there exists an $x_0 \in \left[0, \frac{2\pi}{k}\right)$ such that

$$e^{ikx_0} = -\frac{|b_k|}{b_k}.$$

Hence,

$$\begin{aligned} q\left(re^{ix_0}\right) &= 1 + b_k r^k e^{ikx_0} + b_{k+1} r^{k+1} e^{i(k+1)x_0} + \cdots + b_n r^n e^{inx_0} \\ &= 1 - r^k|b_k| + b_{k+1} r^{k+1} e^{i(k+1)x_0} + \cdots + b_n r^n e^{inx_0}. \end{aligned}$$

Then, if $r^k|b_k| < 1$, we obtain

$$\left|q(re^{ix_0})\right| \leq 1 - r^k(|b_k| - r|b_{k+1}| - \cdots - r^{n-k}|b_n|).$$

Consequently, for sufficiently small r, we obtain $\left|q(re^{ix_0})\right| < 1$, which contradicts (∗). This contradiction implies that the assumption that $m \neq 0$ cannot hold, and that $p(z_0) = 0$.■

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explicitly stated the Fundamental Theorem of Algebra in the following (equivalent) form: An nth-degree polynomial has n roots. Moreover, in 1629, in his *Invention Nouvelle en l'Algèbre*, Albert Girard, stated that every algebraic equation has as many roots (solutions) as the exponent of the highest term indicates. Significant attempts at proving the theorem were made by Leonhard Euler (1749), Daviet de Foncenex (1759), Joseph-Louis Lagrange (1772), and Pierre-Simon de Laplace (1795). The first rigorous proof of this theorem was given by the German mathematician and physicist Johann Carl Friedrich Gauss in 1799, and, in 1816, Gauss published a second proof and a third one.

Number of Zeros Theorem[459]*:* Every polynomial of degree $n \geq 1$ has exactly n complex zeros, where zeros of multiplicity k are counted k times.

Proof: Let $y = p(x)$ be a polynomial function of degree $n \geq 1$ with leading coefficient a_n. Then the Fundamental Theorem of Algebra guarantees that $y = p(x)$ has at least one complex zero, say c_1. By the Factor Theorem (section 2.6), since c_1 is a zero, $x - c_1$ is a factor of $p(x)$, and, therefore,

$$p(x) = (x - c_1)q_1(x),$$

where the polynomial$q_1(x)$ has degree $n - 1$ and leading coefficient a_n. If the degree of $q_1(x)$ is at least 1, then once again the Fundamental Theorem of Algebra guarantees that $y = q_1(x)$ has at least one zero, say c_2. Hence,

$$q_1(x) = (x - c_2)q_2(x),$$

where the polynomial$q_2(x)$ has degree $n - 2$ and leading coefficient a_n. Then

$$p(x) = (x - c_1)(x - c_2)q_2(x).$$

Repeating this process n times, until $q_n(x) = a_n$, we realize that $p(x)$ can be factored into n linear factors and written as follows:

$$p(x) = a_n(x - c_1)(x - c_2) \dots (x - c_n),$$

and, therefore, by the Factor Theorem, $y = p(x)$ has n zeros, namely, $c_1, c_2, \dots, c_n$. Moreover, no other number, say v, distinct from $c_1, c_2, \dots, c_n$ can be a zero of $p(x)$, because

$$p(v) = a_n(v - c_1)(v - c_2) \dots (v - c_n) \neq 0,$$

since none of the factors is zero. As a conclusion, every polynomial function of degree $n \geq 1$ has exactly n (not necessarily distinct) complex zeros.■

Conjugate-Pair Theorem[460]*:* If a complex number $a + bi$ is a zero of a polynomial function of degree $n \geq 1$ with real number coefficients, then the conjugate $a - bi$ is also a zero.

Proof: Let

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

be a polynomial of degree $n \geq 1$ with real number coefficients. If $z = a + bi$ is a zero of $y = p(x)$, then $p(z) = 0$, so that

[459] Ibid.
[460] Ibid.

$0 = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0.$

Then we take the conjugate of both sides of the equation:

$$0 = \overline{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0}$$
$$= a_n \overline{z^n} + a_{n-1} \overline{z^{n-1}} + \cdots + a_1 \bar{z} + a_0$$
$$= a_n (\bar{z})^n + a_{n-1} (\bar{z})^{n-1} + \cdots + a_1 \bar{z} + a_0.$$

The last equation implies that $p(\bar{z}) = 0$, and, therefore, $\bar{z}$ is also a zero of $y = p(x)$.■

2.9. The Birth and the Development of Infinitesimal Calculus

In general, the term "calculus" means a set of objects endowed with rules for their manipulation. "Infinitesimal calculus," in particular, is a branch of mathematical analysis that concerns itself with the systematic study of the concept of an "infinitely small function," namely, a function of a variable x whose absolute value, $|f(x)|$, becomes and remains smaller than any given number as a result of variation of x. The method of the "infinitesimals," that is, the "infinitely small" quantities, was originally used by ancient Greek mathematicians, who determined areas and volumes by the so-called "method of exhaustion," in which infinitesimal quantities are used in order to prove that two given magnitudes (or two ratios between given magnitudes) are equal.[461]

The method of exhaustion was originally developed in the fifth century B.C. by the Athenian scholar Antiphon, and it was put in a rigorous scientific setting shortly afterwards by the Greek mathematician and astronomer Eudoxus of Cnidus, who used it in order to calculate areas and volumes. The Greek mathematician Euclid, the acknowledged father of "Euclidean geometry," and the Greek mathematician, physicist, and engineer Archimedes made extensive use of the method of exhaustion in order to prove several mathematical propositions. For instance, Archimedes used the method of exhaustion in order to compute the area of a circle by approximating the area of a circle from above and below by circumscribing and inscribing regular polygons of an increasingly larger number of sides (so that sides become "infinitesimals," namely, infinitely small): each of the polygons is a union of triangles, and, therefore, it is easily verified that the area of a circle of radius r and circumference C is equal to the area of a triangle whose altitude is equal to r and whose base is equal to $C = 2\pi r$. Then, given that the area of a triangle is equal to half of the product of its base and altitude, we obtain the formula for the computation of the area of a circle: $\frac{1}{2}(rC) = \frac{1}{2}(r2\pi r) = \pi r^2$. Moreover, Archimedes was able to calculate the length of various tangents to the spiral (i.e., to a curve emanating from a point moving farther away as it revolves around the point).

In few words, infinitesimal calculus, or simply calculus, is concerned with two kinds of problem: (i) problems of tangents to curves, and (ii) problems of areas or volumes of regions.

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461 See: Kline, *Mathematical Thought*.

Thus, having studied both of these kinds of problem in a rigorous and systematic way, Archimedes can be considered to be the first pioneer of calculus. Some other great pioneers of calculus are the Flemish Jesuit and mathematician Gregory of Saint Vincent (1584–1667), the Dutch-French philosopher and mathematician René Descartes (1596–1650), the Italian mathematician and Jesuate Bonaventura Francesco Cavalieri (1598–1647), the French lawyer and amateur mathematician Pierre de Fermat (1607–65), the English clergyman and mathematician John Wallis (1616–1703), the English Christian theologian and mathematician Isaac Barrow (1630–77), and the Scottish mathematician and astronomer James Gregory (1638–75).

Infinitesimal calculus, or simply calculus, is primarily aimed at solving problems concerning "change." Thus, infinitesimal calculus is used in many fields, including physics, engineering, biology, economics, statistics, the mathematical modelling of social, political, military, and psychological problems, etc. In the seventeenth century, infinitesimal calculus was erected as a rigorous framework of science as a result of and in the context of the revolutionary achievements that took place in the scientific discipline of celestial mechanics, whose protagonists were Nicolaus Copernicus, Galileo Galilei, Tycho Brahe, Johannes Kepler, and Isaac Newton. In its contemporary rigorous form, calculus was formulated independently in England by Sir Isaac Newton and in Germany by Gottfried Wilhelm Leibniz in the last quarter of the seventeenth century, using the algebraic set-up and, especially, the Cartesian set-up, which had been introduced and developed by their predecessors. Calculus consists of "differential calculus" (which is concerned with problems of tangents to curves) and "integral calculus" (which is concerned with problems of areas or volumes of regions).

2.10. Differential Calculus

2.10.1. Derivative

Let a function $y = f(x)$ be defined at points x and x_1. The difference $x_1 - x$ is called the "increment of the argument," and it is denoted by Δx. The difference $f(x_1) - f(x)$ is called the "increment of the function," and it is denoted by Δf or Δy. Hence, $\Delta x = x_1 - x \Leftrightarrow x_1 = x + \Delta x$, and $\Delta f = f(x_1) - f(x) = f(x + \Delta x) - f(x)$.

For instance, the increment of the function $y = x^2$ when changing from the point x to the point $x + \Delta x$ is $\Delta f = f(x + \Delta x) - f(x) = (x + \Delta x)^2 - x^2 = 2x\Delta x + (\Delta x)^2$. Using this formula, we can compute the value of Δf for any given x and Δx. For instance, for $x = 3$ and $\Delta x = 0.1$, $\Delta f = 0.61$.

Notice that, for any linear function $y = kx + b$, it holds that $\frac{\Delta y}{\Delta x} = k$. The geometric significance of this equation was explained in section 2.2.6, and it is also illustrated in Figure 2.19.

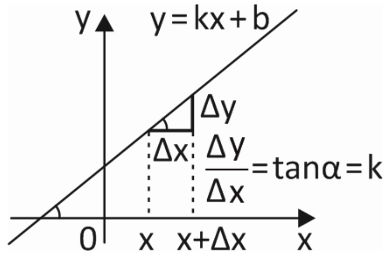

Figure 2.19. The Slope of a Straight Line and the Increment of a Linear Function.

Theorem[462]*:* A function $y = f(x)$ is continuous at the point $x = a$ if and only if $lim_{\Delta x \to 0} \Delta f = 0$, where $\Delta x = x - a$ and $\Delta f = f(x) - f(a)$.

Proof: This theorem follows directly from the definition of continuity (see section 2.7). Indeed, the function $y = f(x)$ is continuous at the point $x = a$ if and only if $lim_{x \to a} f(x)$ exists finitely and $lim_{x \to a} f(x) = f(a)$, or, equivalently, if

$$lim_{x-a \to 0}(f(x) - f(a)) = 0 \Leftrightarrow lim_{\Delta x \to 0} \Delta f = 0. \blacksquare$$

Drawing a Tangent Line to the Graph of a Function[463]

In differential calculus, a main objective is to try to understand tangents to curves, as illustrated in Figure 2.20. Hence, it is important to define a tangent line to an arbitrary plane curve in a rigorous way. A tangent line cannot be rigorously defined as a straight line having only one common point with the corresponding curve. Indeed, the axis of the parabola $y = x^2$ has only one point in common with the curve, but it does not touch the parabola. However, the straight line $y = 1$ has infinitely many common points with the sinusoid $y = sinx$, and it touches the sinusoid at each of these points. The requirement that the curve be located on one side of the straight line (e.g., when the axis of abscissas touches the curve $y = x^3$ at the point $(0,0)$, even though at this point the curve intersects the axis of abscissas) is not a rigorous definition of a tangent line either.

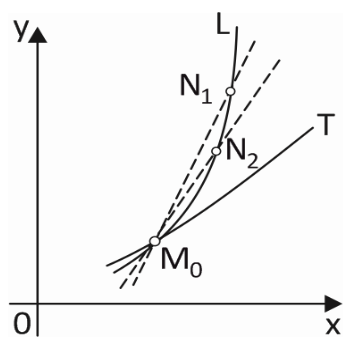

Figure 2.20. A Tangent Line to a Curve.

[462] See: Abbott, *Understanding Analysis*; Apostol, *Mathematical Analysis*; Courant, *Differential and Integral Calculus*; Dieudonné, *Treatise on Analysis*; DuChateau, *Advanced Calculus*; Fraleigh, *Calculus with Analytic Geometry*; Haaser and Sullivan, *Real Analysis*; Hardy, *A Course of Pure Mathematics*; Kolmogorov and Fomin, *Introductory Real Analysis*; Nikolski, *A Course of Mathematical Analysis*; Rudin, *Principles of Mathematical Analysis*, and his *Real and Complex Analysis*; Spivak, *Calculus*.

[463] Ibid.

In order to define a tangent line to an arbitrary plane curve in a rigorous way, we must use the concept of a limit (studied in sections 2.4, 2.5, and 2.6). Let L be an arc of some curve, and M_0 be a point of this curve. We draw a secant M_0N through the point M_0. If the point N, moving in the curve, approaches the point M_0, then the secant M_0N turns about the point M_0. Thus, it may so happen that, as the point N approaches M_0, the secant tends to a certain limit position M_0T, so that M_0T is referred to as the "secant" to the curve L at the point M_0, as illustrated in Figure 2.20. Then the "tangent line" to the curve L at the point M_0 is defined as the limit position of the secant M_0N as $N \to M_0$. The limit position is independent of the direction from where the point N approaches M_0.

However, the secant may not have a limit position, and then we say that it is impossible to draw a tangent line to the curve L at the point M_0. Usually, this is the case when M_0 is a cusp point, a break, a self-intersection point, etc. for the curve L. For most scientifically useful curves, a tangent line can, indeed, be drawn almost at all points.

Let us try to compute the slope of the tangent line for the case when the curve L is the graph of a certain function $y = f(x)$. Let M_0 be a point of the graph with abscissa x_0 and ordinate $y_0 = f(x_0)$. Assuming that the tangent line to the curve L at the point M_0 does exist, we take one more point $N(x_0 + \Delta x, y_0 + \Delta y)$ on the curve, as illustrated in Figure 2.21, and we draw a straight line through the points M_0 and N. If φ is the slope of this secant to the positive direction of the x-axis, then

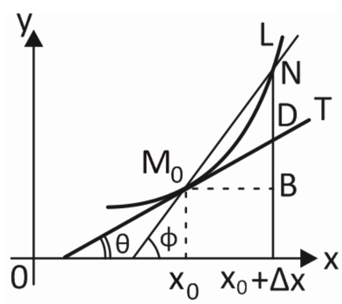

Figure 2.21. The Slope of a Tangent Line.

$|BN| = \Delta y$, $|M_0B| = \Delta x$, and $tan\varphi = \frac{|BN|}{|M_0B|} = \frac{\Delta y}{\Delta x}$,

so that the slope of this secant is $k_{tan} = lim_{N \to M_0} tan\varphi = lim_{\Delta x \to 0} tan\varphi$.

If we denote the slope of the tangent line to the axis of abscissas with θ, then the slope of the tangent line is

$$k_{tan} = tan\theta = lim_{\Delta x \to 0} tan\varphi = lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

Therefore, in order to draw a non-vertical tangent line to the graph of the function $y = f(x)$ at a point with abscissa x_0, it is necessary and sufficient that, at this point, the limit $lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ exists finitely, and, in fact, this limit is equal to the slope of the tangent line. In other words, we create an infinite sequence of slopes, and then we say that the slope of the given tangent line is the infinite limit of this sequence. Hence, infinitesimal calculus provides us with abstract objects (such a tangent to a curve) at which only infinite tasks can arrive through the concept of a limit.

As I have already explained, the concept of a limit has a deep philosophical significance, because it secures the theoretical convenience of being able to do an infinite number of tasks through a theoretical concept—namely, that of a limit—without actually doing each one of them, which would be practically impossible. This abstraction underpins the foundations of calculus as it was articulated by Isaac Newton and Gottfried Wilhelm von Leibniz in the seventeenth century. If, for a function $y = f(x)$, at a fixed point x, there exists the limit of the ratio of the increment $\Delta f(x)$ of the function to the increment Δx of the argument provided that $\Delta x \to 0$, then the function $y = f(x)$ is said to be "differentiable at the point x," and this limit is called the "derivative of the function" $y = f(x)$ at the point x. If $y = f(x)$, then all the following are equivalent notations for the derivative:

$$f'(x) \equiv y' \equiv \frac{df}{dx} \equiv \frac{dy}{dx} \equiv \frac{d}{dx}(f(x)) \equiv Df(x) \equiv \dot{y},$$

where $f'(x) = lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x} = lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$.

If $y = f(x)$, then all the following are equivalent notations for the derivative evaluated at $x = a$:

$$f'(a) \equiv y'|_{x=a} \equiv \frac{df}{dx}|_{x=a} \equiv \frac{dy}{dx}|_{x=a} \equiv Df(a) \equiv \dot{y}|_{x=a}.$$

The process of finding the derivative of a function is called "differentiation." As I have already explained, if a tangent line can be drawn to the graph of a function $y = f(x)$ at a point $M_0(x_0, f(x_0))$, then the slope of the tangent line is equal to the limit $lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, and this limit is the value of the derivative of the function $y = f(x)$ at the point x_0. Hence, the slope of the tangent line to the graph of a function $y = f(x)$ is equal to the value of the derivative at the point of tangency, symbolically, $k_{tan} = f'(x_0)$. This is the geometric significance of the derivative.

In the seventeenth century, Isaac Newton introduced the concept of a "fluxion" to describe what modern mathematicians call a "time derivative," namely, the instantaneous rate of change, or "slope," or "gradient" of a "fluent" (i.e., a time-varying quantity, or function) at a given point. Moreover, in his *Analyst*, published in 1734, the English philosopher and Bishop George Berkeley provided an intuitive exposition of mathematical analysis as follows:

> The Method of Fluxions is the general key by help whereof the modern mathematicians unlock the secrets of Geometry, and consequently of Nature . . . Lines are supposed to be generated by the motion of points, planes by the motion of lines, and solids by the motion of planes. And whereas quantities generated in equal times are greater or lesser according to the greater or lesser velocity wherewith they increase and are generated, a method hath been found to determine quantities from the velocities of their generating motions. And such velocities are called fluxions: and the quantities generated are called flowing quantities. These fluxions are said to be nearly as the increments of the flowing quantities, generated in the least equal particles of time; and to be accurately in the first proportion of the nascent, or in the last of the evanescent increments. Sometimes, instead of velocities, the momentaneous increments

or decrements of undetermined flowing quantities are considered, under the appellation of moments.[464]

The essential idea behind the Method of Fluxions is captured in Figures 2.19, 2.20, and 2.21, which show how to calculate the slope of a straight line, which is the graph of a linear function, and, especially, how to calculate the slopes of tangents to curves. In order to understand the physical, or mechanical, significance of Newton's idea, let us consider the problem of computing the instantaneous velocity of a particle in rectilinear motion.

Let $s = s(t)$ denote the distance travelled by a point moving in a straight line on which a reference point, the unit of measurement, and the direction are chosen (notice that $s(t)$ is the position of the point on the straight line at instant t). In physics, the "average velocity" of motion during a time interval is defined as the ratio of the net displacement to the elapsed time, that is, the average velocity during the time interval from t_1 to t_2 is expressed by the quantity

$$v_{av} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}.$$

If we set $t_1 = t$ and $t_2 - t_1 = \Delta t$, then we obtain

$$v_{av} = \frac{s(t + \Delta t) - s(t)}{\Delta t} = \frac{\Delta s}{\Delta t}.$$

Suppose that the average velocity of a particle is measured for a number of different time intervals, and that it is not constant. In other words, the particle under consideration is moving with varying velocity. Then we have to compute the velocity of the particle at any given instant of time, namely, the instantaneous velocity. The (numerical value of) the "instantaneous velocity," namely, the (numerical value of) the velocity at instant t, is defined as the limit of the average velocity of motion during the time interval $[t, t + \Delta t]$ provided that $\Delta t \to 0$, symbolically,

$$v_{inst} = lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t},$$

which is the derivative of displacement $s = s(t)$ with respect to time, namely,

$$v_{inst} = \frac{ds(t)}{dt}.$$

Similarly, we can compute the instantaneous rate of change of any other physical or (quantifiable) socio-economic phenomenon with respect to its independent variable. For instance, in economics, inflation is defined as the derivative of price (as a function of time) with respect to time; the rate of change of demand with respect to price is defined as the derivative of the quantity demanded (as a function of price) with respect to price; the point price-elasticity of demand, which measures the degree to which the desire for something changes as its price changes within the same demand curve, is equal to the absolute value of

[464] Berkeley, "Analyst and Its Effect upon the Calculus," pp. 628–29.

the derivative of the quantity demanded with respect to price multiplied by the point's price divided by its quantity; etc.[465]

The Formal Definition of the Derivative of a Function[466]

Let $f: X \to \mathbb{R}$ be a function, and let $a \in X$ be an accumulation point of X. Then, $\forall x \in X - \{a\}$, we can define the function

$$F(x) = \frac{f(x)-f(a)}{x-a},$$

and a is an accumulation point of $X - \{a\}$. Hence, it is meaningful to consider the limit of F as $x \to a$.

If $f: X \to \mathbb{R}$, where $X \subseteq \mathbb{R}$, and a is an accumulation point of X, and if

$lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ exists,

then the function f is said to be "differentiable" at a, and the number

$f'(a) = lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ is said to be the "derivative" of f at a.

We require that $a \in X$ in order for the concept of $f(a)$ to be meaningful. Moreover, we require that a be an accumulation point of X, because, as already stated, the limit of F is defined at a only if a is an accumulation point of X.

If in the aforementioned definition, we set $x - a = h$, so that $x = a + h$ and $h \to 0$ when $x \to a$, provided, of course, that $a + h \in X$, then we obtain the following formula for the derivative of f at $a \in X$:

$$f'(a) = lim_{h \to 0} \frac{f(a+h)-f(a)}{h}.$$

If the derivative of a function exists, then it is unique. The uniqueness of the derivative of a function follows directly from the uniqueness of the limit of a function (proven in section 2.6).

At the point $a \in X$, the function F may not have a limit, but it may have one-sided limits. Then we define the one-sided derivatives of f at a as follows: (i) If $lim_{x \to a^-} F(x)$ exists, then we say that the function f is "differentiable at a from the left," and the number $f'_-(a)$ is called the "derivative of f at a from the left" (or the "left-hand derivative of f at a"). (ii) By analogy, if $lim_{x \to a^+} F(x)$ exists, then we say that the function f is "differentiable at a from the right," and the number $f'_+(a)$ is called the "derivative of f at a from the right" (or the "right-hand derivative of f at a"). According to what we already know regarding the limit of

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[465] See: Lovell, *Economics with Calculus*.

[466] See: Abbott, *Understanding Analysis*; Apostol, *Mathematical Analysis*; Courant, *Differential and Integral Calculus*; Dieudonné, *Treatise on Analysis*; DuChateau, *Advanced Calculus*; Fraleigh, *Calculus with Analytic Geometry*; Haaser and Sullivan, *Real Analysis*; Hardy, *A Course of Pure Mathematics*; Kolmogorov and Fomin, *Introductory Real Analysis*; Nikolski, *A Course of Mathematical Analysis*; Piskunov, *Differential and Integral Calculus*; Rudin, *Principles of Mathematical Analysis*, and his *Real and Complex Analysis*; Spivak, *Calculus*.

a function (studied in section 2.6), f is differentiable at $a \in X$ if and only if its one-sided derivatives at a exist and are equal to each other.

If a function f is differentiable at every point of $A \subseteq X$, then f is said to be "differentiable on the set A." Then the function f', whose value is defined to be $f'(x)\ \forall x \in A$, is said to be the "derivative of f in the set A." If f is differentiable on X, namely, on its domain, then we simply say that f is a differentiable function. If, in particular, $X = [a, b]$, then the statement that f is a differentiable function implies the following: (i) $f'(x)$ exists $\forall x \in (a, b)$, (ii) $f'_+(a)$ exists, and (iii) $f'_-(b)$ exists.

Recalling the definition of a convergent sequence (studied in section 2.4), we can articulate the following equivalent definition of the derivative of f at $a \in X$: A function f is differentiable at $a \in X$ if and only if, for every sequence $x_n \in X - \{a\}$ with $x_n \to a$, it holds that

$$lim_{x_n \to a} \frac{f(x_n) - f(a)}{x_n - a} = f'(a).$$

Furthermore, the aforementioned definition can take the following form, which is based on the concept of a Cauchy sequence (studied in section 2.4): A number $f'(a)$ is said to be the derivative of a function $f: X \to \mathbb{R}$ at $a \in X$ if, $\forall \varepsilon > 0$, there exists a $\delta \equiv \delta(\varepsilon) > 0$ such that, $\forall x \in X$ with $0 < |x - a| < \delta$, it holds that

$$\left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| < \varepsilon. \qquad (*)$$

Let us define the function $\lambda_a: X - \{a\} \to \mathbb{R}$ with

$$\lambda_a(x) = \frac{f(x) - f(a)}{x - a} - f'(a). \qquad (**)$$

Then, by $(*)$, $lim_{x \to a} \lambda_a(x) = 0$.

It can be easily verified that, if $f(x) = c$ is a constant function, then $f'(x) = (c)' = 0$.

In view of the aforementioned definitions of the derivative of a function, we can prove the following characterization of derivatives:

Characterization of Derivatives[467]*:* A function $f: X \to \mathbb{R}$ is differentiable at $a \in X$, where a is an accumulation point of X, if and only if there exist a real number $f'(a) \in \mathbb{R}$, a $\delta > 0$, and a function $\lambda_a: X - \{a\} \to \mathbb{R}$ with $lim_{x \to a} \lambda_a(x) = 0$ such that

$$\forall x \in X, 0 < |x - a| < \delta \Rightarrow f(x) - f(a) = f'(a)(x - a) + \lambda_a(x)(x - a). \qquad (***)$$

Proof: Assume that $(***)$ is true. Then $(**)$ implies that

$$f'(a) = lim_{x \to a} \frac{f(x) - f(a)}{x - a} \text{ exists.}$$

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[467] Ibid.

Conversely, if $f'(a)$ exists, then $(*)$ holds, and, therefore, the function $\lambda_a(x)$ defined by $(**)$ satisfies $(***)$.■

Notice that the derivative of a function f at a is a real number, but the derivative of f is a function (specifically, an operator).

Theorem[468]*:* If a function $f(x)$ is differentiable at $x = a$ (having a finite derivative), then it is also continuous at $x = a$. But the converse is not necessarily true.

Proof: Assume that f is differentiable at $x = a$. Then the limit

$lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ exists and is equal to the real number $f'(a)$.
Notice that

$$lim_{x \to a}(f(x) - f(a)) = lim_{x \to a}\left(\frac{f(x)-f(a)}{x-a} \cdot (x-a)\right) = lim_{x \to a}\frac{f(x)-f(a)}{x-a} \cdot lim_{x \to a}(x-a) = f'(a) \cdot 0.$$

Therefore,

$$lim_{x \to 0}(f(x) - f(a)) = 0 \text{ or}$$
$$lim_{x \to a} f(x) = f(a),$$

which proves that f is continuous at $x = a$.

In order to prove that the converse is not necessarily true, it suffices to give an example. For instance, consider $f(x) = |x| \; \forall x \in \mathbb{R}$. Then, at $x = 0$, the function is continuous, because $lim_{x \to 0} f(x) = f(0)$, but it is not differentiable at $x = 0$, because $f'_+(0) \neq f'_-(0)$, and, in fact, $f'_+(0) = 1$ while $f'_-(0) = -1$.■

Remark: There exist functions that are continuous over the entire $\mathbb{R}$ without being differentiable at any point of $\mathbb{R}$. Such a function is the so-called "Weierstrass function," which is defined by the series $f(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} cos(3^n x)$, where $x \in \mathbb{R}$.

Using the aforementioned definitions of the derivative, the aforementioned Characterization of Derivatives, and theorems on limits, the following properties of the derivative can be easily verified:

Theorem[469]*:* Let $X \subseteq \mathbb{R}$ be an interval, $a \in X$, and $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ be functions that are differentiable at a. Then:

i. If $k \in \mathbb{R}$, then the function kf is differentiable at a, and $(kf)'(a) = kf'(a)$.
ii. The function $f + g$ is differentiable at a, and

[468] Ibid.
[469] Ibid.

$(f+g)'(a) = f'(a) + g'(b)$.

iii. The function $f \cdot g$ is differentiable at a, and
$(f \cdot g)'(a) = f'(a)g(a) + f(a)g'(a)$.

iv. If $g(a) \neq 0$, then the function $\frac{f}{g}$ is differentiable at a, and
$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}$.

Theorem (Power Rule)[470]*:* $\frac{d}{dx}x^n = nx^{n-1}$, where n is a positive integer.

Proof: If $f(x) = x^n$, then, by definition,

$$f'(x) = lim_{h\to 0}\frac{(x+h)^n - x^n}{h}.$$

Given that

$$(x+h)^n = x^n + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \cdots + nxh^{n-1} + h^n,$$

we obtain

$$(x+h)^n - x^n = nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \cdots + nxh^{n-1} + h^n.$$

Hence,

$$\frac{(x+h)^n - x^n}{h} = \frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \cdots + nxh^{n-1} + h^n}{h},$$

which implies that

$$\frac{(x+h)^n - x^n}{h} = nx^{n-1} + \binom{n}{2}x^{n-2}h + \binom{n}{3}x^{n-3}h^2 + \cdots + nxh^{n-2} + h^{n-1}.$$

Therefore,

$$f'(x) = lim_{h\to 0}\left[nx^{n-1} + \binom{n}{2}x^{n-2}h + \binom{n}{3}x^{n-3}h^2 + \cdots + nxh^{n-2} + h^{n-1}\right] = nx^{n-1}.\blacksquare$$

Theorem (the derivative of a composite function: the Chain Rule)[471]*:* Consider two functions $f: D_f \to \mathbb{R}$ and $g: R_f \to \mathbb{R}$ ($D_f \subseteq \mathbb{R}$), where D_f is the domain of f, and R_f is the range of f. If a is an accumulation point of D_f, if $f(a) = b$ is an accumulation point of R_f, and if $f'(a)$ and $g'(b)$ exist, then the function $h = g\circ f: D_f \to \mathbb{R}$ is differentiable at a, and its derivative is given by the formula

[470] Ibid.
[471] Ibid.

$(g \circ f)'(a) = g'(f(a))f'(a),$

that is, the derivative of $g \circ f$ is the derivative of $g(x)$ evaluated at $f(x)$ times the derivative of $f(x)$.

Proof: Given that $f'(a)$ and $g'(b)$ exist, the aforementioned Characterization of Derivatives, namely, $(***)$, implies that there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$f(x) - f(a) = f'(a)(x-a) + \lambda_a(x)(x-a) \text{ and} \tag{1}$$

$$g(y) - g(b) = g'(b)(y-b) + \mu_b(y)(y-b), \tag{2}$$

$\forall x \in D_f$ with $0 < |x-a| < \delta_1$ and $\forall y \in D_g$ with $0 < |y-b| < \delta_2$, where $lim_{x \to a}\lambda_a(x) = 0$ and $lim_{y \to b}\mu_b(y) = 0$. Let

$$\mu_b(b) = 0. \tag{3}$$

Because $g(b) = g(f(a)) = h(a)$, and, for $y = f(x)$, we obtain $g(y) = g(f(x)) = h(x)$, equations (1) and (2) imply the following:

$$\begin{aligned} & h(x) - h(a) = g(y) - g(b) = g'(b)(y-b) + \mu_b(y)(y-b) \\ & = g'(b)(f(x) - f(a)) + \mu_b(y)(f(x) - f(a)) \\ & = g'(b)[f'(a)(x-a) + \lambda_a(x)(x-a)] + \mu_b(y)[f'(a)(x-a) + \lambda_a(x)(x-a)] \\ & = g'(b)f'(a)(x-a) + [g'(b)\lambda_a(x) + \mu_b(y)f'(a) + \mu_b(y)\lambda_a(x)](x-a). \end{aligned}$$

Notice that

$$\begin{aligned} & lim_{x \to a}[g'(b)\lambda_a(x) + \mu_b(y)f'(a) + \mu_b(y)\lambda_a(x)] \\ & = lim_{x \to a}\left[g'\big(f(a)\big)\lambda_a(x) + \mu_b(f(x))f'(a) + \mu_b(f(x))\lambda_a(x)\right] = 0, \end{aligned}$$

since: $lim_{x \to a}\lambda_a(x) = 0$, and, because f is continuous at a (since f is differentiable at a), and due to (3), it holds that

$$lim_{x \to a}\mu_b\big(f(x)\big)f'(a) = \mu_b(f(a))f'(a) = \mu_b(b)f'(a) = 0.$$

Therefore, according to the aforementioned Characterization of Derivatives, namely, $(***)$, $h'(a)$ exists, and $h'(a) = g'\big(f(a)\big)f'(a)$.■

Remark: A simple application of the chain rule is the following problem: If Mr. X runs 4 times as fast as Mr. Y, and Mr. Y runs 3 times as fast as Mr. Z, how many times as fast as Mr. Z does Mr. X run? It is clear that

$$\frac{dX}{dZ} = \frac{dX}{dY} \cdot \frac{dY}{dZ} = 4 \times 3 = 12.$$

Implicit Differentiation: Whenever the dependent variable y is a function of the independent variable x, we express y in terms of x. If this is the case, then we say that y is an "explicit function" of x. For instance, in the equation $y = 3x + 2$, y is defined explicitly in terms of x. But, whenever the relation between the function y and the variable x is expressed by an equation where y is not expressed entirely in terms of x, we say that the equation defines y "implicitly" in terms of x. An equation defines a function implicitly if the function satisfies the given equation. For instance, the equation $y - x^2 = 3$ defines the function $y = x^2 + 3$ implicitly. Implicit differentiation allows us to find slopes of tangents to curves that are not functions (namely, to curves that fail the vertical line test[472]). *Algorithm for implicit differentiation:*

i. Differentiate both sides of the equation, keeping in mind that y is a function of x (and apply the Chain Rule where necessary).
ii. Rewrite the equation so that every term containing $\frac{dy}{dx}$ is on the left, and every term that does not contain $\frac{dy}{dx}$ is on the right.
iii. Factor out $\frac{dy}{dx}$ on the left.
iv. Solve for $\frac{dy}{dx}$ by dividing both sides of the equation by an appropriate algebraic expression.

Implicit differentiation is illustrated in the following example: Find $\frac{dy}{dx}$ if $x^4 + y^3 = 7$. Since $y = f(x)$, $(x^4)' + (y^3)' = (7)' \Leftrightarrow 4x^3 + 3y^2\frac{dy}{dx} = 0 \Leftrightarrow 3y^2\frac{dy}{dx} = -4x^3 \Leftrightarrow \frac{dy}{dx} = -\frac{4x^3}{3y^2}$.

If $f(x) = lnx$, then $f'(x)$ can be found through implicit differentiation: Since $y = lnx$, $e^y = x \Leftrightarrow e^y\frac{dy}{dx} = 1 \Leftrightarrow \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$. Hence, $\frac{d(lnx)}{dx} = \frac{1}{x}, x > 0$. In case u is a differentiable function of x, we apply the chain rule: $\frac{d(lnu)}{dx} = \frac{1}{u} \cdot \frac{du}{dx}, u > 0$.

If $f(x) = log_a x$, then the method of implicit differentiation can be applied, too: Since $y = log_a x$, $a^y = x \Leftrightarrow (lna) \cdot a^y \cdot \frac{dy}{dx} = 1 \Leftrightarrow \frac{dy}{dx} = \frac{1}{lna} \cdot \frac{1}{a^y} = \frac{1}{lna} \cdot \frac{1}{x}, x > 0$. In case, u is a differentiable function of x, we apply the chain rule: $\frac{d(log_a u)}{dx} = \frac{1}{lna} \cdot \frac{1}{u} \cdot \frac{du}{dx}, u > 0$.

One more important application of implicit differentiation is to finding derivatives of inverse functions. Notice that the graph of f^{-1} is the reflection of the graph of f across the line $y = x$, and, therefore, if $\frac{dy}{dx}$ is the slope of a line tangent to the graph of f, then $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$ is the slope of a line tangent to the graph of f^{-1}.

[472] Ibid (notice that every vertical line intersects the graph of a function at most once).

Theorem (the derivative of an inverse function)[473]*:* Let $I \subseteq \mathbb{R}$ be an interval, and let $f: I \to \mathbb{R}$ be a strictly monotonic function continuous over I. If f is differentiable at $a \in I$ and $f'(a) \neq 0$, then the inverse function $f^{-1}: J = f(I) \to \mathbb{R}$ is differentiable at $b = f(a)$, and

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}.$$

Proof: In section 2.7, it was proved that, if$f: I = [a, b] \to \mathbb{R}$ is strictly monotonic and continuous on the interval I, then f has an inverse function that is strictly monotonic and continuous on $f(I)$, and, in fact, f^{-1} has the same kind of monotonicity as f. Hence, in this case, f^{-1} exists and is continuous on $f(I)$, namely,

$lim_{y \to b} f^{-1}(y) = f^{-1}(b)$, where $y \in f(I)$.

Moreover, f^{-1} is strictly monotonic, and it has the same kind of monotonicity as f. If $y = f(x)$ and $b = f(a)$, then $y \neq b \Rightarrow f^{-1}(y) \neq f^{-1}(b)$, and $y \to b \Rightarrow f^{-1}(y) \to f^{-1}(b) \Rightarrow x \to a$. Therefore,

$$(f^{-1})'(b) = lim_{y \to b} \frac{f^{-1}(y) - f^{-1}(b)}{y - b} = lim_{x \to a} \frac{x - a}{f(x) - f(a)} = lim_{x \to a} \frac{1}{\frac{f(x) - f(a)}{x - a}} = \frac{1}{f'(a)}. \blacksquare$$

Remark: The requirement that $f'(a) \neq 0$ secures that f^{-1} is differentiable at $b = f(a)$. If $f'(a) = 0$, then f^{-1} is not differentiable at $b = f(a)$; indeed, if f^{-1} were differentiable at b, then, since f is the inverse of f^{-1}, the application of the aforementioned theorem to f^{-1} implies that f is differentiable at $a = f^{-1}(b)$, and that $1 = f'(a)(f^{-1})'(b) = 0$, which is absurd. For instance, consider the function $f(x) = x^3$ with $x \in \mathbb{R}$ in case $a = 0$.

Notice that, if $f(x)$ is differentiable on an interval I such that $f'(x) = w(x)\ \forall x \in I$, then the function f is called the "primitive" of the function w. If $f(x) = g(x) + r\ \forall x \in I$, and if both f and g are differentiable on I, then $f'(x) = g'(x) = w(x)\ \forall x \in I$, and, therefore, the function g is also a primitive of w. Consequently, if a function w has a primitive over an interval, it has any number of primitives over that interval, and any two of such primitives differ by some real number r.

Many of the basic laws of the physical sciences, the biological sciences, and the social sciences are formulated in terms of mathematical equations involving certain known and unknown quantities and their derivatives. Such equations are called "differential equations" (see section 2.20).

[473] Ibid.

Higher Order Derivatives

It is evident that the first derivative $\frac{dy}{dx}$ expresses the rate of change of y with respect to x (e.g., velocity). Then $\frac{d}{dx}\left(\frac{dy}{dx}\right) \equiv \frac{d^2y}{dx^2} \equiv y''$ expresses the rate of change of the first derivative of y with respect to x (e.g., acceleration), and $\frac{d^3y}{dx^3} \equiv y'''$ expresses the rate of change of the second derivative of y with respect to x (e.g., jerk). Of course, we can compute the nth derivative of $y = f(x)$, denoted by $\frac{d^ny}{dx^n} \equiv y^{(n)}$, where n is called the order of the derivative. The nth derivative $f^{(n)}(x)$, where $n \in \mathbb{N}$, is defined as the derivative of the derivative of order $(n-1)$, symbolically, $f^{(n)}(x) = \left(f^{(n-1)}(x)\right)'$. The process of obtaining higher order derivatives is called "successive differentiation."

Table of the Derivatives of Elementary Functions[474]

Using the definition of the derivative of a function and the aforementioned theorems and methods of differentiation, we obtain the following formulas:

$\frac{d}{dx}(c) = 0$, where c is any constant, $x \in \mathbb{R}$;
$\frac{d}{dx}(x) = 1$, $x \in \mathbb{R}$;
$\frac{d}{dx}(e^x) = e^x$, $x \in \mathbb{R}$;
$\frac{d}{dx}(a^x) = a^x ln(a)$, $x \in \mathbb{R}$, $a > 0$;
$\frac{d}{dx}(ln|x|) = \frac{1}{x}, x \in \mathbb{R} - \{0\}$;
$\frac{d}{dx}\left(log_a(x)\right) = \frac{1}{xlna}, x > 0$;
$\frac{d}{dx}(sinx) = cosx$, $x \in \mathbb{R}$, because:

$(sinx)' = lim_{h\to 0}\frac{sin(x+h)-sinx}{h} = lim_{h\to 0}\frac{2sin\frac{h}{2}cos\left(x+\frac{h}{2}\right)}{h} = lim_{h\to 0}\frac{sin\frac{h}{2}}{\frac{h}{2}} \cdot cos\left(x+\frac{h}{2}\right) =$
$cosx$, since $lim_{x\to 0}\frac{sinx}{x} = 1$;

$\frac{d}{dx}(cosx) = -sinx$, $x \in \mathbb{R}$, because:
$(cosx)' = \left[sin\left(\frac{\pi}{2}-x\right)\right]' = cos\left(\frac{\pi}{2}-x\right)\cdot\left(\frac{\pi}{2}-x\right)' = cos\left(\frac{\pi}{2}-x\right)\cdot(-1) = -sinx$;
$\frac{d}{dx}(tanx) = \frac{1}{cos^2x} = sec^2x$, $x \in \mathbb{R} - \left\{x|x = k\pi + \frac{\pi}{2}\right\}$;
$\frac{d}{dx}(cotx) = -\frac{1}{sin^2x} = -csc^2x$, $x \in \mathbb{R} - \{x|x = k\pi\}$,
$\frac{d}{dx}(secx) = secxtanx$,
$\frac{d}{dx}(cscx) = -cscxcotx$,
$\frac{d}{dx}(sin^{-1}x) \equiv \frac{d}{dx}(arcsinx) = \frac{1}{\sqrt{1-x^2}}$, $x \in (-1,1)$,
$\frac{d}{dx}(cos^{-1}x) \equiv \frac{d}{dx}(arccosx) = -\frac{1}{\sqrt{1-x^2}}$, $x \in (-1,1)$,
$\frac{d}{dx}(tan^{-1}x) \equiv \frac{d}{dx}(arctanx) = \frac{1}{1+x^2}$, $x \in \mathbb{R}$,

474 Ibid.

$\frac{d}{dx}(cot^{-1}x) \equiv \frac{d}{dx}(arccotx) = -\frac{1}{1+x^2}, x \in \mathbb{R},$
$\frac{d}{dx}(sinhx) = coshx, x \in \mathbb{R},$
$\frac{d}{dx}(coshx) = sinhx, x \in \mathbb{R},$
$\frac{d}{dx}(tanhx) = \frac{1}{cosh^2 x}, x \in \mathbb{R},$
$\frac{d}{dx}(cothx) = -\frac{1}{sinh^2 x}, x \in \mathbb{R} - \{0\},$
$\frac{d}{dx}(sechx) = -tanhxsechx,$
$\frac{d}{dx}(cschx) = -cothxcschx,$
$\frac{d}{dx}(arcsinhx) = \frac{1}{\sqrt{x^2+1}}, x \in \mathbb{R},$
$\frac{d}{dx}(arccoshx) = \frac{1}{\sqrt{x^2-1}}, x \in (1, +\infty),$
$\frac{d}{dx}(arctanhx) = \frac{1}{1-x^2}, x \in (-1,1),$
$\frac{d}{dx}(arccothx) = \frac{1}{1-x^2}, x \in (-\infty, -1) \cup (1, +\infty).$

The Differential of a Function[475]

If the function $y = f(x)$ is differentiable on the interval $[a, b]$, then the derivative of y at some $x \in [a, b]$ is

$$f'(x) = lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

As $\Delta x \to 0$, the ratio $\frac{\Delta y}{\Delta x}$ approaches a definite number $f'(x)$, and, therefore, it differs from the derivative $f'(x)$ by an infinitesimal:

$$\frac{\Delta y}{\Delta x} = f'(x) + q, \qquad (*)$$

where $q \to 0$ as $\Delta x \to 0$. Multiplying every term of $(*)$ by Δx, we obtain

$$\Delta y = f'(x)\Delta x + q\Delta x. \qquad (**)$$

If $f'(x) \neq 0$, for a constant x and a variable $\Delta x \to 0$, then the product $f'(x)\Delta x$ is an infinitesimal of the first order with respect to Δx. However, the product $q\Delta x$ is always an infinitesimal of higher order with respect to Δx, since

$$lim_{\Delta x \to 0} \frac{q\Delta x}{\Delta x} = lim_{\Delta x \to 0} q = 0.$$

Hence, the increment Δy of the function consists of two terms, and the first of these terms (when $f'(x) \neq 0$) is called the "principal part" of the increment, and it is linear with respect to x. The product $f'(x)\Delta x$ is called the "differential" of the function, and it is denoted by dyor $df(x)$, so that $dy = f'(x)\Delta x$. Therefore, the derivative $f'(x)$ can be construed as the

[475] Ibid.

ratio of the differential of the function $y = f(x)$ to the differential of the independent variable x.

A Note about Complex Derivatives

In case of a function g that takes real inputs and gives complex outputs, the derivative with respect to its real input consists in taking the derivatives of the real and the imaginary parts separately, namely:

$$\frac{dg}{dx} = \frac{dRe(g)}{dx} + i\frac{dIm(g)}{dx},$$

where $i = \sqrt{-1}$, $Re(g)$ is the real part of g, and $Im(g)$ is the imaginary part of g.

2.10.2. The Basic Theorems of Differential Calculus

Rolle's Theorem[476]*:* Let $f: [a, b] \to \mathbb{R}$ be a function satisfying the following conditions:

i. f is continuous on the closed interval $[a, b]$,
ii. f is differentiable on the open interval (a, b), and
iii. $f(a) = f(b)$.

Then there exists at least one point $c \in (a, b)$ such that $f'(c) = 0$.

Geometric interpretation: Under the above conditions, there exists a point c at which the tangent line to the graph of $y = f(x)$ is parallel to the x-axis, as shown in Figure 2.22. In particular, conditions (i) and (ii) imply that the curve $y = f(x)$ is continuous from $x = a$ to $x = b$, and it has a definite tangent at each point between $x = a$ and $x = b$; and condition (iii) implies that the ordinates at the endpoints a and b are equal.

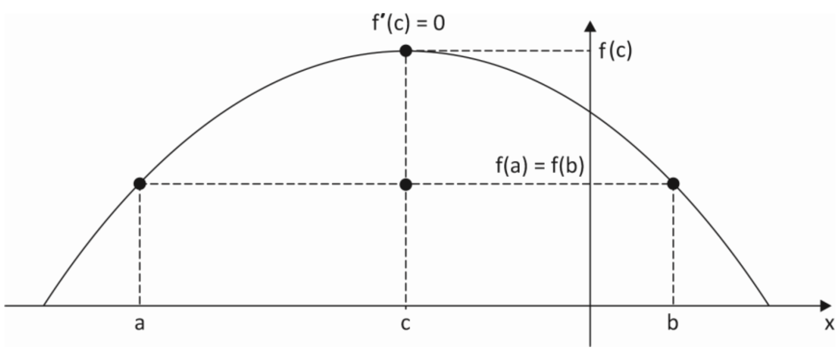

Figure 2.22. Rolle's Theorem.

Algebraic interpretation: Since, according to condition (iii), $f(a) = f(b)$, let $f(a) = f(b) = 0$. Then Rolle's Theorem means that, if $f(x)$ is a polynomial in x, and if a and b are

[476] Ibid.

two roots of the equation $f(x) = 0$, then the equation $f'(x) = 0$ has at least one root between a and b. In fact, the French mathematician Michel Rolle, after whom the above theorem is named, proved the given theorem in 1691 only in the case of polynomial functions, and a general proof of this theorem was achieved and published by Augustin-Louis Cauchy in 1823. The name "Rolle's Theorem" was first used by the German mathematician, logician, psychologist, and philosopher Moritz Wilhelm Drobisch in the 1830s.

Proof: Since f is continuous on the closed interval $[a, b]$, it is bounded and attains its supremum (least upper bound) and its infimum (greatest lower bound) in $[a, b]$, as it was proved in section 2.7. Let $\inf(f) = m$, $\sup(f) = M$, and $f(a) = f(b) = k$. Then it must hold that $m \leq k \leq M$.

First case: If $m = k = M$ (i.e., if f is a constant function), then $f(x) = k$, and, therefore, $f'(x) = 0\ \forall k \in (a, b)$.

Second case: If $m \neq M$, then $m < k$ or $k < M$. Suppose that $k < M$. There exists a $c \in (a, b)$ such that $f(c) = M$, since, in section 2.7, it was proved that, if f is continuous on the closed interval $[a, b]$, then it attains its supremum and its infimum in $[a, b]$. Moreover, $f'(c)$ exists, because $a < c < b$. Notice that $f(x) \leq M\ \forall x \in [a, b]$. Therefore, if $a \leq x < c$, then $\frac{f(x)-f(c)}{x-c} = \frac{f(x)-M}{x-c} \geq 0$, so that $lim_{x \to c^-} \frac{f(x)-f(c)}{x-c} \geq 0 \Leftrightarrow f'_-(c) \geq 0$. If $c < x \leq b$, then $\frac{f(x)-f(c)}{x-c} = \frac{f(x)-M}{x-c} \leq 0$, so that $lim_{x \to c^+} \frac{f(x)-f(c)}{x-c} \leq 0 \Leftrightarrow f'_+(c) \leq 0$. Consequently, $0 \leq f'_-(c) = f'(c) = f'_+(c) \leq 0 \Rightarrow f'(c) = 0$. We can work similarly in order to prove the theorem for $m < k$.■

In mathematical analysis, the mean value theorems play a very important role, because they examine the relationship between the values of a function and the values of the derivative of the given function.

Lagrange's Mean Value Theorem[477]*:* If $f: [a, b] \to \mathbb{R}$ is a function continuous on $[a, b]$ and differentiable on (a, b), then there exists a $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Geometric interpretation: As shown in Figure 2.23, Lagrange's Mean Value Theorem implies that the slope of the chord passing through the points of the graph corresponding to the ends of the segment a and b is equal to $k = tan\theta = \frac{f(b)-f(a)}{b-a}$, and then there exists a point $x = c$ inside the closed interval $[a, b]$ such that the tangent to the graph at $x = c$ is parallel to the chord.

Proof: Let $g(x)$ be a function whose domain is $D_f = [a, b]$, and let

$$g(x) = f(x) - \left[f(a) + \frac{f(b)-f(a)}{b-a}(x-a)\right] = [f(x) - f(a)] - \left[\frac{f(b)-f(a)}{b-a}(x-a)\right].$$

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477 Ibid.

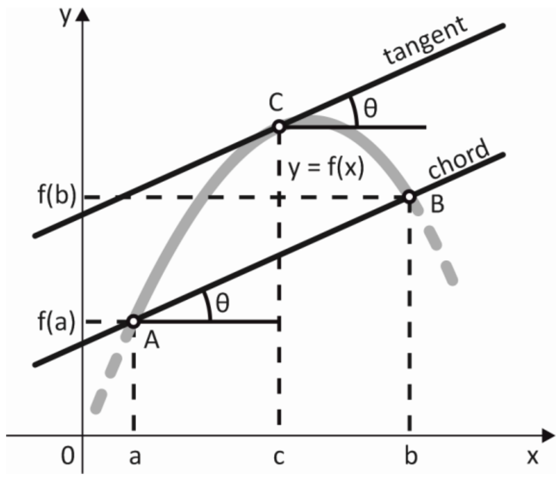

Figure 2.23. Lagrange's Mean Value Theorem.

Because f is continuous on $[a,b]$ and differentiable on (a,b), g is also, and $g(a) = g(b) = 0$. Therefore, due to Rolle's Theorem, $\exists c \in (a,b) | g'(c) = 0$, and $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a} \Rightarrow g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$.■

For instance, given $f(x) = x^2 + x + 1$, if we are asked to find the point c at which $f'(x)$ gets its mean value in $[0,2]$, then we work as follows: we confirm that the hypotheses of Lagrange's Mean Value Theorem are satisfied, and, therefore, $\exists c \in (a,b) | \frac{f(b)-f(a)}{b-a} = f'(c) \Rightarrow \frac{f(2)-f(0)}{2-0} = 3 = f'(c) = 2c + 1 \Rightarrow c = 1$.

Given a differentiable function $f(x)$ with, say, $f(0) = 1$ and $|f'(x)| \leq 4$ for $0 \leq x \leq 1$, if we are asked to find the bounds on $f(x)$ in $[0,1]$, we work as follows: $\forall x \neq 0, \frac{f(x)-f(0)}{x-0} = \frac{f(x)-1}{x} = f'(c), 0 < c < x \,\&\, |f'(c)| \leq 4$. Moreover, because x is positive, we have $\left|\frac{f(x)-1}{x}\right| = \frac{|f(x)-1|}{x} \leq 4 \Leftrightarrow |f(x) - 1| \leq 4x \Leftrightarrow -4x \leq f(x) - 1 \leq 4x \Leftrightarrow 1 - 4x \leq f(x) \leq 1 + 4x$. Since $x \in [0,1]$, we obtain $1 \leq f(x) \leq 5$.

Augustin-Louis Cauchy has generalized Lagrange's Mean Value Theorem, so that there is Cauchy's Mean Value Theorem, too.

Cauchy's Mean Value Theorem[478]*:* Let f and g be two functions continuous on $[a,b]$ and differentiable on (a,b) with $g'(x) \neq 0 \; \forall x \in (a,b)$. Then there exists a $c \in (a,b)$ such that $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$.

Proof: Suppose that $h(x) = [g(b) - g(a)] \cdot [f(x) - f(a)] - [g(x) - g(a)] \cdot [f(b) - f(a)]$. Then $h(a) = 0 = h(b)$. Because h is continuous on $[a,b]$ and differentiable on (a,b), Rolle's Theorem implies that $\exists c \in (a,b) | h'(c) = 0$. Hence, $h'(c) = [g(b) - g(a)] \cdot [f'(c)] - [g'(c)] \cdot [f(b) - f(a)] \Rightarrow [f'(c)] \cdot [g(b) - g(a)] = [g'(c)] \cdot [f(b) - f(a)]$. Dividing by $g'(c) \neq 0$ and by $[g(b) - g(a)]$, we obtain $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$.■

[478] Ibid.

L'Hôpital's Rule[479]*:* Assume that the functions f and g are continuous on $[a, b]$ and differentiable on (a, b) with $f(a) = g(a) = 0$ and $g'(x) \neq 0\ \forall x \in (a, b)$. Then, if $lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L$ where $L \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, it holds that $lim_{x \to a^+} \frac{f(x)}{g(x)} = L$.

Proof: If $lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L$, then,

$$\forall \varepsilon > 0, \exists \delta > 0 | a < x < a + \delta \Rightarrow \left| \frac{f'(x)}{g'(x)} - L \right| < \varepsilon.$$

Due to Cauchy's Mean Value Theorem,

$$\forall x\ with\ a < x < a + \delta, \exists p \in (a, x) | \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(p)}{g'(p)} \Rightarrow \frac{f(x)}{g(x)} = \frac{f'(p)}{g'(p)}.$$

Thus, since $a < p < x < a + \delta$,

$$\left| \frac{f(x)}{g(x)} - L \right| = \left| \frac{f'(p)}{g'(p)} - L \right| < \varepsilon,$$

which holds $\forall x\ with\ a < x < a + \delta$, so that

$$lim_{x \to a^+} \frac{f(x)}{g(x)} = L.$$

If $L = +\infty$, then $lim_{x \to a^+} \frac{f'(x)}{g'(x)} = +\infty$ implies that,

$$\forall t > 0, \exists \delta > 0 | a < x < a + \delta \Rightarrow \frac{f'(x)}{g'(x)} > t.$$

For every such x, Cauchy's Mean Value Theorem implies that there exists a $p \in (a, x)$ such that

$$\frac{f(x)}{g(x)} = \frac{f'(p)}{g'(p)} > t,$$

and, therefore, $lim_{x \to a^+} \frac{f(x)}{g(x)} = +\infty$. In case $L = -\infty$, work similarly.■

Remark: L'Hôpital's Rule also holds if we are considering left-hand limits. Moreover, if f and g are two functions such that $lim_{x \to a} f(x) = \infty$ and $lim_{x \to a} g(x) = \infty$, then $lim_{x \to a} \frac{f(x)}{g(x)} = lim_{x \to a} \frac{f'(x)}{g'(x)}$.

In other words, L'Hôpital's Rule, named after the French mathematician Guillaume Marquis de l'Hôpital (1661–1704), tells us that, if we have an indeterminate form $0/0$ or ∞/∞, all we need to do is to differentiate both the numerator and the denominator and then

479 Ibid.

compute the limit. For instance, $lim_{x\to\infty}\frac{lnx}{x} = lim_{x\to\infty}\frac{(lnx)'}{(x)'} = lim_{x\to\infty}\frac{\frac{1}{x}}{1} = 0$. Similarly, in order to compute $lim_{n\to\infty}\sqrt[n]{n}$, we set $f(x) = x^{\frac{1}{x}} \Leftrightarrow f(x) = e^{\frac{1}{x}lnx}$, so that $lim_{x\to\infty}x^{\frac{1}{x}} = e^{lim_{x\to\infty}\frac{lnx}{x}} = e^{lim_{x\to\infty}\frac{\frac{1}{x}}{1}} = e^0 = 1$.

The Formulas of Taylor and MacLaurin[480]*:* In mathematical analysis, we can study the behavior of functions by approximating them through polynomials. According to Lagrange's Mean Value Theorem,

$$f'(c) = \frac{f(b)-f(a)}{b-a} \Rightarrow f(b) - f(a) = (b-a)f'(c) \Rightarrow f(b) = f(a) + \frac{b-a}{1!}f'(c).$$

Hence, if $f: \mathbb{R} \to \mathbb{R}$ is a continuous function such that f has continuous derivatives of all orders at $x = c$, then $f(x)$ can be expanded into a power series as follows:

$$f(x) = f(c) + \frac{x-c}{1!}f'(c) + \frac{(x-c)^2}{2!}f''(c) + \cdots + \frac{(x-c)^n}{n!}f^{(n)}(c) + \cdots,$$

which is known as Taylor's Formula, and it is named after the English mathematician Brook Taylor, who stated a version of it in 1715. For $c = 0$, the aforementioned power series becomes

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \cdots,$$

which is known as MacLaurin's Formula, and it is named after the Scottish mathematician Colin MacLaurin, who discovered it in the 1730s.

Specifically, we can work as follows: Let us assume that the function $y = f(x)$ is differentiable of order $(n+1)$ on some interval containing the point $x = a$. We shall find a polynomial $y = P_n(x)$ of degree at most equal to n, the value of which at $x = a$ is equal to the value of the function $f(x)$ at this point, and the values of its derivatives up to the nth order at $x = a$ are equal to the values of the corresponding derivatives of the function $f(x)$ at this point, symbolically:

$$P_n(a) = f(x) \text{ and} \tag{1}$$

$$P_n'(a) = f'(a), P_n''(a) = f''(a), \dots, P_n^{(n)}(a) = f^{(n)}(a). \tag{2}$$

Such a polynomial is "close" to the function $f(x)$. Let us think of this polynomial as a polynomial in degrees of $(x-a)$ with undetermined coefficients, namely,

$$P_n(x) = A_0 + A_1(x-a) + A_2(x-a)^2 + \cdots + A_n(x-a)^n. \tag{3}$$

The undetermined coefficients $A_0, A_1, A_2, \dots, A_n$ must satisfy conditions (1) and (2).

[480] Ibid.

The derivatives of $P_n(x)$ are the following:

$$\left.\begin{array}{c} P_n'(x) = A_1 + 2A_2(x-a) + 3A_3(x-a)^2 + \cdots + nA_n(x-a)^{n-1} \\ P_n''(x) = 2A_2 + 3\cdot 2A_3(x-a) + \cdots + n(n-1)A_n(x-a)^{n-2} \\ \vdots \\ P_n^{(n)}(x) = n(n-1)\ldots 2\cdot 1\cdot A_n \end{array}\right\}. \tag{4}$$

If we substitute into the left-hand and the right-hand sides of (3) and (4) the value of a in place of x, and if, according to (1) and (2), we replace $P_n(a)$ with $f(a)$, $P_n'(a)$ with $f'(a)$, etc., then we obtain

$$\begin{aligned} &f(a) = A_0, \\ &f'(a) = A_1, \\ &f''(a) = 2\cdot 1A_2, \\ &f'''(a) = 3\cdot 2\cdot 1A_3, \\ &\vdots \\ &f^{(n)}(a) = n(n-1)(n-2)\ldots 2\cdot 1A_n, \end{aligned}$$

so that:

$$\begin{aligned} &A_0 = f(a), \\ &A_1 = f'(a), \\ &A_2 = \tfrac{1}{2\cdot 1}f''(a), \\ &A_3 = \tfrac{1}{3\cdot 2\cdot 1}f'''(a), \\ &\vdots \\ &A_n = \tfrac{1}{n(n-1)(n-2)\ldots 2\cdot 1}f^{(n)}(a). \end{aligned}$$

Substituting the aforementioned values of $A_0, A_1, A_2, \ldots, A_n$ into (3), we obtain the polynomial

$$P_n(x) = f(a) + \tfrac{x-a}{1}f'(a) + \tfrac{(x-a)^2}{1\cdot 2}f''(a) + \tfrac{(x-a)^3}{1\cdot 2\cdot 3}f'''(a) + \cdots + \tfrac{(x-a)^n}{1\cdot 2\cdot 3\ldots n}f^{(n)}(a).$$

Let

$$R_n(x) = f(x) - P_n(x) \Leftrightarrow f(x) = P_n(x) + R_n(x),$$

namely, $R_n(x)$ is the difference of the values of the given function $f(x)$ and of the constructed polynomial $P_n(x)$, so that:

$$f(x) = f(a) + \tfrac{x-a}{1!}f'(a) + \tfrac{(x-a)^2}{2!}f''(a) + \tfrac{(x-a)^3}{3!}f'''(a) + \cdots + \tfrac{(x-a)^n}{n!}f^{(n)}(a) + R_n(x). \tag{5}$$

The expression $R_n(x)$ is called the "remainder," and, for those values of x that the remainder is sufficiently small, the polynomial $P_n(x)$ yields a satisfactory approximation of the function $f(x)$. In other words, formula (5), that is, Taylor's Formula, enables one to replace the original function $y = f(x)$ with the polynomial $y = P_n(x)$ to an appropriate degree of accuracy equal to the remainder $R_n(x)$.

Theorem (Taylor's Formula)[481]*:* Let $f\colon [a,b] \to \mathbb{R}$ be a function such that $f^{(n-1)}(x)$ is continuous on $[a,b]$, and $f^{(n)}$ exists in (a,b). If $c \in [a,b]$, then, for every $x \in [a,b]$, there exists a number p between x and c such that

$$f(x) = f(c) + \frac{x-c}{1!} f'(c) + \frac{(x-c)^2}{2!} f''(c) + \cdots + \frac{(x-c)^{n-1}}{(n-1)!} f^{(n-1)}(c) + R_n(x),$$

where

$$R_n(x) = \frac{(x-p)^{n-m} \cdot (x-c)^m}{m(n-1)!} f^{(n)}(p), \text{ and } m \text{ is a positive integer.}$$

Proof: Notice that the continuity of $f^{(n-1)}(x)$ implies the existence of the derivatives, $f', f'', \dots, f^{(n-1)}$ and their continuity on $[a,b]$. Let J be the closed interval whose endpoints are x and c (we do not know which of these two numbers is greater than the other). Consider the function $g\colon J \to \mathbb{R}$ with

$$g(t) = f(t) + (x-t)f'(t) + \cdots + \frac{(x-t)^{n-1}}{(n-1)!} f^{(n-1)}(t) + A(x-t)^m,$$

where A is a constant, which we choose in such a way that $g(x) = g(c)$. Then

$$g(x) = f(x) \text{ and}$$
$$g(c) = f(c) + (x-c)f'(c) + \cdots + \frac{(x-c)^{n-1}}{(n-1)!} f^{(n-1)}(c) + A(x-c)^m,$$

so that

$$R_n(x) = A(x-c)^m. \qquad (*)$$

We know that

$f, f', f'', \dots, f^{(n-1)}$ are continuous on $[a,b]$,

the function g is continuous on J,

$f, f', f'', \dots, f^{(n-1)}$ and $(x-t)^m$ are differentiable on (a,b), and
$g(x) = g(c)$.

[481] Ibid.

Then, due to Rolle's Theorem, there exists a p between x and c such that $g'(p) = 0$. But

$$g'(t) = f'(t) + [-f'(t) + (x-t)f''(t)] + \left[-(x-t)f''(t) + \frac{(x-t)^2}{2}f'''(t)\right] + \cdots + \left[-\frac{(x-t)^{n-2}}{(n-2)!}f^{(n-1)}(t) + \frac{(x-t)^{n-1}}{(n-1)!}f^{(n)}(t)\right] - Am(x-t)^{m-1}.$$

Therefore, by simplifying the first term in each parenthesis with its previous one, we obtain

$$0 = g'(p) = \frac{(x-p)^{n-1}}{(n-1)!}f^{(n)}(p) - Am(x-p)^{m-1}.$$

Hence, $A = \frac{(x-p)^{n-m}}{m(n-1)!}f^{(n)}(p)$, and $(*)$ implies that

$$R_n(x) = \frac{(x-p)^{n-m}}{m(n-1)!}(x-c)^m f^{(n)}(p),$$

which proves the theorem.■

Remark: In the above theorem, $R_n(x)$ is called the "Schlömilch and Röche Remainder." In particular:

i. for $m = 1$, we obtain $R_n(x) = \frac{(x-c)(x-p)^{n-1}}{(n-1)!}f^{(n)}(p)$, which is known as the "Cauchy Remainder";
ii. for $m = n$, we obtain $R_n(x) = \frac{(x-c)^n}{n!}f^{(n)}(p)$, which is known as the "Lagrange Remainder."

Because every number p between 0 and x can be expressed as $p = kx$ where $0 < k < 1$, the aforementioned remainders can be expressed, respectively, as follows:

i. $R_n(x) = \frac{x^n(1-k)^{n-m}}{m(n-1)!}f^{(n)}(kx)$, $0 < k < 1$ (Schlömilch and Röche);
ii. $R_n(x) = \frac{x^n(1-k)^{n-1}}{(n-1)!}f^{(n)}(kx)$, $0 < k < 1$ (Cauchy);
iii. $R_n(x) = \frac{x^n}{n!}f^{(n)}(kx)$, $0 < k < 1$ (Lagrange).

Using MacLaurin's formula, we can find a power series expansion of $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^x$ at the neighborhood of the origin as follows: Because

$f(x) = e^x, f'(x) = e^x, f''(x) = e^x, \dots, f^{(n)}(x) = e^x$, and
$f(0) = 1, f'(0) = 1, f''(0) = 1, \dots, f^{(n)}(0) = 1$,

MacLaurin's series expansion implies that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

and, for $x = 1$, we obtain

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots,$$

which was originally discovered by Leonhard Euler. Therefore, $\forall x \in \mathbb{R}$,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

and, given that $f(x) = e^x$, $f^{(n)}(x) = e^x$, and $f^{(n)}(0) = 1$, it holds that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n-1}}{(n-1)!} + R_n(x),$$

where the Lagrange Remainder $R_n(x) = \frac{x^n}{n!} e^{kx}$ with $0 < k < 1$.

Furthermore, we can prove that

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots$$

is irrational in the following way, which was originally proposed by the French mathematician and physicist Joseph Fourier in 1815: Because

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

it holds that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!} e^{kx}, \text{ where } 0 < k < 1.$$

For $x = 1$, we obtain

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!} + \frac{1}{n!} e^k,\ 0 < k < 1.$$

In order to prove that e is irrational, it suffices to prove that there exist no positive integers p and q such that $e = \frac{p}{q}$. For the sake of contradiction, let

$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!} + \frac{e^k}{n!} = \frac{p}{q}$ with $n \geq q + 1$ and $n \geq 3$. Multiplying by $(n-1)!$, we obtain

$(n-1)!\frac{p}{q} - (n-1)!\left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!}\right) = \frac{e^k}{n}$.

But this equation is not valid, because the first part is an integer, whereas the second part is not an integer, since $0 < k < 1 \Rightarrow e^0 < e^k < e \Rightarrow 1 < e^k < 3$, that is, $0 < \frac{e^k}{n} < 1$. The aforementioned contradiction implies that e is an irrational number.

In addition, by defining $sinx$, $cosx$, $tanx$, and $cotx$ via exponential functions, we can verify Euler's theorem according to which $e^{ix} = cosx + isinx$, where $i = \sqrt{-1}$, and we can prove that $i^i \in \mathbb{R}$, since $i^i = \left(e^{\frac{i\pi}{2}}\right)^i = e^{i^2\frac{\pi}{2}} = e^{-\frac{\pi}{2}}$ ($i = e^{i\frac{\pi}{2}}$ derives from the representation $e^{ix} = \cos x + isinx$, which, for $x = \frac{\pi}{2}$, gives $e^{i\frac{\pi}{2}} = cos\frac{\pi}{2} + isin\frac{\pi}{2} = 0 + i \cdot 1 = i$).

2.10.3. Monotonicity, Critical Points, and Extreme Points of a Function

Theorem[482]*:* If a function $y = f(x)$ is differentiable on an interval (a, b), then:

i. f is increasing on the interval (a, b) if and only if $f'(x) \geq 0 \; \forall \; x \in (a, b)$;
ii. f is decreasing on the interval (a, b) if and only if $f'(x) \leq 0 \; \forall \; x \in (a, b)$.

Geometric significance: A differentiable function increases where its graph has positive slopes, and decreases where its graph has negative slopes.

Proof: (i) Let f be differentiable and increasing on the interval (a, b). Consider an arbitrary point $x_0 \in (a, b)$. If a function $y = f(x)$ is increasing on (a, b), then, by definition, the following conditions hold:
$\forall x \in (a, b), x > x_0 \Rightarrow f(x) > f(x_0)$, and
$\forall x \in (a, b), x < x_0 \Rightarrow f(x) < f(x_0)$.

It can be easily verified that, in both cases, the following inequality holds:

$\frac{f(x)-f(x_0)}{x-x_0} \geq 0$, where $x \neq x_0$.

In the limit $lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} \geq 0$, the left-hand side of the inequality is equal to the derivative of the function at the point x_0. Therefore, $lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} \geq 0 \Leftrightarrow f'(x_0) \geq 0$, and this inequality holds for an arbitrary point $x_0 \in (a, b)$.

Conversely, suppose that $f'(x) \geq 0 \; \forall \; x \in (a, b)$. Hence, Lagrange's Mean Value Theorem implies that, if x_1 and x_2 are two arbitrary points of the interval (a, b) with $x_1 < x_2$, then

$\exists c \in (x_1, x_2) | f(x_2) - f(x_1) = (x_2 - x_1) \cdot f'(c)$.

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[482] Ibid.

Because $f'(c) \geq 0$ and $x_2 - x_1 > 0$, it holds that $f(x_2) - f(x_1) \geq 0$. Therefore, $f(x_2) \geq f(x_1)$, meaning that f is increasing on the interval (a, b).

(ii) Its proof is similar to the proof of (i).■

Remark: If $f: A \to \mathbb{R}$ is differentiable on the interval A, then, if $f'(x) > 0$ (resp. $f'(x) < 0$) $\forall\, x \in A$, f is strictly increasing (resp. strictly decreasing) on A. Moreover, notice that, if $f'(x) = 0$ for all $x \in A$, then $f(x)$ is constant on the interval A.

Example 1: Let us investigate the monotonicity of $f(x) = x^2 - 2x + 10$. Because $f'(x) = 2x - 2$, $f(x)$ increases in $(1, +\infty)$ and decreases in $(-\infty, 1)$. Similarly, we can investigate the monotonicity of $f(x) = sinx$ as follows: given that $f'(x) = cosx$, and that $cosx$ is positive in the first and the fourth quadrants, and negative in the second and the third quadrants, $f(x) = sinx$ is increasing on the intervals $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, 2\pi\right)$ and decreasing on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

Example 2: We can prove that, if $x < y$, then $x + cosx < y + cosy$ as follows: If $f(w) = w + cosw$, then the fact that $f'(w) = 1 - sinw \geq 0$implies that $f(w)$ is increasing. Therefore, for $x < y$, $f(x) < f(y) \Rightarrow x + cosx < y + cosy$.

The points at which the derivative of a function f is equal to zero or does not exist are called the "critical points" of f.

A function $f: A \to \mathbb{R}$, where $A \subseteq \mathbb{R}$, has a "local maximum" (resp. "local minimum") at $a \in A$ if and only if, there exists a $\delta > 0$ such that $f(a) \geq f(x)$ (resp. $f(a) \leq f(x)$) for every $x \in N_\delta(a) \cap A$, that is, for every $x \in (a - \delta, a + \delta) \cap A$. Then the real number $f(a)$ is called a "local maximum" (resp. "local minimum") of f at $x = a \in A$. The local maximum and the local minimum of f are called the "local extrema," or the "local extreme values," of f. Thus, at a point $a \in A$ at which $f: A \to \mathbb{R}$ has a local extremum (i.e., a local maximum or a local minimum), the difference $f(x) - f(a)$ maintains the same sign for all x in the neighborhood $N_\delta(a) = (a - \delta, a + \delta)$. If the function $f: A \to \mathbb{R}$ is continuous, and if A is a closed interval, say $A = [u, v]$, then, by the theorems on continuous functions that were proved in section 2.7, there exist x_k and x_l in $[u, v]$ such that $f(x_k) = \min(f(x))$ and $f(x_l) = \max(f(x))$ with $x \in [u, v]$. Then $f(x_k) \leq f(x)\ \forall x \in [u, v]$, and $f(x_l) \geq f(x)\ \forall x \in [u, v]$. The value $f(x_k)$, which is the least value of f for $x \in [u, v]$, is called a "global minimum," or simply a "minimum" of f. By analogy, then, $f(x_l)$ is called a "global maximum," or simply a "maximum" of f.

A necessary condition for the existence of an extremum is determined by the following theorem:

Fermat's Extreme Value Theorem[483]*:* Assume that $f: A \to \mathbb{R}$, where $A \subseteq \mathbb{R}$, has a local extremum at $a \in A$, where a is an interior point of A. Moreover, assume that $f'(a)$ exists. Then $f'(a) = 0$.

[483] Ibid.

Proof: In fact, Fermat's Extreme Value Theorem states that, if a continuous function f has a local maximum at a point a, then , before that point, f is increasing, and, after that point, f is decreasing. By analogy, we can show that, if a continuous function f has a local minimum at a point a, then , before that point, f is decreasing, and, after that point, f is increasing. Thus, exactly at the point of a local extremum, the tangent to the graph of f is parallel to the x-axis (and, hence, its derivative is equal to zero).

We can start by assuming that a is a local maximum, and then prove that the derivative vanishes. According to the definition of a local maximum, $f(x) \leq f(a)\ \forall x \in A$, and, therefore,

$f(x) - f(a) \leq 0$. Hence:

$$\left.\begin{cases} x_n \to a \\ x_n < a \\ x_n \in A \end{cases}\right\} \Rightarrow lim_{x_n \to a} \frac{f(x_n)-f(a)}{x_n - a} \geq 0 \Leftrightarrow f'_{-}(a) \geq 0,$$

and

$$\left.\begin{cases} z_n \to a \\ z_n > a \\ z_n \in A \end{cases}\right\} \Rightarrow lim_{z_n \to a} \frac{f(z_n)-f(a)}{z_n - a} \leq 0 \Leftrightarrow f'_{+}(a) \leq 0.$$

Because f is differentiable at a, it follows that $f'_{-}(a) = f'_{+}(a) = f'(a) = 0$, since $f'_{+}(a) \leq 0$ and $f'_{-}(a) \geq 0$.

By analogy, we can prove that, if a is a local minimum, then $f'(a) = 0$.■

First Derivative Test: If $x = a$ is a critical point of $f(x)$, namely, a point at which the function is continuous, and the derivative is either equal to zero or does not exist, then $x = a$ is:

i. a local maximum of $f(x)$ if $f'(x) > 0$ to the left of $x = a$, and $f'(x) < 0$ to the right of $x = a$;
ii. a local minimum of $f(x)$ if $f'(x) < 0$ to the left of $x = a$, and $f'(x) > 0$ to the right of $x = a$;
iii. not a local extremum of $f(x)$ if $f'(x)$ is of the same sign on both sides of $x = a$.

Second Derivative Test: If $x = a$ is a critical point of $f(x)$ such that $f'(a) = 0$, then $x = a$:

i. is a local maximum of $f(x)$ if $f''(a) < 0$;
ii. is a local minimum of $f(x)$ if $f''(a) > 0$;
iii. may be a local maximum, or a local minimum, or neither if $f''(a) = 0$ (i.e., the test is inconclusive). In this case, we can use a test based on substituting into the given equation a value a little less and a value a little greater than $x = a$ and examining the direction of the graph of $f(x)$.

Hence, in order to find the relative extrema (and/or classify the critical points) of $f(x)$, we work as follows:

Step 1: We find all critical points of $f(x)$.
Step 2: We use the First Derivative Test or the Second Derivative Test on each critical point.

In order to find the global extrema of the continuous function $f(x)$ on the interval $[a, b]$, we work as follows:

Step 1: We find all critical points of $f(x)$ in $[a, b]$.
Step 2: We evaluate $f(x)$ at all points found in Step 1.
Step 3: We evaluate $f(a)$ and $f(b)$.
Step 4: We identify the global maximum (i.e., the greatest value of $f(x)$) and the global minimum (i.e., the smallest value of $f(x)$) from the evaluations obtained in Steps 2 and 3.

Example 1: Assume that, in order to transfer commodities from an area A to a different area B as shown in Figure 2.24, we have to use two different means of transportation, and that the point P denotes the corresponding transit station. The transportation cost from point A to point P is c_1 dollars per tonne per kilometer, and the transportation cost from point P to point B is c_2 dollars per tonne per kilometer.

We can find the optimum location of P, for which the total transportation cost for the transfer of one tonne of commodities from A to B is the minimum possible as follows: The total transportation cost is $c_1(a-x)+c_2\sqrt{x^2+h^2}$. Hence, the problem reduces to the computation of the minimum of the function

$$f(x)=c_1(a-x)+c_2\sqrt{x^2+h^2}, x \in [0, a].$$

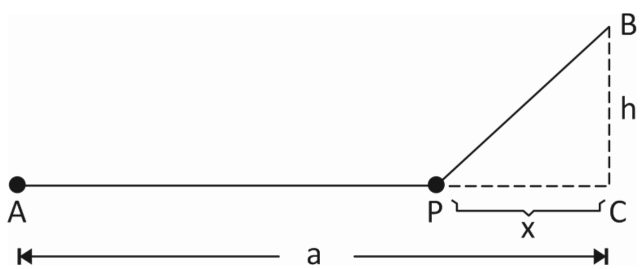

Figure 2.24. A Transportation Problem.

Then $f'(x)=-c_1+c_2 \cdot \frac{x}{\sqrt{x^2+h^2}}$, which vanishes at the point $x_0=\frac{c_1 h}{\sqrt{c_2^2-c_1^2}}$, $c_2>c_1$. It is easily verified that $f''(x_0)$ is positive. Therefore, f has a global maximum at x_0, and, in order for $x \in [0, a]$, the following condition must be satisfied:

$$c_1 h \leq a\sqrt{c_2^2-c_1^2}.$$

Example 2: We can prove that the rectangle of maximum perimeter inscribed in a given circle is a square, as shown in Figure 2.25.

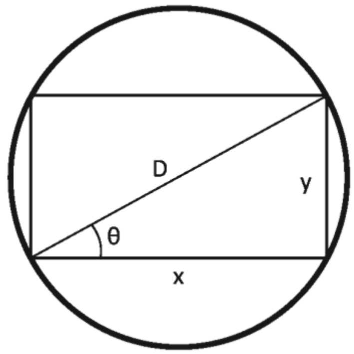

Figure 2.25. Largest Rectangle Inscribed in a Circle.

The diameter D is constant (since the circle is given). Hence,

$$x^2 + y^2 = D \Rightarrow 2x + 2yy' = 0 \Rightarrow y' = -\frac{x}{y}.$$

Regarding the perimeter P of the inscribed rectangle, we have:

$$P = 2x + 2y \Rightarrow \frac{dP}{dx} = 2 + 2y' = 0 \Rightarrow 2 + 2\left(-\frac{x}{y}\right) = 0 \Rightarrow y = x,$$

which proves that the largest rectangle inscribed in a given circle is a square.

Example 3: We can compute the shortest distance from a point $(1 + n, 0)$ to the curve $y = x^n$, where n is an arbitrary positive integer, as follows: The distance between the given point and the curve is given by

$$d = \sqrt{[x - (1 + n)]^2 + y^2} \Rightarrow d = \sqrt{[x - (1 + n)]^2 + x^{2n}}.$$

Hence, the derivative of d is

$$\frac{dd}{dx} = \frac{2[x-(1+n)]+2nx^{2n-1}}{2\sqrt{[x-(1+n)]^2+x^{2n}}} = 0 \Rightarrow x - 1 - n + nx^{2n-1} = 0,$$

so that $x + nx^{2n-1} = 1 + n$, and, therefore, $x = 1$. Consequently,

$d = \sqrt{[1 - (1 + n)]^2 + 1^{2n}} \Rightarrow d = \sqrt{n^2 + 1^{2n}}$, so that the required distance is $d = \sqrt{1 + n^2}$.

Example 4: Assume that, in order to produce x units of product A a company spends $C(x) = ax^2 + bx$ dollars, where $a, b \in \mathbb{R}$. If the product is sold at p dollars per unit, then we can compute the sales volume at which profit is maximized as follows: When this company sells x units of product A, its revenue is

$$R(x) = px,$$

and, therefore, the company's profit is

$$P(x) = R(x) - C(x) = px - (ax^2 + bx) = (p - b)x - ax^2.$$

Then $P'(x) = [(p - b)x - ax^2]' = p - b - 2ax$
$\Rightarrow P'(x) = 0 \Rightarrow p - b - 2ax = 0 \Rightarrow x = \frac{p-b}{2a}$.

Hence, $P''(x) = (p - b - 2ax)' = -2a < 0$,

and, because the second derivative is negative, the point $x = \frac{p-b}{2a}$ is the maximum point, that is, the given company will maximize its profit at this point.

Miscellaneous Examples: Maxima and minima can be used in economics in order to maximize the beneficial values, such as profit, efficiency, output of a company, etc., and in order to minimize negative values, such as expenses, efforts, etc., as well as in order to study inventory control, economic order quantity, etc. Thus, computing maxima and minima through differential calculus has important applications to linear programming and game theory, too. Furthermore, in other cases, the shape of a container may be determined by minimizing the amount of material required to manufacture it. For instance, the design of piping systems is determined by the minimization of pressure drop, which, in turn, minimizes required pump sizes and reduces cost. Moreover, the shapes of steel beams are determined by the maximization of strength.

2.10.4. Concave-Up and Concave-Down Functions

A function is said to be "concave up" (or "concave upward," or "convex") if its slope increases, as shown, for instance, in Figure 2.26 (i.e., it "opens" up). A function is said to be "concave down" (or "concave downward," or simply "concave") if its slope decreases, as shown, for instance, in Figure 2.27 (i.e., it "opens" down).

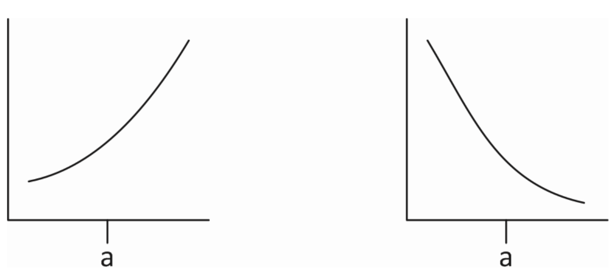

Figure 2.26. A Concave-Up Function.

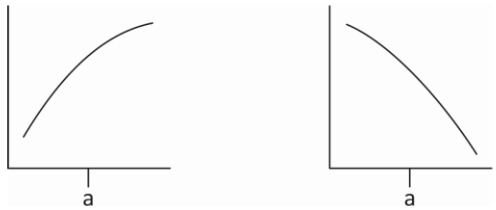

Figure 2.27. A Concave-Down Function.

Let f be a function differentiable on (a, b). (i) If f' is increasing (or if $f''(x) > 0$ on (a, b)), then f is concave up on (a, b). (ii) If f' is decreasing (or if $f''(x) < 0$ on (a, b)), then f is concave down on (a, b).

If $f: (a, b) \rightarrow \mathbb{R}$ changes its direction of concavity at x_0, then the point $(x_0, f(x_0))$ is said to be a "point of inflection." In other words, x_0 is a point of inflection if $x_0 \in (a, b)$, so that either f is concave down in (a, x_0) and concave up in (x_0, b), or f is concave up in (a, x_0) and concave down in (x_0, b).

Remark: If $f: (a, b) \rightarrow \mathbb{R}$ is a function two times differentiable and $(a, b) \in \mathbb{R}$, then the point $(x_0, f(x_0))$ is a point of inflection of f if either $f''(x_0)$ does not exists, or $f''(x_0) = 0$ and $f''(x_0 - h) \cdot f''(x_0 + h) < 0$ for $x \neq 0$ and $x_0 - h, x_0 + h \in (a, b)$.

For instance, we can determine the concavity and the points of inflection of $f(x) = 5x^3 - 2x + 7$ as follows: $f'(x) = 15x^2 - 2$, and $f''(x) = 30x = 0 \Rightarrow x = 0$. Because $30x < 0$ if $x < 0$, the graph of f is concave down for $x < 0$. Because $30x > 0$ if $x > 0$, the graph of f is concave up for $x > 0$. For $x = 0$, $f(x) = 7$, and the point $(0,7)$ is a point of inflection.

2.10.5. Asymptotes of a Function

When we study a function and, especially, when we try to sketch its graph, it is often necessary to know the behavior of the given function at a point or when it approaches some straight line. For this reason, we study the asymptotes of a function.

A straight line l is called an "asymptote" of a function f if the distance from a changing point of the curve (the graph of f) to l tends to zero when this point approaches infinity moving along a branch of the curve. There are three kinds of asymptotes:

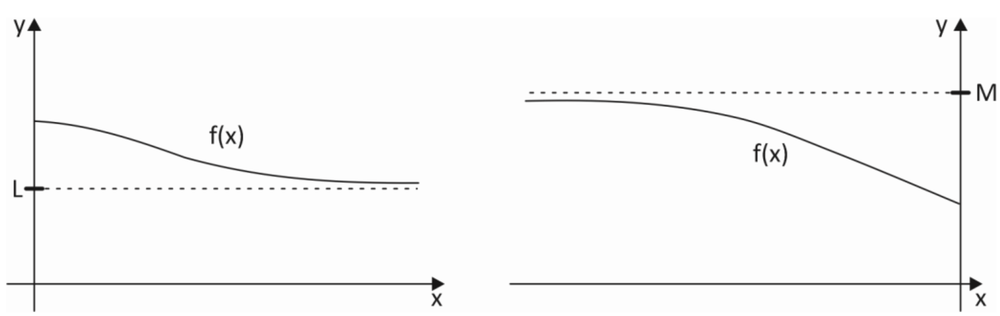

Figure 2.28. Horizontal Asymptote.

i. Horizontal asymptote, shown, for instance in Figure 2.28: a straight line $y = l$ is said to be a horizontal asymptote of a function $f(x)$ if $lim_{x\to+\infty}f(x) = lim_{x\to-\infty}f(x) = l$, provided that both $+\infty$ and $-\infty$ are accumulation points of the domain of $f(x)$.

ii. Vertical asymptote, shown, for instance, in Figure 2.29: a straight line $x = p$ such that $p \in \mathbb{R}$ is an accumulation point of the domain of a function $f(x)$ is said to be a vertical asymptote of $f(x)$ if $lim_{x\to p}f(x) = +\infty\ or\ -\infty$.

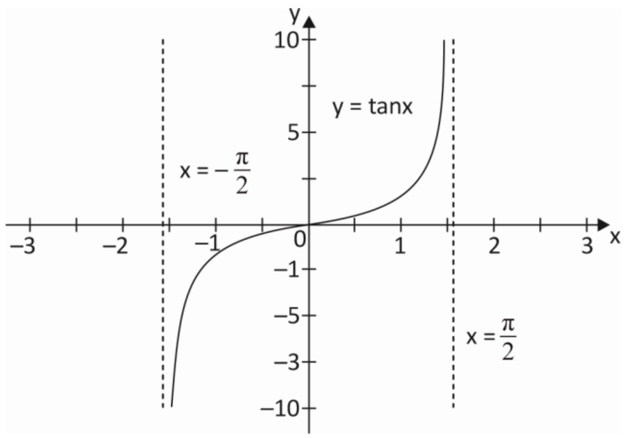

Figure 2.29. Vertical Asymptote.

iii. Oblique asymptote, as shown, for instance, in Figure 2.30: a straight line $y = ax + b$ is said to be an oblique asymptote of a function $f(x)$ if

(a) $a = lim_{x\to+\infty}\frac{f(x)}{x} and b = lim_{x\to+\infty}[f(x) - ax]$, or

(b) $a = lim_{x\to-\infty}\frac{f(x)}{x} and b = lim_{x\to-\infty}[f(x) - ax]$.

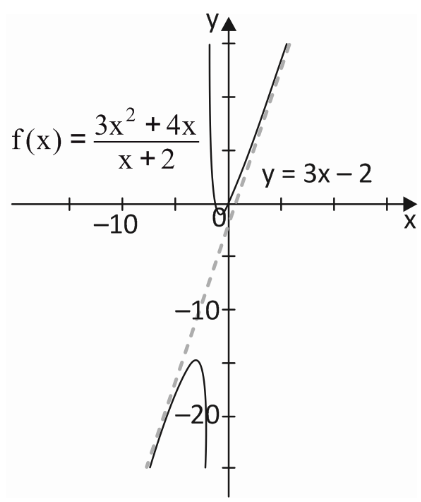

Figure 2.30. An Oblique Asymptote.

2.10.6. Steps for Function Investigation and Curve Sketching

1. Determine the domain D_f of the given function f.
2. Examine if f is continuous.
3. Examine if f is periodic, or even, or odd. Then the investigation of a function can take place in a subset of D_f. If f is periodic (i.e., if it repeats its values at regular intervals), then its investigation takes place in an one-period interval. If f is even or odd, then its investigation takes place in $D_f \cap \mathbb{R}^+$. If f is even, then it is symmetric about the y-axis. If f is odd, then its center of symmetry is the origin of the coordinate system.
4. Examine if f is two times differentiable.
5. Determine the monotonicity and the extrema of f.
6. Find the points of inflection of f and the intervals wherein f is concave up or concave down.
7. Find the asymptotes of f.

2.10.7. Curvature and Radius of Curvature[484]

By the term "curvature," we refer to the measure of how sharply a curve bends. Let us consider a plane curve defined by the equation $y = f(x)$. Moreover, let us assume that $f(x)$ has a continuous second derivative. We draw tangents to the curve at the points P and P_1 with abscissas x and $x + \Delta x$, respectively, and we denote by φ and $\varphi + \Delta\varphi$ the angles of inclination of these tangents. In particular, Figure 2.31 shows the tangent to the curve at a point $P(x, y)$, and this tangent forms an angle φ with the horizontal axis (x-axis).

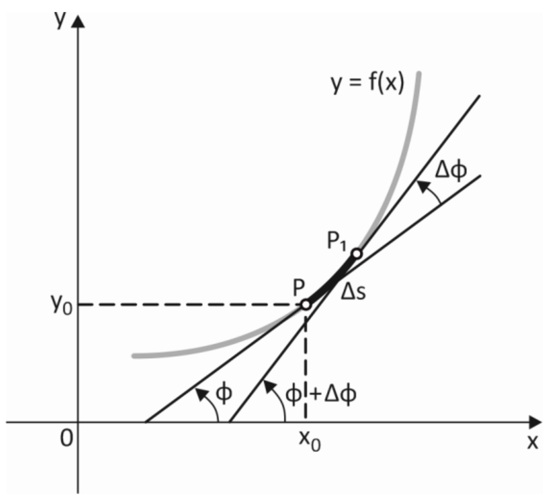

Figure 2.31. Curvature.

[484] Ibid.

As a consequence of the displacement Δs along the arc of the curve, the point P moves to the point P_1, and, therefore, the position of the tangent changes. In particular, the angle of inclination of the tangent to the positive x-axis at the point P_1 is $\varphi + \Delta\varphi$. Hence, as the point moves by the distance Δs, the tangent rotates by the angle $\Delta\varphi$, and we assume that the angle φ increases when rotating clockwise. Then the absolute value

$$\left|\frac{\Delta\varphi}{\Delta s}\right|$$

is said to be the "mean curvature" of the arc PP_1. The "curvature of the curve" at the point P is defined as

$$K = \lim_{\Delta s \to 0} \left|\frac{\Delta\varphi}{\Delta s}\right|$$

where φ, $\Delta\varphi$, and Δs are defined as previously (i.e., K characterizes the speed of rotation of the tangent to the curve at the given point). Hence, we have the following formulas:

i. If $y = f(x)$ is a plane curve, then the curvature at any point $P(x, y)$ is expressed in terms of the first and the second derivatives of the function $f(x)$ by the formula $K = \frac{|f''(x)|}{[1+(f'(x))^2]^{\frac{3}{2}}}$.
ii. If a curve is defined in parametric form by the equations $x = x(t)$ and $y = y(t)$, then its curvature at an arbitrary point $P(x, y)$ is given by the formula $K = \frac{|x'y''-y'x''|}{[(x')^2+(y')^2]^{\frac{3}{2}}}$.
iii. If a curve is given by the polar equation $r = r(\theta)$, the curvature is given by the formula $K = \frac{|r^2+2(r')^2-rr''|}{[r^2+(r')^2]^{\frac{3}{2}}}$.

The "radius of curvature" of a curve at a point $P(x, y)$ is the inverse of the curvature K of the curve at the given point, namely:

$$R = \frac{1}{K}.$$

Therefore, if $y = f(x)$ is a plane curve, then the radius of curvature at any point $P(x, y)$ is expressed in terms of the first and the second derivatives of the function $f(x)$ by the formula

$$R = \frac{[1+(f'(x))^2]^{\frac{3}{2}}}{|f''(x)|},$$

and R characterizes the radius of the circular arc that best approximates the curve at the given point.

Notice that: (i) positive curvature means that the contour will be convergent, because it is concave up (or convex), and, thus, if we have a triangle in positive curvature (e.g., in the case of geometry on the sphere), the sum of its angles is greater than π (180°); (ii) negative curvature means that the contour will be divergent, because it is concave down (or simply concave), and, thus, if we have a triangle in negative curvature (e.g., in the case of hyperbolic geometry), the sum of its angles is less than π (180°); and (iii) zero curvature means that the contour is straight, that is, neither convergent nor divergent, and, thus, if we have a triangle in zero curvature (e.g., in the case of Euclidean geometry), the sum of its angles is equal to π (180°).

2.10.8. Differentiation of Multivariable Functions[485]

So far, we have studied exclusively functions of a single (independent) variable x, but we can also apply the concept of differentiation to functions of several variables $x, y, \ldots$ Notice that, if $f: A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^n$, is a function, then its "limit" at the point $P_0\left(x_{1_0}, x_{2_0}, \ldots, x_{n_0}\right) \in D(A)$, where $D(A)$ is the derived set of A (i.e., the set containing all the accumulation points of A), is the real number L if and only if, $\forall \varepsilon > 0$, there exist $\delta_i = \delta_i(\varepsilon) > 0$, where $i = 1,2, \ldots, n$, so that, $\forall P(x_1, x_2, \ldots, x_n) \in A$ with $0 < \left|x_i - x_{i_0}\right| < \delta_i$, where $i = 1,2, \ldots, n$, the following condition is satisfied: $|f(x_1, x_2, \ldots, x_n) - L| < \varepsilon$; and then we write

$$lim_{P \to P_0} f(P) = L \equiv lim_{(x_1, x_2, \ldots, x_n) \to (x_{1_0}, x_{2_0}, \ldots, x_{n_0})} f(x_1, x_2, \ldots, x_n) = L.$$

A function $f: A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^n$, is said to be continuous at the point $P_0 \in A \cap D(A)$ (where $D(A)$ is the derived set of A) if $lim_{P \to P_0} f(P) = f(P_0)$. The properties of the limit and the continuity of multivariable functions are analogous to the properties of the limit and the continuity of functions of one variable.

Suppose that $f(x, y)$ is a function of two real variables x and y, and that the limits

$$lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} \text{ and } lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

exist for all values of x and y in question, so that $f(x, y)$ possesses a derivative $\frac{df}{dx}$ with respect to x and a derivative $\frac{df}{dy}$ with respect to y. These derivatives are called the "partial derivatives" of f, and they are denoted by

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \text{ or}$$
$$f_x(x, y), f_y(x, y).$$

[485] Ibid.

In general, when calculating partial derivatives, we treat all independent variables other than the variable with respect to which we differentiate as constants. For instance, if $f(x,y) = x^2 - 3xy - 5$, then

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 - 3xy - 5) = \frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial x}(3xy) - \frac{\partial}{\partial x}(5) = 2x - 3y, \text{ and}$$
$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 - 3xy - 5) = \frac{\partial}{\partial y}(x^2) - \frac{\partial}{\partial y}(3xy) - \frac{\partial}{\partial y}(5) = -3x.$$

Partial differentiation of $f: \mathbb{R}^2 \to \mathbb{R}$ at $P(x_0, y_0)$:

$$\frac{\partial f(x,y)}{\partial x}\Big|_{(x_0,y_0)} = lim_{x\to x_0} \frac{f(x,y_0) - f(x_0,y_0)}{x - x_0},$$
$$\frac{\partial f(x,y)}{\partial y}\Big|_{(x_0,y_0)} = lim_{y\to y_0} \frac{f(x_0,y) - f(x_0,y_0)}{y - y_0}.$$

Given a function $f: \mathbb{R}^3 \to \mathbb{R}$ of three variables x, y, and z, the partial derivative of f with respect to x is defined as

$$\frac{\partial f(x,y,z)}{\partial x} = lim_{\Delta x\to 0} \frac{f(x+\Delta x,y,z) - f(x,y,z)}{\Delta x},$$

the partial derivative of f with respect to y is defined as

$$\frac{\partial f(x,y,z)}{\partial y} = lim_{\Delta y\to 0} \frac{f(x,y+\Delta y,z) - f(x,y,z)}{\Delta y},$$

and , the partial derivative of f with respect to z is defined as

$$\frac{\partial f(x,y,z)}{\partial z} = lim_{\Delta z\to 0} \frac{f(x,y,z+\Delta z) - f(x,y,z)}{\Delta z}.$$

Generalization: If $f: \mathbb{R}^n \to \mathbb{R}$ is a function, that is,

$$\mathbb{R}^n \ni (x_1, x_2, \dots, x_n) \to f(x_1, x_2, \dots, x_n) \in \mathbb{R},$$

then

$$\frac{\partial f(x_1, x_2, \dots, x_i, \dots, x_n)}{\partial x_i}\Big|_{(x_{1_0},x_{2_0},\dots,x_{i_0},\dots,x_{n_0})}$$
$$= lim_{\Delta x_i\to 0} \frac{f\big(x_{1_0}, x_{2_0}, \dots, x_{i_0} + \Delta x, \dots, x_{n_0}\big) - f\big(x_{1_0}, x_{2_0}, \dots, x_{i_0}, \dots, x_{n_0}\big)}{\Delta x_i}$$

is the partial derivative of $f(x_1, x_2, \dots, x_n)$ with respect to x_i, where $i = 1,2, \dots, n$, at the point $\big(x_{1_0}, x_{2_0}, \dots, x_{i_0}, \dots, x_{n_0}\big)$.

Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a function defined over D_f, and let f be differentiable at (x_0, y_0). Then $\frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}$ and $\frac{\partial f}{\partial y}\Big|_{(x_0,y_0)}$ exist. Moreover,

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x^2},$$
$$\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right)=\frac{\partial^3 f}{\partial x^3},$$
$$\vdots$$
$$\frac{\partial}{\partial x}\left(\frac{\partial^{n-1} f}{\partial x^{n-1}}\right)=\frac{\partial^n f}{\partial x^n}.$$

By analogy,

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y^2},$$
$$\frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y}\right)=\frac{\partial^3 f}{\partial y^3},$$
$$\vdots$$
$$\frac{\partial}{\partial y}\left(\frac{\partial^{n-1} f}{\partial y^{n-1}}\right)=\frac{\partial^n f}{\partial y^n}.$$

Notice that $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$ can be written as $f_{xx}(x,y)$ and $f_{yy}(x,y)$, respectively. Moreover, $f_{xy}\equiv\frac{\partial^2 f}{\partial y\partial x}\equiv\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$, and $f_{yx}\equiv\frac{\partial^2 f}{\partial x\partial y}\equiv\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$.

The geometric significance of $\frac{\partial f}{\partial x}|_{(x_0,y_0)}$ and $\frac{\partial f}{\partial y}|_{(x_0,y_0)}$ is illustrated in Figure 2.32. Let us consider a function $z=f(x,y)$, whose graph in $\mathbb{R}^3$ is a surface. We suppose that $P(x_0,y_0)$ is an arbitrary point of the domain of f. Notice that, in $\mathbb{R}^3$, the equation $y=y_0$ represents a plane Π that is perpendicular to the y-axis. This plane intersects the surface $z=f(x,y)$ by a curve C whose equation is $z=f(x,y_0)$. If $Q(x_0,y_0,z_0)$ is a point belonging to C, so that its orthogonal projection to the plane xOy is the point P, then the slope of the tangent to the curve C at Q is equal to $\frac{\partial f}{\partial x}|_{(x_0,y_0)}=tan\varphi$, where φ is the angle formed by the x-axis and the tangent to the curve C at Q, as shown in Figure 2.32. In the same way, we can show that the slope of the tangent to the curve C at Q is equal to $\frac{\partial f}{\partial y}|_{(x_0,y_0)}=tan\theta$, where θ is the angle formed by the y-axis and the tangent to the curve C at Q.

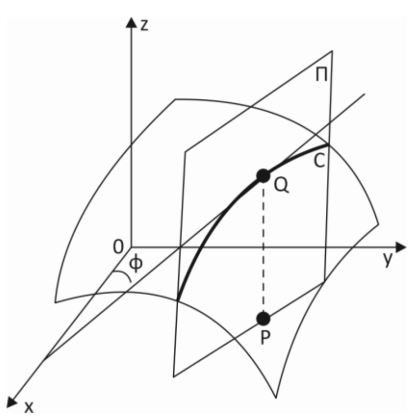

Figure 2.32. Partial Derivative $\frac{\partial f}{\partial x}|_{(x_0,y_0)}=tan\varphi$.

Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a function of two real variables such that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist. Then the "total differential" of $f(x, y)$ is denoted by df, and it is defined by

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy,$$

meaning that any infinitely small change in (x, y) consists of a change dx in x and a change dy in y, so that the change in $f(x, y)$ resulting from the change in x is $\frac{\partial f}{\partial x} dx$, the change in $f(x, y)$ resulting from the change in y is $\frac{\partial f}{\partial y} dy$, and the total change in $f(x, y)$ resulting from the change in (x, y) is $\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$.

In general, if $f: \mathbb{R}^n \to \mathbb{R}$ is a function of n real variables, $x_1, x_2, \dots, x_n$, then the total differential of f is

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \dots + \frac{\partial f}{\partial x_n} dx_n,$$

provided that $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}$ exist.

If $f(x_1, x_2, \dots, x_n) = c$ for all $x_1, x_2, \dots, x_n$ belonging to the domain of f, and if $c \in \mathbb{R}$, then $df = 0$. For instance, the 1-sphere $S^1 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$ is a constant function of 2 variables, and the 2-sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$ is a constant function of 3 variables.

The mixed partial derivatives f_{xy} and f_{yx} are equal to each other in the following cases:

i. if f, f_x, f_y, and f_{xy} are continuous in the neighborhood of a point of the domain of f; or
ii. if f_{xy} and f_{yx} are continuous in the neighborhood of a point of the domain of f.

Differentiation of Composite Functions, Harmonic Functions, and Homogeneous Functions[486]

If $z = f(x, y)$ is a differentiable function such that x and y are functions of two variables r and s, namely, $x = g(r, s)$ and $y = h(r, s)$, then it holds that

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \text{ and}$$
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}.$$

By analogy, we can generalize for a function $z = f(x_1, x_2, \dots, x_n)$, whose variables are functions of k variables, namely,$x_1 = g_1(r_1, r_2, \dots, r_k), \dots, x_n = g_n(r_1, r_2, \dots, r_k)$, as follows:

$$\frac{\partial z}{\partial r_i} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial r_i} + \frac{\partial z}{\partial x_2}\frac{\partial x_2}{\partial r_i} + \dots + \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial r_i}, \text{ where } i = 1,2, \dots, k.$$

[486] Ibid.

Obviously, in case $x_1, x_2, \dots, x_n$ depend on only one variable r,

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial r} + \cdots + \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial r}.$$

For instance, if $z = x^2 + y^2$, $x = rcos\theta$, and $y = rsin\theta$, then

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = 2xcos\theta + 2ysin\theta = 2(rcos\theta)cos\theta + 2(rsin\theta)sin\theta =$$
$2r(cos^2\theta + sin^2\theta) = 2r$, and
$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = 2x(-rsin\theta) + 2y(rcos\theta) = -2r^2 sin\theta cos\theta + 2r^2 sin\theta cos\theta = 0.$$

Let $f(x,y)$ be a function where $x = x(r)$ and $y = y(r)$. Then it holds that

$$\frac{\partial^2 f}{\partial r^2} = \frac{\partial^2 f}{\partial x^2}\left(\frac{dx}{dr}\right)^2 + 2\frac{\partial^2 f}{\partial x \partial y}\frac{dx}{dr}\frac{dy}{dr} + \frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dr}\right)^2 + \frac{\partial f}{\partial x}\frac{d^2 f}{dr^2} + \frac{\partial f}{\partial y}\frac{d^2 y}{dr^2} = \left(\frac{\partial f}{\partial x}\frac{dx}{dr} + \frac{\partial f}{\partial y}\frac{dy}{dr}\right)^2 + \frac{\partial f}{\partial x}\frac{d^2 x}{dr^2} + \frac{\partial f}{\partial y}\frac{d^2 y}{dr^2}.$$

Similarly, if $f(x,y)$ is a function with $x = x(r,s)$ and $y = y(r,s)$, it holds that

$$\frac{\partial^2 f}{\partial r^2} = \frac{\partial^2 f}{\partial x^2}\left(\frac{\partial x}{\partial r}\right)^2 + 2\frac{\partial^2 f}{\partial x \partial y}\frac{\partial x}{\partial r}\frac{\partial y}{\partial r} + \frac{\partial^2 f}{\partial y^2}\left(\frac{\partial y}{\partial r}\right)^2 + \frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial r^2} + \frac{\partial f}{\partial y}\frac{\partial^2 y}{\partial r^2} = \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}\right)^2 + \frac{\partial f}{\partial x}\frac{\partial^2 x}{\partial r^2} + \frac{\partial f}{\partial y}\frac{\partial^2 y}{\partial r^2},$$

and

$$\frac{\partial^2 f}{\partial s^2} = \frac{\partial^2 f}{\partial x^2}\left(\frac{\partial x}{\partial s}\right)^2 + 2\frac{\partial^2 f}{\partial x \partial y}\frac{\partial x}{\partial s}\frac{\partial y}{\partial s} + \frac{\partial^2 f}{\partial y^2}\left(\frac{\partial y}{\partial s}\right)^2 + \frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial s^2} + \frac{\partial f}{\partial y}\frac{\partial^2 y}{\partial s^2} = \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}\right)^2 + \frac{\partial f}{\partial x}\frac{\partial^2 x}{\partial s^2} + \frac{\partial f}{\partial y}\frac{\partial^2 y}{\partial s^2}.$$

Moreover,

$$\frac{\partial^2 f}{\partial r \partial s} = \frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial r}\frac{\partial x}{\partial s} + \frac{\partial^2 f}{\partial x \partial y}\left(\frac{\partial x}{\partial r}\frac{\partial y}{\partial s} + \frac{\partial x}{\partial s}\frac{\partial y}{\partial r}\right) + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial r}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial x}\frac{\partial^2 x}{\partial r \partial s} + \frac{\partial f}{\partial y}\frac{\partial^2 y}{\partial r \partial s}.$$

A function $f(x,y)$ of two arbitrary real variables x and y is said to be “harmonic” if its value at any point is equal to the average of its value along any circle around that point, provided that $f(x,y)$ is defined within the circle. For instance, in physics, harmonic functions describe those conditions of equilibrium such as the temperature of electric charge distribution over a region in which the value at each point is constant. In general, given a function $f: \mathbb{R}^n \to \mathbb{R}$ of n real variables $x_1, x_2, \dots, x_n$, we say that $f: \mathbb{R}^n \to \mathbb{R}$ is a “harmonic function” if it satisfies the “equation of Laplace,” namely: $\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = 0$.

For instance, the function $f(x,y) = x^2 - y^2$ is harmonic, because: $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial^2 f}{\partial x^2} = 2$, $\frac{\partial f}{\partial y} = -2y$, and $\frac{\partial^2 f}{\partial y^2} = -2$, so that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$.

If a function is such that, whenever all its arguments are multiplied by a factor, its value is multiplied by some power of this factor, then it is said to be "homogeneous." Symbolically: If, for a parameter λ and a constant n,

$$f(\lambda x_1, \lambda x_2, \dots, \lambda x_n) = \lambda^n f(x_1, x_2, \dots, x_n),$$

then $f: \mathbb{R}^n \to \mathbb{R}$ is a "homogeneous function of degree n."

Euler's Theorem on Homogeneous Functions[487]*:* If $f(x_1, x_2, \dots, x_n)$ is a homogeneous function of degree n, then

$$x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + \dots + x_n \frac{\partial f}{\partial x_n} = nf.$$

Proof: Let us prove that, if $f(\lambda x, \lambda y) = \lambda^n f(x,y)$, then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$. Set $\lambda x = a$ and $\lambda y = b$, so that

$$f(a,b) = \lambda^n f(x,y) \Rightarrow \frac{\partial f(a,b)}{\partial \lambda} = \frac{\partial f}{\partial a}\frac{\partial a}{\partial \lambda} + \frac{\partial f}{\partial b}\frac{\partial b}{\partial \lambda} = \frac{\partial f}{\partial a}x + \frac{\partial f}{\partial b}y.$$

Moreover, $\frac{\partial[\lambda^n f(x,y)]}{\partial \lambda} = n\lambda^{n-1} f(x,y)$.

Taking advantage of the fact that λ is an arbitrary real number, we can choose an element that facilitates our calculations. In particular, if $\lambda = 1$, then $n\lambda^{n-1} f(x,y)$ implies that $a = x$ and $b = y$, so that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$.■

For instance, we can prove that $f(x,y) = x^3 y ln\frac{x}{y} \Rightarrow x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 4f(x,y)$ as follows: Because $f(\lambda x, \lambda y) = \lambda^3 x^3 \lambda y ln\frac{x}{y} = \lambda^4 x^3 y ln\frac{x}{y} = \lambda^4 f(x,y)$. The given expression is just a direct consequence of Euler's theorem on homogeneous functions.

Differentiation of Implicit Functions[488]

A function of the form $f(x,y,z)$ where $z = g(x,y)$, that is, $f[x,y,g(x,y)]$, is called "implicit," while $z = g(x,y)$ is an explicit function. An equation of the form $F(x,y) = 0$ determines an implicit function $y = f(x)$ in $A \subseteq \mathbb{R}$ if and only if, $\forall x \in A$, $F[x, f(x)] = 0$.

Remarks: Let $F(x,y) = 0$ be an equation defined in $A \subseteq \mathbb{R}^2$, and let (x_0, y_0) be a point of A. We assume that F_x and F_y exist and are continuous, $F(x_0, y_0) = 0$, and $F_y(x_0, y_0) \neq 0$. Then there exists a neighborhood of x_0, say $N(x_0)$, in the interior of which the equation

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[487] Ibid.
[488] Ibid.

$F(x,y)=0$ determines a continuous function $y=f(x)$ in a unique way, so that F_x is continuous, $y_0=f(x_0)$, and $\frac{\partial y}{\partial x}=-\frac{F_x}{F_y}$. Notice that:

$$\frac{\partial^2 y}{\partial x^2}=-\frac{F_{xx}F_y^2-2F_{xy}F_xF_y+F_{yy}F_x^2}{F_y^3}.$$

For instance, using the aforementioned formula, we can find the second-order derivative of the function $y=y(x)$ that is determined by the equation $F(x,y)=y^3-yx=0$, by calculating $F_x=-y$, $F_y=3y^2-x$, $F_{xx}=0$, $F_{yy}=6y$, and $F_{xy}=-1$, so that $\frac{\partial^2 y}{\partial x^2}=-\frac{F_{xx}F_y^2-2F_{xy}F_xF_y+F_{yy}F_x^2}{F_y^3}=\frac{2y(3y^2-x)-6y^3}{(3y^2-x)^3}$.

An equation of the form $F(x,y,z)=0$ determines an implicit function $z=z(x,y)$ in $A\subseteq\mathbb{R}^2$ if and only if, $\forall(x,y)\in A$, it holds that $F[x,y,f(x,y)]=0$. For instance, the equation $x^2+y^2-z=0$ determines the implicit function $z=x^2+y^2$.

Remarks: Let $F(x,y,z)=0$ be an equation defined in $A\subseteq\mathbb{R}^3$, and let (x_0,y_0,z_0) be a point of the interior of A. We assume that F_x, F_y, and F_z exist and are continuous, $F(x_0,y_0,z_0)=0$, and $F_z(x_0,y_0,z_0)\neq 0$. Then there exists a neighborhood of (x_0,y_0), say $N(x_0,y_0)$, in the interior of which the equation $F(x,y,z)=0$ determines a continuous implicit function $z=z(x,y)$ in a unique way, so that F_x and F_y are continuous and $z_0=z(x_0,y_0)$. Moreover, notice that:

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z},\text{ and }\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}. \qquad (*)$$

The points (x_0,y_0,z_0) of the surface $F(x,y,z)=0$ for which $\frac{\partial F}{\partial z}|_{(x_0,y_0,z_0)}\neq 0$ and $F(x_0,y_0,z_0)=0$ are called "ordinary points." The points of the surface $F(x,y,z)=0$ for which $F_x=F_y=F_z=0$ are called "singular points," and, obviously, $(*)$ does not hold at singular points. The total differential of an implicit function $z=z(x,y)$ that is determined by the equation $F(x,y,z)=0$ is given by

$$\frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial y}dy+\frac{\partial F}{\partial z}dz=0.$$

Jacobian (or Functional) Determinant[489]

Let f and g be functions of x and y connected by an identical relation

$$h(f,g)=0. \qquad (*)$$

Differentiating $(*)$ with respect to x and y, we obtain

$$\frac{\partial h}{\partial f}\frac{\partial f}{\partial x}+\frac{\partial h}{\partial g}\frac{\partial g}{\partial x}=0\text{ and }\frac{\partial h}{\partial f}\frac{\partial f}{\partial y}+\frac{\partial h}{\partial g}\frac{\partial g}{\partial y}=0,$$

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[489] Ibid.

so that, eliminating the derivatives of h, we obtain

$$J = \begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix} = f_x g_y - f_y g_x = 0,$$

where f_x, g_x, f_y, g_y are the derivatives of f and g with respect to x and y. The aforementioned condition is necessary and sufficient for the existence of a relation such as $(*)$. Two functions f and g are said to be "dependent" or "independent" according as they are or are not connected by such a relation as $(*)$, respectively. It is usual to call J the "Jacobian determinant" (or the "functional determinant") of f and g with respect to x and y, and to write

$$J = \frac{\partial(f,g)}{\partial(x,y)}.$$

The Jacobian determinant of n functions, $f_1, f_2, \dots, f_n$, of n real variables, $x_1, x_2, \dots, x_n$, with respect to $x_1, x_2, \dots, x_n$ is defined by

$$J = \frac{\partial(f_1, f_2, \dots, f_n)}{\partial(x_1, x_2, \dots, x_n)} = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix}.$$

It is named after the nineteenth-century German mathematician Carl Gustav Jacob Jacobi. Notice that, if $x_1 = x_1(r_1, r_2, \dots, r_n), \dots, x_n = x_n(r_1, r_2, \dots, r_n)$, then

$$\frac{\partial(f_1, f_2, \dots, f_n)}{\partial(r_1, r_2, \dots, r_n)} = \frac{\partial(f_1, f_2, \dots, f_n)}{\partial(x_1, x_2, \dots, x_n)} \cdot \frac{\partial(x_1, x_2, \dots, x_n)}{\partial(r_1, r_2, \dots, r_n)}.$$

Jacobians are useful in order to compute the partial derivatives of implicit functions:

Case I: Let $\begin{Bmatrix} f(x,y,z) = 0 \\ g(x,y,z) = 0 \end{Bmatrix}$ be a system such that

the functions f and g have continuous fist-order partial derivatives, $f(x_0, y_0, z_0) = 0 = g(x_0, y_0, z_0)$, and $\frac{\partial(f,g)}{\partial(y,z)}\big|_{(x_0,y_0,z_0)} \neq 0$. Every system of this form has a unique solution $y = y(x)$ and $z = z(x)$, where $y(x)$ and $z(x)$ are two functions whose derivatives (with respect to x) are continuous on a neighborhood of x_0, so that $y_0 = y(x_0)$ and $z_0 = z(x_0)$. Then

$$\frac{dy}{dx} = -\frac{\frac{\partial(f,g)}{\partial(x,z)}}{\frac{\partial(f,g)}{\partial(y,z)}} \text{ and } \frac{dz}{dx} = -\frac{\frac{\partial(f,g)}{\partial(y,x)}}{\frac{\partial(f,g)}{\partial(y,z)}}.$$

Equivalently, we can write:

$$\frac{dx}{\begin{vmatrix} f_y & f_z \\ g_y & g_z \end{vmatrix}} = \frac{dy}{\begin{vmatrix} f_z & f_x \\ g_z & g_x \end{vmatrix}} = \frac{dz}{\begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix}}.$$

Case II: Let $\begin{Bmatrix} f(x,y,z,t) = 0 \\ g(x,y,z,t) = 0 \end{Bmatrix}$ be a system such that

the functions f and g have continuous fist-order partial derivatives, $f(x_0, y_0, z_0, t_0) = 0 = g(x_0, y_0, z_0, t_0)$, and $\frac{\partial(f,g)}{\partial(z,t)}|_{(x_0,y_0,z_0,t_0)} \neq 0$. Every system of this form has a unique solution $z = z(x,y)$ and $t = t(x,y)$, where $z(x,y)$ and $t(x,y)$ are two functions whose partial derivatives are continuous on a neighborhood of (x_0, y_0), so that $z_0 = z(x_0, y_0)$ and $t_0 = t(x_0, y_0)$. Then

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial(f,g)}{\partial(x,t)}}{\frac{\partial(f,g)}{\partial(z,t)}}, \frac{\partial z}{\partial y} = -\frac{\frac{\partial(f,g)}{\partial(y,t)}}{\frac{\partial(f,g)}{\partial(z,t)}}, \frac{\partial t}{\partial x} = -\frac{\frac{\partial(f,g)}{\partial(z,x)}}{\frac{\partial(f,g)}{\partial(z,t)}}, \text{ and } \frac{\partial t}{\partial y} = -\frac{\frac{\partial(f,g)}{\partial(z,y)}}{\frac{\partial(f,g)}{\partial(z,t)}}.$$

Mean Value Theorems[490]

First Mean Value Theorem[491]*:* If a function $f: A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^2$, is differentiable at the points of the straight line segment $\overline{ab}$, where $a = (a_1, a_2)$ and $b = (b_1, b_2)$, then there exists a number θ with $0 < \theta < 1$ such that

$$f'[a + \theta(b - a)] = \frac{f(b) - f(a)}{b - a},$$

where f' is the partial derivative of $f(x,y)$ with respect to x.

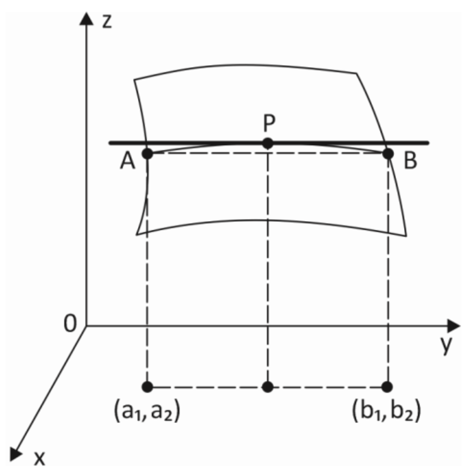

Figure 2.33. Mean Value Theorem.

[490] Ibid.
[491] Ibid.

Geometric interpretation: As shown in Figure 2.33, if $A\big(a_1, a_2, f(a_1, a_2)\big)$ and $B\big(b_1, b_2, f(b_1, b_2)\big)$ are two points of the curve $z = f(x, y)$ that correspond to points a and b, then there exists a point P of the curve

$$\left\{\begin{matrix} x = a_1 + t(b_1 - a_1) \\ y = a_2 + t(b_2 - a_2) \\ z = f\big(a + t(b - a)\big) \end{matrix}\right\}, \text{ where } t \in [0,1],$$

of the surface $z = f(x, y)$ such that the tangent at P is parallel to the chord AB.

Equivalent formulation: If a function $f: \mathbb{R}^2 \to \mathbb{R}$ is differentiable, $X_0 = (x_0, y_0)$, and $X = (x_0 + h, y_0 + k)$, then there exists a C that lies on the line joining X_0 and X such that

$$f(X) = f(X_0) + f'(C)(X - X_0),$$

namely, there exists a $c \in (0,1)$ such that

$$f(x_0 + h, y_0 + k) = f(x_0, y_0) + hf_x(C) + kf_y(C),$$

where $C = (x_0 + ch, y_0 + ck)$.

Proof: Let $F: [0,1] \to \mathbb{R}$ be defined by

$$F(t) = f(x_0 + th, y_0 + tk), \text{ where } t \in [0,1].$$

Thus, $F(t)$ is differentiable, and

$$F'(t) = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = hf_x + kf_y.$$

According to Lagrange's Mean Value Theorem, there exists a $c \in (0,1)$ such that

$$F(1) - F(0) = F'(c),$$

which proves the theorem.■

Remark: The result can be extended to any number of variables.

Taylor's Theorem of the Mean for Multivariable Functions[492]*:* If all the nth partial derivatives of $f(x, y)$ are continuous in a closed region, and the $(n + 1)$st partial derivatives exist in the open region, namely, in some neighborhood of a point (x_0, y_0) in the domain of f, then

[492] Ibid.

$$f(x_0+h,y_0+k)=f(x_0,y_0)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x_0,y_0)+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x_0,y_0)+\cdots+\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(x_0,y_0)+R_n,$$

where R_n is the remainder given by

$$R_n=\frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1} f(x_0+\theta h,y_0+\theta k), \text{ where } \theta\in(0,1).$$

Proof: For simplicity, let $n=2$ (i.e., terms up to order 3). Moreover, let $x=x_0+th$, and $y=y_0+tk$, where $t\in[0,1]$. We define a function

$F(t)=f(x_0+th,y_0+tk)$, so that
$$F'(t)=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x_0+th,y_0+tk).$$

Similarly, we can compute the second-order derivative:

$$F''(t)=h\left(\frac{\partial^2 f}{\partial x^2}h+\frac{\partial^2 f}{\partial y\partial x}k\right)+k\left(\frac{\partial^2 f}{\partial x\partial y}h+\frac{\partial^2 f}{\partial y^2}k\right),$$

since h corresponds to $\frac{dx}{dt}$, k corresponds to $\frac{dy}{dt}$, $\left(\frac{\partial^2 f}{\partial x^2}h+\frac{\partial^2 f}{\partial y\partial x}k\right)=\frac{d}{dt}\left(\frac{\partial f}{\partial x}\right)$, and $\left(\frac{\partial^2 f}{\partial x\partial y}h+\frac{\partial^2 f}{\partial y^2}k\right)=\frac{d}{dt}\left(\frac{\partial f}{\partial y}\right)$. Hence,

$$F''(t)=h^2\frac{\partial^2 f}{\partial x^2}+2hk\frac{\partial^2 f}{\partial x\partial y}+k^2\frac{\partial^2 f}{\partial y^2}=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x_0+th,y_0+tk).$$

Similarly, we can compute the third-order derivative:
$$F'''(t)=h^2\left(\frac{\partial^3 f}{\partial x^3}h+\frac{\partial^3 f}{\partial y\partial x^2}k\right)+2hk\left(\frac{\partial^3 f}{\partial x^2\partial y}h+\frac{\partial^3 f}{\partial x\partial y^2}k\right)+k^2\left(\frac{\partial^3 f}{\partial x\partial y^2}h+\frac{\partial^3 f}{\partial y^3}k\right)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^3 f(x_0+th,y_0+tk).$$

By Taylor's Theorem for $F(t)$ about 0, we have
$$F(t)=F(0)+tF'(0)+\frac{t^2}{2!}F''(0)+\frac{t^3}{3!}F'''(\theta t), \text{ where } \theta\in(0,1).$$

Thus,

$$F(1)=F(0)+F'(0)+\frac{1}{2!}F''(0)+\frac{1}{3!}F'''(\theta), \text{ where } \theta\in(0,1).$$

In view of the foregoing results, we obtain:

$f(x_0+h,y_0+k)=f(x_0,y_0)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x_0,y_0)+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x_0,y_0)+\frac{1}{3!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^3 f(x_0+\theta h,y_0+\theta k)$.

Consequently, in general (inductively), we have:

$f(x_0+h,y_0+k)=f(x_0,y_0)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x_0,y_0)+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x_0,y_0)+\cdots+\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(x_0,y_0)+\frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1} f(x_0+\theta h,y_0+\theta k)$, where $\theta\in(0,1)$.■

Remark: Equivalently, we can write:

$f(x,y)=f(x_0,y_0)+\left((x-x_0)\frac{\partial}{\partial x}+(y-y_0)\frac{\partial}{\partial y}\right)f(x_0,y_0)+\cdots+\frac{1}{(n+1)!}\left((x-x_0)\frac{\partial}{\partial x}+(y-y_0)\frac{\partial}{\partial y}\right)^{n+1} f\big(x_0+\theta(x-x_0),y_0+\theta(y-y_0)\big)$,

where $\theta\in(0,1)$. If $lim_{n\to\infty}R_n=0\ \forall(x,y)$ in a region, then we can obtain the infinite series expansion of the function $f(x,y)$ in powers of $(x-x_0)$ and $(y-y_0)$ convergent in this region. Notice that, if we want to write the infinite series expansion of the function $f(x,y)$ in the neighborhood of the point $(x_0,y_0)=(0,0)$, then we apply MacLaurin's formula:

$f(x,y)=f(0,0)+\frac{1}{1!}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)f(0,0)+\cdots+\frac{1}{n!}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^n f(0,0)+R_n$,
$R_n=\frac{1}{(n+1)!}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^{n+1} f(\theta x,\theta y),\ 0<\theta<1$.

Maxima, Minima, and Saddle Points[493]

A function $f: A\to\mathbb{R}$, where $A\subseteq\mathbb{R}^2$, is said to have a "local minimum" (resp. a "local maximum") at a point $(x_0,y_0)\in A$ if there exists a neighborhood $N(x_0,y_0)$ in A such that $f(x_0,y_0)\le f(x,y)$ (resp. $f(x_0,y_0)\ge f(x,y)$) for every $(x,y)\in N(x_0,y_0)$. The local minimum and the local maximum are called the "local extrema" of f. If the local extrema refer to the entire domain A, then they are called the "global extrema" of f.

Theorem[494]*:* If a function $z=f(x,y)$ has an extremum at (x_0,y_0), then each first-order partial derivative of z either vanishes at (x_0,y_0) or does not exist.

Proof: If $y=y_0$, then $f(x,y_0)$ is a function of one variable, x. Since $z=f(x,y)$ has an extremum (maximum or minimum) at $x=x_0$, $\frac{\partial z}{\partial x}\big|_{(x_0,y_0)}$ is either equal to zero or does not exist. Similarly, we can prove that $\frac{\partial z}{\partial y}\big|_{(x_0,y_0)}$ is either equal to zero or does not exist.■

[493] Ibid.
[494] Ibid.

Remark: The aforementioned theorem is a necessary condition of an extremum, but it is not sufficient for investigating the extrema of a function. For instance, the function $z = y^2 - x^2$ has derivatives $\frac{\partial z}{\partial x} = -2x$ and $\frac{\partial z}{\partial y} = 2y$, which vanish at $(0,0)$, but, at $(0,0)$, this function has neither a maximum nor a minimum. In particular, at the origin, this function is equal to zero, and it takes on both positive and negative values at points arbitrarily close to the origin, for which reason $(0,0)$ is neither a maximum nor a minimum.

The points at which $\frac{\partial z}{\partial x} = 0$ or does not exist *and* $\frac{\partial z}{\partial y} = 0$ or does not exist are said to be the "critical points" of the function $z = f(x,y)$; and, whenever a function reaches an extremum at some point, this occurs only at a critical point.

Theorem of Local Extrema[495]*:* If $P(x_0, y_0)$ is a critical point of $f(x,y)$, and if

$$D = \frac{\partial^2 f(x_0,y_0)}{\partial x^2} \cdot \frac{\partial^2 f(x_0,y_0)}{\partial y^2} - \left[\frac{\partial^2 f(x_0,y_0)}{\partial x \partial y}\right]^2, \qquad (*)$$

then:

i. (x_0, y_0) is a local maximum if $D > 0$ and $\frac{\partial^2 f(x_0,y_0)}{\partial x^2} < 0$;
ii. (x_0, y_0) is a local minimum if $D > 0$ and $\frac{\partial^2 f(x_0,y_0)}{\partial x^2} > 0$;
iii. $f(x,y)$ has neither a maximum nor a minimum if $D < 0$;
iv. (x_0, y_0) may or may not be an extremum if $D = 0$ (in this case, additional investigation is required).

Proof: We shall prove only case (i), because the proof of each of the other cases is analogous. Taylor's Theorem of the Mean for Multivariable Functions implies that, for $f_x(x_0, y_0) = 0 = f_y(x_0, y_0)$, $f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) = \frac{1}{2!}\left[(\Delta x)^2 f_{xx} + 2(\Delta x)(\Delta y) f_{xy} + (\Delta y)^2 f_{yy}\right]$, where $f_{xx}(x_0 + \theta\Delta x, y + \theta\Delta y)$ and $f_{yy}(x_0 + \theta\Delta x, y + \theta\Delta y)$ are defined, and $0 < \theta < 1$. Then

$$f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) = \frac{1}{2} f_{xx}\left[\left(\Delta x + \frac{f_{xy}}{f_{xx}}\Delta y\right)^2 + \left(\frac{f_{xx}f_{yy} - f_{xy}^2}{f_{xx}^2}\right)(\Delta y)^2\right].$$

By hypothesis, there exists a neighborhood of (x_0, y_0) such that $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$. Therefore, $f(x_0 + \Delta x, y_0 + \Delta y) \le f(x_0, y_0)$ for every sufficiently small Δx and Δy, meaning that $f(x,y)$ attains a local maximum at (x_0, y_0).■

Remark: In $(*)$, if $D < 0$, then (x_0, y_0) is called a "saddle point," and it is neither a local maximum nor a local minimum of $f(x,y)$. A saddle point resembles the center point of a horse saddle or the low point of a ridge joining two peaks, and, therefore, a saddle point is that peculiar point on a surface which is simultaneously a peak along a path on the given surface and a dip along another path on the given surface. Moreover, if, in $(*)$, $D = 0$, then:

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[495] Ibid.

i. (x_0, y_0) is a local maximum if and only if $f(x,y) - f(x_0, y_0) \leq 0$ for every (x,y) that belongs to a neighborhood of (x_0, y_0).
ii. (x_0, y_0) is a local minimum if and only if $f(x,y) - f(x_0, y_0) \geq 0$ for every (x,y) that belongs to a neighborhood of (x_0, y_0).

The global extrema of a multivariable function can be found in the following way: If $f: A \to \mathbb{R}$, where $A \subset \mathbb{R}^n$, is continuous on a compact set $A \subset \mathbb{R}^n$, then, due to Weierstrass's Theorem, f has a global minimum and a global maximum in A, that is, there exist at least one point $p_1 \in A$ and at least one point $p_2 \in A$ as well as real numbers m, M with $0 \leq m \leq M$ such that the following holds:

$m \leq f(x_1, x_2, \dots, x_n) \leq M$ with $f(p_1) = m$ and $f(p_2) = M$.

If the partial derivatives of f exist, if $Int(A)$ denotes the interior of A, and if $A - Int(A)$ is the boundary of A, then the global extrema of f are located (and should be pursued) among those points of $Int(A)$ at which the partial derivatives of f become equal to zero and among those points of $A - Int(A)$ which are the local extrema of $f(x_1, x_2, \dots, x_n)$ with $(x_1, x_2, \dots, x_n) \in A - Int(A)$.

The aforementioned results about local and global extrema can be extended to any number of variables.

In order to find the local extrema of a function whose independent variables must satisfy at least one specific condition (constraint), we can apply different methods. The major methods that can be used in order to find constrained local extrema are the following:

Method I: Suppose that we want to find the local extrema of a function $f(x_1, x_2, \dots, x_n)$ that is defined on an open set $A \subseteq \mathbb{R}^n$. Moreover, suppose that the independent variables $x_1, x_2, \dots, x_n$ of f satisfy the following m conditions, where $m < n$:

$$\left\{\begin{matrix} g_1(x_1, x_2, \dots, x_n) = 0 \\ g_2(x_1, x_2, \dots, x_n) = 0 \\ \vdots \\ g_m(x_1, x_2, \dots, x_n) = 0 \end{matrix}\right\}.$$

Then, by the aforementioned system of m equalities, we determine the m variables via the rest $n - m$ ones, and we make the pertinent substitutions in $f(x_1, x_2, \dots, x_n)$ in order to obtain a function of $n - m$ variables whose local extrema can be found more easily.

Method II: If we want to find the local extrema of $f(x_1, x_2, x_3)$ that is subject to the constraints $g_1(x_1, x_2, x_3) = 0$ and $g_2(x_1, x_2, x_3) = 0$, then we can work as follows:

First, we form the auxiliary function

$h = f + \lambda_1 g_1 + \lambda_2 g_2$, where λ_1 and λ_2 are parameters called Lagrange multipliers.

Second, we solve the system

$$\begin{Bmatrix} h_{x_1} = 0 \\ h_{x_2} = 0 \\ h_{x_3} = 0 \\ g_1 = 0 \\ g_2 = 0 \end{Bmatrix}$$

for $x_1, x_2, x_3, \lambda_1, \lambda_2$. Let one solution of the given system be $x_1 = a_1, x_2 = a_2, x_3 = a_3, \lambda_1 = b_1, \lambda_2 = b_2$.

Third, we calculate the matrix

$$\begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_3} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial x_3} \end{bmatrix}$$

for $x_1 = a_1, x_2 = a_2, x_3 = a_3$. Then we calculate the 2-square sub-determinants of the given matrix (2 is the number of constraints). If at least one of the sub-determinants of this matrix is different from zero, then, possibly, f attains a local extremum at the point (a_1, a_2, a_3).

Four, because there are three independent variables (x_1, x_2, x_3) and two constraints (g_1, g_2), we form a determinant D of order $3 + 2 = 5$, so that

$$D = \begin{vmatrix} h_{x_1x_1} & h_{x_1x_2} & h_{x_1x_3} & \frac{\partial g_1}{\partial x_1} & \frac{\partial g_2}{\partial x_1} \\ h_{x_2x_1} & h_{x_2x_2} & h_{x_2x_3} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_2}{\partial x_2} \\ h_{x_3x_1} & h_{x_3x_2} & h_{x_3x_3} & \frac{\partial g_1}{\partial x_3} & \frac{\partial g_2}{\partial x_3} \\ \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_3} & 0 & 0 \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial x_3} & 0 & 0 \end{vmatrix}$$

for $x_1 = a_1, x_2 = a_2, x_3 = a_3, \lambda_1 = b_1, \lambda_2 = b_2$. If $D(a_1, a_2, a_3, b_1, b_2) > 0$, then f attains a local minimum. If $D(a_1, a_2, a_3, b_1, b_2) < 0$, then f attains a local maximum.

Method III: If we want to find the local extrema of $f(x, y)$ under the condition that $g(x, y) = 0$, then we may use Lagrange multipliers in order to determine the possible positions of extrema, and, subsequently, we may apply either the method of differentiating implicit functions (in which case, the problem reduces to finding the extrema of a function of one variable) or the definition of local extrema (we work similarly if we are given a function of three variables and two constraints). Alternatively, we can parametrize the constraint $g(x, y) = 0$ by setting $x = acost$ and $y = bsint$, where $t \in [0,2\pi]$, and then we have to find the local extrema of $f(acost, bsint)$, which is a function of one variable.

Remark: Now, let us find the local extrema of an implicit function. Given an equation $f(x,y)=0$ where $f(x,y)$ has continuous first-order and second-order partial derivatives, suppose that (x_0,y_0) is a solution to the system

$$\begin{Bmatrix} f(x,y)=0 \\ f_x(x,y)=0 \end{Bmatrix},$$

and that $f_y(x_0,y_0)\neq 0$. Then, according to the method of implicit differentiation that has been already discussed, the equation $f(x,y)=0$ determines a function, say $y=y(x)$, where $x\in A\subset\mathbb{R}$. We assume that $x_0\in Int(A)$. Then:

i. if $f_y(x_0,y_0)f_{xx}(x_0,y_0)>0$, then $y=y(x)$ attains a local maximum at x_0, $y(x_0)=y_0$;
ii. if $f_y(x_0,y_0)f_{xx}(x_0,y_0)<0$, then $y=y(x)$ attains a local minimum at x_0, $y(x_0)=y_0$;
iii. if $f_y(x_0,y_0)f_{xx}(x_0,y_0)=0$ (i.e., if $f_{xx}(x_0,y_0)=0$, given that, by hypothesis, $f_y(x_0,y_0)\neq 0$), then this method is inconclusive.

In case of a function $f(x,y,z)$, we work in an analogous way: If (x_0,y_0,z_0) is a solution to the system

$$\begin{Bmatrix} f(x,y,z)=0 \\ f_x(x,y,z)=0 \\ f_y(x,y,z)=0 \end{Bmatrix},$$

and $f_z(x_0,y_0,z_0)\neq 0$, then, according to the method of implicit differentiation that has been already discussed, the equation $f(x,y,z)=0$ determines a function, say $z=z(x,y)$, where $(x,y)\in A\subset\mathbb{R}^2$. We assume that $(x_0,y_0)\in Int(A)$. Then:

i. if $f_z(x_0,y_0,z_0)f_{xx}(x_0,y_0,z_0)>0$ and
$\begin{vmatrix} f_{xx}(x_0,y_0,z_0) & f_{xy}(x_0,y_0,z_0) \\ f_{yx}(x_0,y_0,z_0) & f_{yy}(x_0,y_0,z_0) \end{vmatrix}>0,$
then (x_0,y_0) is the location of a local maximum, $z(x_0,y_0)=z_0$;
ii. if $f_z(x_0,y_0,z_0)f_{xx}(x_0,y_0,z_0)<0$ and
$\begin{vmatrix} f_{xx}(x_0,y_0,z_0) & f_{xy}(x_0,y_0,z_0) \\ f_{yx}(x_0,y_0,z_0) & f_{yy}(x_0,y_0,z_0) \end{vmatrix}>0,$
then (x_0,y_0) is the location of a local minimum, $z(x_0,y_0)=z_0$;
iii. if
$\begin{vmatrix} f_{xx}(x_0,y_0,z_0) & f_{xy}(x_0,y_0,z_0) \\ f_{yx}(x_0,y_0,z_0) & f_{yy}(x_0,y_0,z_0) \end{vmatrix}=0,$
then this method is inconclusive.

Curvilinear Coordinates and Transformations[496]

[496] Ibid.

Let us consider an arbitrary domain in a Euclidean space $\mathbb{R}^n$. By the term "domain," we mean an arbitrary set A in a Euclidean space such that every point p of A is contained in A together with a ball with center p and sufficiently small radius. Additionally, let us consider a second copy of the Euclidean space, say $\mathbb{R}_1^n$. In order to define the coordinates of the point p in the domain A, we have to associate with this point a set of numbers, called coordinates, in such a way that distinct sets of numbers (coordinates) should correspond to different points of the domain. The operation of associating with each point p of a domain A a set of n real numbers gives rise to a set of n functions $x_1(p), x_2(p), \ldots, x_n(p)$ defined in the domain A. In this case, the functions $x_1, x_2, \ldots, x_n$ are coordinates in the Euclidean space $\mathbb{R}_1^n$. These functions are usually required to be continuous and even smooth (almost everywhere differentiable) in the domain A, in the sense that a small change in the position of p should lead to a small change in its coordinates, and a smooth deformation of p should yield a smooth variation of its coordinates.

Assume that we have two copies of a Euclidean space: $\mathbb{R}^n$ with Cartesian coordinates $y_1, y_2, \ldots, y_n$ and $\mathbb{R}_1^n$ with Cartesian coordinates $x_1, x_2, \ldots, x_n$ ($\mathbb{R}_1^n$ is an "arithmetic" Euclidean space, in the sense that it identifies its points with real sequences of length n). A "continuous coordinate system" in a domain A of a Euclidean space $\mathbb{R}^n$ is defined as a system of functions $x_1(y_1, y_2, \ldots, y_n), \ldots, x_n(y_1, y_2, \ldots, y_n)$ that map the domain A continuously and bijectively onto a certain domain B of $\mathbb{R}_1^n$ (i.e., the system of functions $x_1(y_1, y_2, \ldots, y_n), \ldots, x_n(y_1, y_2, \ldots, y_n)$ determines a homeomorphism of A onto B).

Let $f: A \to B$ be a smooth mapping defined by a set of functions $x_1(y_1, y_2, \ldots, y_n), \ldots, x_n(y_1, y_2, \ldots, y_n)$. A "curvilinear coordinate system" in a domain A of a Euclidean space $\mathbb{R}^n$ is defined as a system of smooth functions $x_1(y_1, y_2, \ldots, y_n), \ldots, x_n(y_1, y_2, \ldots, y_n)$ that map bijectively the domain A onto a certain domain B of $\mathbb{R}_1^n$ and are such that the Jacobian $J(f)$ is not zero at all points of A. The condition that the Jacobian $J(f)$ is not zero at all points of A means that the inverse mapping f^{-1} is not only continuous but also smooth, according to the Implicit Function Theorem. In other words, a curvilinear coordinate system is defined by two smooth mutually inverse mappings establishing a homeomorphism between a domain A and a domain B.

Let us consider a system of n functions $f_1, f_2, \ldots, f_n$ of n coordinates $x_1, x_2, \ldots, x_n$:

$$\left\{\begin{array}{c} y_1 = f_1(x_1, x_2, \ldots, x_n) \\ y_2 = f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ y_n = f_n(x_1, x_2, \ldots, x_n) \end{array}\right\}. \tag{S}$$

The system (S) determines a mapping F from $\mathbb{R}^n$ to $\mathbb{R}^n$, and this mapping is often called a "transformation" of $\mathbb{R}^n$. Moreover, the system (S) may be "locally invertible," namely, we may be able to define a system of functions

$$\left\{\begin{array}{c} x_1 = g_1(y_1, y_2, \ldots, y_n) \\ x_2 = g_2(y_1, y_2, \ldots, y_n) \\ \vdots \\ x_n = g_n(y_1, y_2, \ldots, y_n) \end{array}\right\}, \tag{S^*}$$

such that its equations are the unique solutions to (S) in the neighborhood of some point. The system (S^*), consisting of the functions that solve (S), determines a transformation that is the inverse of F. If the functions $f_1, f_2, \dots, f_n$ have continuous first-order partial derivatives in a neighborhood of a point $(p_1, p_2, \dots, p_n) \in \mathbb{R}^n$, if the Jacobian determinant

$$\frac{\partial(f_1,f_2,\dots,f_n)}{\partial(x_1,x_2,\dots,x_n)}\Big|_{(p_1,p_2,\dots,p_n)} \neq 0,$$

and $a_1 = f_1(p_1, p_2, \dots, p_n), a_2 = f_2(p_1, p_2, \dots, p_n), \dots, a_n = f_n(p_1, p_2, \dots, p_n)$, then the system (S) has a unique solution in terms of $x_1, x_2, \dots, x_n$, and the functions $x_i = g_i(y_1, y_2, \dots, y_n)$, where $i = 1,2, \dots, n$, which solve (S), have continuous first-order derivatives in a neighborhood of the point $(p_1, p_2, \dots, p_n)$, and $p_i = f_i(a_1, a_2, \dots, a_n)$.

For instance, let us consider the system $\begin{Bmatrix} x = rcos\theta \\ y = rsin\theta \end{Bmatrix}$, where $r \geq 0$ and $0 \leq \theta < 2\pi$. This system determines a transformation that is one-to-one and continuously differentiable, since

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} cos\theta & sin\theta \\ -rsin\theta & rcos\theta \end{vmatrix} = r.$$

Hence, for $r \neq 0$, the given transformation is invertible. In particular, the inverse of the given transformation is the transformation $\begin{Bmatrix} r = \sqrt{x^2 + y^2} \\ \theta = arctan\frac{y}{x} \end{Bmatrix}$. Notice that the given transformation is not one-to-one at the origin, $r = 0$, where the Jacobian $J = 0$: as (x, y) tends to $(0,0)$ along the line $y = kx$, then (r, θ) tends to $(0, arctank)$, and then $(x, y) = (0,0)$ corresponds to $(r, \theta) = (0, arctank)$ for all k, and the transformation is not one-to-one near this point.

Cylindrical Coordinates[497]*:* We can locate a point $P(x, y, z)$ in $\mathbb{R}^3$ via the real numbers r, θ, z. As we can see in Figure 2.34, $r = OP'$, where P' is the projection of the point P on the plane Oxy, and θ is the angle $\angle AOP'$. Hence, we have the following transformation:

$x = rcos\theta$, $y = rsin\theta$, and $z = z$,

and these relations imply that $r^2 = x^2 + y^2$, $\theta = arctan\frac{y}{x}$, and $z = z$. The real numbers r, θ, z are the "cylindrical coordinates" of the point $P(x, y, z) \in \mathbb{R}^3$, and we write $P(r, \theta, z)$ with $r = k > 0$, $\theta \in [0,2\pi]$, and $z \in \mathbb{R}$. We use the term "cylindrical coordinates," because the locus of the points $P(r, \theta, z)$ in $\mathbb{R}^3$, for some $r = k > 0$, is a cylinder, so that this cylinder is defined by the equation $x^2+y^2 = k^2$. Notice that, in cylindrical coordinates, the Jacobian of the given transformation is

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} cos\theta & -rsin\theta & 0 \\ sin\theta & rcos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

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[497] Ibid.

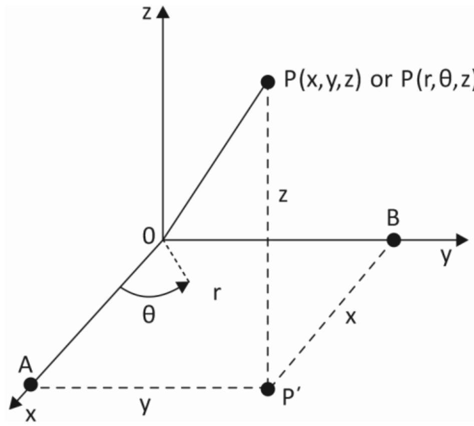

Figure 2.34. Cylindrical Coordinates.

Spherical Coordinates[498]*:* We can locate a point $P(x,y,z)$ in $\mathbb{R}^3$ via the real numbers r,θ,φ. As we can see in Figure 2.35, $r = OP$, θ is the angle $\angle AOP'$, and φ is the angle $\angle COP$. Hence, we have the following transformation:

$x = rcos\theta sin\varphi$, $y = rsin\theta sin\varphi$, and $z = rcos\varphi$,

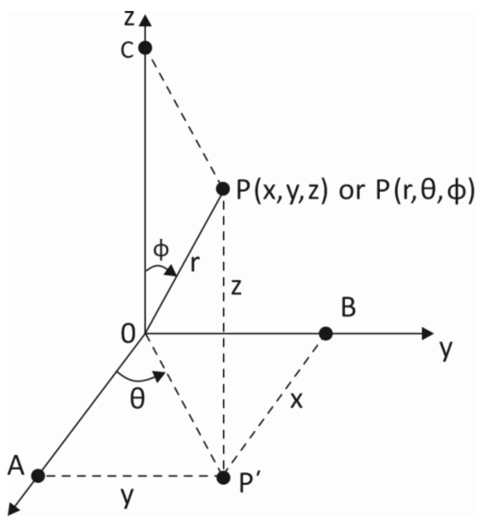

Figure 2.35. Spherical Coordinates.

and these relations imply that $r^2 = x^2 + y^2 + z^2$, $\theta = arctan\frac{y}{x}$, and $\varphi = arccos\frac{z}{\sqrt{x^2+y^2+z^2}}$. The real numbers r,θ,φ are the "spherical coordinates" of $P(x,y,z) \in \mathbb{R}^3$, and we write $P(r,\theta,\varphi)$ with $r \geq 0$, $\theta \in [0,2\pi]$, and $\varphi \in [0,\pi]$. We use the term "spherical coordinates," because the locus of the points $P(r,\theta,\varphi)$ for some $r = k > 0$, $\theta \in [0,2\pi]$, and $\varphi \in [0,\pi]$ is a sphere defined by the equation $x^2+y^2+z^2 = k^2$. Notice that, in spherical coordinates, the Jacobian of the given transformation is

[498] Ibid.

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = \begin{vmatrix} sin\varphi cos\theta & -rsin\varphi sin\theta & rcos\varphi cos\theta \\ sin\varphi sin\theta & rsin\varphi cos\theta & rcos\varphi sin\theta \\ cos\varphi & 0 & -rsin\varphi \end{vmatrix} = -r^2 sin\varphi.$$

2.11. INTEGRAL CALCULUS

From a rather elementary perspective, integration can be construed as the inverse of differentiation. Let $f: I \to \mathbb{R}$ be a function, where I is an interval; in fact, I may have one of the following forms:

$$[a,b], [a,b), (a,b], (a,b), [a,+\infty), (a,+\infty), (-\infty,b], (-\infty,b), (-\infty,+\infty).$$

If $F: I \to \mathbb{R}$ is a function such that $F'(x) = f(x)\ \forall x \in I$, then F is called the "antiderivative" of f in I, and it is denoted by

$$F(x) = \int f(x)dx, x \in I$$

according to Leibniz's notation.[499] In other words, $\int f(x)dx = F(x) + c$ if and only if $[F(x) + c]' = f(x)$. The aforementioned definition implies that the "indefinite integral" of a given function with respect to x is a new function plus a constant if and only if the derivative of the new function and of the constant equals the given function. Thus, differentiation can be used in order to verify the result of an integral.

Examples:

i. $\int x^2\, dx = \frac{x^3}{3} + c$, because $\left(\frac{x^3}{3} + c\right)' = x^2$;
ii. $\int e^x\, dx = e^x + c$, because $(e^x + c)' = e^x$;
iii. $\int \frac{dx}{x} = ln|x| + c$, because, $\forall x \in \mathbb{R}_+^*$, $(lnx)' = \frac{1}{x}$; and, in general, $\int x^n\, dx = \begin{cases} \frac{x^{n+1}}{n+1} + cifn \neq -1 \\ ln|x| + cifn = -1 \end{cases}$.

Theorem[500]: Let F_1 and F_2 be two antiderivatives of f. Then

$F_2(x) = F_1(x) + c\ \forall x \in I$ (c is a contant).

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[499] See: Apostol, *Calculus*; Courant and John, *Introduction to Calculus and Analysis*; Edwards, *A Treatise on the Integral Calculus*; Fraleigh, *Calculus with Analytic Geometry*; Gillespie, *Integration*; Haaser and Sullivan, *Real Analysis*; McLeod, *The Generalized Riemann Integral*; Nikolski, *A Course of Mathematical Analysis*; Piskunov, *Differential and Integral Calculus*; Rudin, *Principles of Mathematical Analysis*; Spivak, *Calculus*; Taylor, *General Theory of Functions and Integration*.

[500] Ibid.

Proof: Let us set $F(x) = F_2(x) - F_1(x) \Rightarrow F'(x) = F_2'(x) - F_1'(x) = f(x) - f(x) = 0\ \forall x \in I$. Hence, F is constant in I, that is, $F(x) = c$ or $F_2(x) = F_1(x) + c$.■

Remark: Similarly, we can prove that, if F_1 is an antiderivative of f in I, and if F_2 is defined on I, so that $F_2(x) = F_1(x) + c$ for all $x \in I$ and for any $c \in \mathbb{R}$, then F_2 is an antiderivative of f in I.

Theorem[501]*:* Let $f: I \to \mathbb{R}$ and $g: I \to \mathbb{R}$ be two functions. If their indefinite integrals exist in I, then there is the indefinite integral of $af + bg$, too, and

$$\int [af(x) + bg(x)]dx = a\int f(x)dx + b\int g(x)dx,$$

where a and b are constants.

Proof: Suppose that $\int f(x)\,dx = F_1(x)$ and $\int g(x)\,dx = F_2(x)$. Then $F_1'(x) = f(x)$ and $F_2'(x) = g(x)$ for all $x \in I$. Therefore, $[aF_1(x) + bF_2(x)]' = aF_1'(x) + bF_2'(x) = af(x) + bg(x)$.■

Examples:

i. $\int \frac{xdx}{x^2+2}$. We observe that $(x^2+2)' = 2x$. Then we transform the numerator of the integrand, so that the integral will be reduced to the form $\int \frac{dx}{x}$. Hence, $\int \frac{xdx}{x^2+2} = \frac{1}{2}\int \frac{2xdx}{x^2+2} = \frac{1}{2}\int \frac{d(x^2+2)}{x^2+2} = \frac{1}{2}ln(x^2+2) + c$.

ii. $\int sinx = -cosx + c$.

iii. $\int cosx = sinx + c$.

iv. $\int tanx = \int \frac{sinx}{cosx}dx = -\int \frac{d(cosx)}{cosx} = -ln|cosx| + c$.

v. $\int cotxdx = \int \frac{cosx}{sinx}dx = \int \frac{d(sinx)}{sinx} = ln|sinx| + c$.

vi. $\int e^{-x}\,dx = -\int e^{-x}d(-x) = -e^{-x} + c$.

vii. $\int \frac{dx}{x^2+1}$. Let us set $x = tant$, so that
$\int \frac{dx}{x^2+1} = \int \frac{d(tant)}{tan^2t+1} = \int \frac{(tant)'dt}{\frac{sin^2t}{cos^2t}+\frac{cos^2t}{cos^2t}} = \int \frac{\frac{1}{cos^2t}dt}{\frac{sin^2t+cos^2t}{cos^2t}} = \int 1dt = t + c,$
that is, t is the arc whose tangent is x, and, therefore, in this case, $\int 1dt = \int \frac{dx}{x^2+1} = arctanx + c$.

viii. $\int \frac{dx}{x^2+k^2}$, where k is a constant. Let us set $x = kt \Rightarrow dx = d(kt) = (kt)'dt = (k't + kt')dt = kdt$. Then
$\int \frac{dx}{x^2+k^2} = \int \frac{kdt}{k^2t^2+k^2} = \int \frac{kdt}{k^2(t^2+1)} = \int \frac{1dt}{k(t^2+1)} = \frac{1}{k}\int \frac{dt}{t^2+1} = \frac{1}{k}arctant + c = \frac{1}{k}arctan\frac{x}{k} + c$.

Remark: The integral $\int \frac{\mathrm{dx}}{\mathrm{ax^2+bx+c}}$, where $D = b^2 - 4ac < 0$, reduces to the form $\int \frac{du}{u^2+k^2}$.

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[501] Ibid.

The "definite integral" is written as

$$\int_a^b f(x)dx$$

and represents the area bounded by the curve $y = f(x)$, the x-axis, and the ordinates $x = a$, and $x = b$ if $f(x) \geq 0$. If $f(x)$ is sometimes positive and sometimes negative, then the definite integral represents the algebraic sum of the areas above and below the x-axis, and, in particular, the areas that are above the x-axis are considered to be positive, whereas the areas that are below the x-axis are considered to be negative.

The Definition of the Integral as the Limit of a Sum[502]

As shown in Figure 2.36, the definite integral $\int_a^b f(x)dx$ can be defined as follows:

We subdivide the interval $[a, b]$ into n subintervals

$$[a, x_1], [x_1, x_2], \dots, [x_{k-1}, x_k], \dots, [x_{n-1}, b] \qquad (*)$$

by means of the points $x_1, x_2, \dots, x_{n-1}$, which have been chosen arbitrarily. Hence, the set of points $P = \{a = x_0, x_1, x_2, \dots, x_{k-1}, x_k, \dots, x_{n-1}, x_n = b\}$ is a "partition" of $[a, b]$. Let Δx_k be the length of the kth subinterval, that is, $\Delta x_k = x_k - x_{k-1}$. Then the "norm" of partition P is denoted by $\|P\|$, and it is equal to $max\{\Delta x_k | k = 1,2, \dots, n\}$.

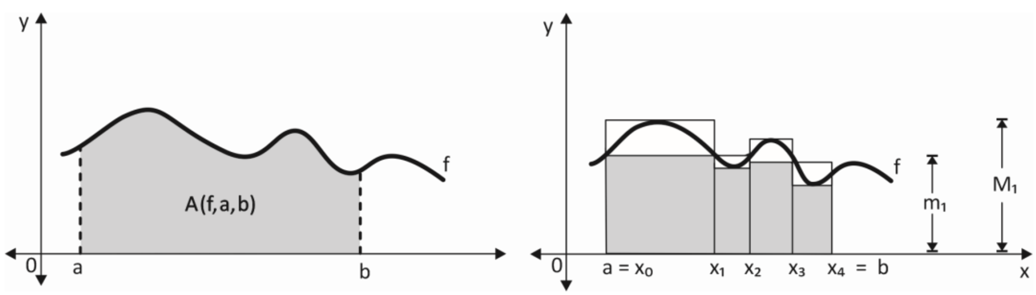

Figure 2.36. The Integral as the Limit of a Sum.

In each of the n subintervals mentioned in $(*)$, we choose points $c_1, c_2, \dots, c_n$ in an arbitrary way, and we form the sum

$$S(P, f, c_k) = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \cdots + f(c_k)\Delta x_k + \cdots + f(c_n)\Delta x_n = \sum_{k=1}^{n} f(c_k)\Delta x_k.$$

Notice that, as the number of subdivisions n increases, $\|P\|$ vanishes, that is, $\|P\| \to 0$ as $n \to \infty$. Hence, if $lim_{\|P\|\to 0} S(P, f, c_k)$ exists and is independent of the mode of subdivision of $[a, b]$, then this limit is said to be the integral of f on $[a, b]$, symbolically,

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[502] Ibid.

$$lim_{\|P\|\to 0} S(P, f, c_k) = \int_a^b f(x)dx$$

where $f(x)dx$ is called the "integrand," $[a, b]$ is called the "range of integration," and a and b are, respectively, called the lower and the upper "limit of integration." Notice that the aforementioned limit exists if $f(x)$ is continuous (or sectionally continuous) on $[a, b]$.

Leibniz symbolized the definite integral of a function $f(x)$ on $[a, b]$ as $\int_a^b f(x)dx$, because the sign $\int$ is an elongated S standing for the word "sum," since, as I have already explained, Leibniz defined $\int_a^b f(x)dx$ as the sum of infinitely many rectangles of height $f(x)$ and infinitesimally small width dx.

The Physical Significance of the Integral

As I have already explained, the development of infinitesimal calculus by Newton and Leibniz is intimately related to the study of celestial mechanics and, generally, physics by them. In the aforementioned context, infinitesimal calculus, known also as the differentiation–integration method, is concerned with the limits of applicability of physical laws. Physical laws are not absolute, and the validity of a law is restricted to the framework of the applicability limits (i.e., certain conditions). However, a physical law can be expanded by changing its form beyond the limits of applicability by means of infinitesimal calculus. This method is based on the following two principles: (i) the principle that a law can be represented in differential form, and (ii) the superposition principle, according to which the quantities that enter into the law are additive. In particular, the "principle of superposition" was first stated by Daniel Bernoulli (1753), and it consists of two properties whose meaning will become better understood later in this chapter: (i) the sum of any number of linearly independent partial solutions of a differential equation is also a solution of the given differential equation; and (ii) any constant multiple of a solution is also a solution.

Suppose that a physical law has the form

$$X = YZ, \tag{$*$}$$

where X, Y, and Z are physical quantities, and, in particular, Y is a constant representing the given law's limits of applicability. We can generalize the given law to the case where Y is not a constant but a function of Z, namely, $Y = Y(Z)$, as follows: As shown in Figure 2.37, we isolate an interval dZ so small that the variation of Z over this interval can be ignored. Hence, in the interval ("infinitesimal") dZ, we can approximately assume that Y is constant, and that the law $(*)$ is valid in this interval. Therefore,

$$dX = Y(Z)dZ, \tag{$**$}$$

where dX is the variation of X over dZ. Due to the superposition principle, that is, by summing the quantities $(**)$ over all the intervals of variation of Z, we obtain an expression for X in the form

$$X = \int_m^M Y(Z)dZ, \tag{$***$}$$

where m and M are the initial and the final values of Z, respectively.

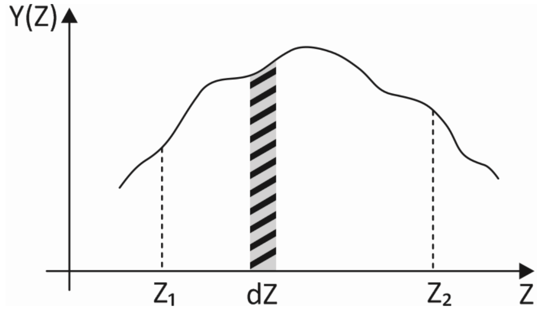

Figure 2.37. The Method of Infinitesimal Calculus.

As a conclusion, the method of infinitesimal calculus consists of two parts: in the first part of the method, we find the differential ($**$) of the quantity under investigation; and, in the second part of the method, we sum, or "integrate," having adequately determined the integration variable and the limits of integration (in order to determine the integration variable, we must analyze the quantities on which the differential of the investigated quantity depends and choose the most important variable; and the limits of integration are the lower and the upper values of the integration variable).

Integration of Complex Functions of One Variable[503]

The integral of a complex function $f(x) = g(x) + ih(x)$, where $i = \sqrt{-1}$, and x is a real variable, between the limits a and b, is defined by

$$\int_a^b f(x)dx = \int_a^b [g(x) + ih(x)]dx = \int_a^b g(x)dx + i\int_a^b h(x)dx.$$

Obviously, the properties of such integrals may be deduced from the properties of the real integrals.

2.12. Standard Integration Techniques[504]

There are two standard integration techniques: integration by substitution and integration by parts.

Integration by Substitution

This technique of integration is based on the following theorem:

[503] Ibid.
[504] Ibid.

Theorem[505]*:* Let A and B be two intervals, and let $f: A \to B$ be a continuous function. Moreover, let $g: B \to \mathbb{R}$ be a differentiable function with $g'(t) \neq 0 \; \forall \; t \in B$ and such that $R_g \subseteq A$ (where R_g denotes the range of g). Then

$$\int f(x)\,dx = \int f\big(g(t)\big)g'\,(t)dt.$$

Proof: Let $F(x) = \int f(x)\,dx$. Then $F'(x) = f(x) \; \forall \; x \in A$. Let $G(t) = F\big(g(t)\big)$, where $t \in B$. Therefore,

$$G'(t) = F'\big(g(t)\big)g'(t) = f\big(g(t)\big)g'(t),$$

meaning that G has an antiderivative in B.■

Case Studies ($a \in \mathbb{R}^*$):

i. Assume that an integral contains the expression $\sqrt{a^2 - x^2}$. We need $a^2 - x^2 > 0 \Rightarrow -x^2 > -a^2 \Rightarrow x^2 < a^2 \Rightarrow \frac{x^2}{a^2} < 1 \Rightarrow \left|\frac{x}{a}\right| < 1$. Hence, we set $x = asint \Rightarrow dx = acostdt$. We can also make the substitution $x = acost \Rightarrow dx = -asintdt$. For instance, let us compute the integral $\int \frac{x}{\sqrt{4-x^2}}dx$, applying the technique of integration by substitution: $4 - x^2 > 0 \Rightarrow -x^2 > -4 \Rightarrow x^2 < 4 \Rightarrow \left(\frac{x}{2}\right)^2 < 1 \Rightarrow \left|\frac{x}{2}\right| < 1$. Let us set $x = 2sint$ and $dx = 2costdt$. Hence,

ii. $\int \frac{x}{\sqrt{4-x^2}}dx = \int \frac{2sint}{\sqrt{4-(2sint)^2}}2costdt = \int \frac{2sint}{2cost}2costdt = 2\int sintdt = -2cost + c = -2\sqrt{1 - sin^2t} + c = -\sqrt{4 - x^2} + c$.

iii. Assume that an integral contains the expression $\sqrt{x^2 - a^2}$. We need $x^2 - a^2 > 0 \Rightarrow x^2 > a^2 \Rightarrow \frac{x^2}{a^2} > 1 \Rightarrow \frac{a^2}{x^2} < 1 \Rightarrow \left|\frac{a}{x}\right| < 1$. Hence, we set $\frac{a}{x} = sint \Rightarrow x = \frac{a}{sint}$ and $dx = a\left(\frac{1}{sint}\right)' = -a(sint)^{-2}costdt$. We can also make the substitution $x = \frac{a}{cost}$ and $dx = a\frac{sint}{cos^2t}dt$.

iv. Assume that an integral contains the expression $\sqrt{a^2 + x^2}$. This expression is defined $\forall x \in \mathbb{R}$, and, therefore, we set $x = atant$ and $dx = \frac{a}{cos^2t}dt$, since tangent is defined in $\mathbb{R}$.

v. Assume that an integral contains the expression $\sqrt{ax + b}$. Then we set $\sqrt{ax + b} = t$. For instance, let us compute the integral $\int \frac{sin\sqrt{x}}{\sqrt{x}}dx$, applying the technique of integration by substitution: Let us set $\sqrt{x} = t$, so that $x = t^2$ and $dx = 2tdt$. Hence, $\int \frac{sin\sqrt{x}}{\sqrt{x}}dx = \int \frac{sint}{t}2tdt = 2\int sintdt = -2cost + c = -2cos\sqrt{x} + c$.

vi. Assume that an integral contains the expression $\sqrt{2ax - ax^2}$, where $a > 0$. Then we set $x = a(1 - cost)$ and $dx = asintdt$.

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[505] Ibid.

Integration by Parts

This technique of integration is based on the following theorem:

Theorem[506]*:* If the functions u and v are differentiable on the interval I, and if the indefinite integral of $u'v$ exists in I, then

$$\int udv = uv - \int vdu.$$

Proof: $(uv)' = u'v + uv' \Rightarrow (uv)'dx = vu'dx + uv'dx$. Then $\int (uv)'dx = \int vu'dx + \int uv'dx$, and $\int d(uv) = \int vdu + \int udv \Rightarrow uv = \int vdu + \int udv$. Therefore, $\int udv = uv - \int vdu$.■

Case Studies: If we have $\int uvdx$, and we must determine which function to put inside the differential (i.e., $\int u'\,vdx$ or $\int uv'\,dx$), we can use the following table that categorizes functions in descending order of preference:

1. Exponential Functions,
2. Trigonometric Functions,
3. Polynomial Functions,
4. Logarithmic Functions.

Examples:

i. Let us compute the integral $\int xcosxdx$, applying the technique of integration by parts: $\int xcosxdx = \int xd(sinx) = xsinx - \int (sinx)x'dx = xsinx - (-cosx) + c = xsinx + cosx + c$.
ii. Let us compute the integral $\int x^n lnxdx$, $n \in \mathbb{N}^*$, applying the technique of integration by parts: $\int x^n lnxdx = \int lnx\left(\frac{x^{n+1}}{n+1}\right)' dx = \frac{x^{n+1}}{n+1} lnx - \int \frac{x^{n+1}}{n+1} (lnx)'dx = \frac{x^{n+1}}{n+1} lnx - \frac{1}{n+1} \int x^{n+1} \frac{1}{x} dx = \frac{x^{n+1}}{n+1} lnx - \frac{1}{n+1} \frac{x^{n+1}}{n+1} + c$.

2.13. Reduction Formulas[507]

The technique of integration by parts helps us to find a reduction formula in order to compute integrals of the form $\int f^n(x)\, g(x)dx$, where $n \in \mathbb{N}$, such as $\int sin^n xdx$, $\int \frac{dx}{(x^2+1)^n}$, etc.

Examples:

[506] Ibid.
[507] Ibid.

i. $I_n = \int sin^n xdx =$
$\int sin^{n-1} xsinxdx = -\int sin^{n-1} xd(cosx) = -\int sin^{n-1}x(cosx)'dx =$
$-sin^{n-1}xcosx + \int cosx(sin^{n-1}x)'dx = -sin^{n-1}xcosx + \int(cosx)(n-1)sin^{n-2}x\,(sinx)'dx = -sin^{n-1}xcosx + (n-1)\int cos^2 xsin^{n-2}xdx =$
$-sin^{n-1}xcosx + (n-1)\int(1-sin^2x)sin^{n-2}xdx = -sin^{n-1}xcosx +$
$(n-1)\int sin^{n-2}xdx - (n-1)\int sin^n xdx$. Hence,
$I_n = -sin^{n-1}xcosx + (n-1)I_{n-2} - (n-1)I_n \Leftrightarrow I_n = -\frac{sin^{n-1}xcosx}{n} + \frac{n-1}{n}I_{n-2}$,
where $n = 2,3, \dots$ If $n = 1$, then $I_n = I_1 \Rightarrow I_1 = \int sinxdx = -cosx + c$.

ii. $J_n = \int cos^n xdx =$
$\int cos^{n-1}xcosxdx = \int cos^{n-1}xd(sinx) = \int cos^{n-1}x(sinx)'\,dx = cos^{n-1}xsinx -$
$\int sinx(cos^{n-1}x)'\,dx = cos^{n-1}xsinx - \int(n-1)cos^{n-2}x(cosx)'sinxdx =$
$cos^{n-1}xsinx + (n-1)\int cos^{n-2}xsin^2xdx = cos^{n-1}xsinx + (n-1)\int cos^{n-2}x(1-cos^2x)dx = cos^{n-1}xsinx + (n-1)\int cos^{n-2}xdx - (n-1)\int cos^{n-2}xcos^2xdx = cos^{n-1}xsinx + (n-1)\int cos^{n-2}xdx - (n-1)\int cos^n xdx \Leftrightarrow \int cos^n xdx + (n-1)\int cos^n xdx = cos^{n-1}xsinx + (n-1)\int cos^{n-2}xdx \Leftrightarrow n\int cos^n xdx = cos^{n-1}xsinx + (n-1)\int cos^{n-2}xdx \Leftrightarrow J_n =$
$\frac{cos^{n-1}xsinx}{n} + \frac{n-1}{n}J_{n-2}$, where $n = 2,3, \dots$ If $n = 1$, then $J_n = J_1 \Rightarrow J_1 =$
$\int cosxdx = sinx + c$.

iii. $Q_n = \int tan^n xdx = \int tan^{n-2} xtan^2xdx = \int tan^{n-2}x\frac{sin^2x}{cos^2x}dx =$
$\int tan^{n-2}x\frac{1-cos^2x}{cos^2x}dx = \int tan^{n-2}x\frac{dx}{cos^2x} - \int tan^{n-2}x\frac{cos^2x}{cos^2x}dx =$
$\int tan^{n-2}xd(tanx) - Q_{n-2} \Rightarrow Q_n = \frac{tan^{n-1}x}{n-1} - Q_{n-2}$.

iv. $R_n = \int\frac{dx}{(a^2+x^2)^n} = \frac{1}{a^2}\int\frac{a^2+x^2-x^2}{(a^2+x^2)^n}dx = \frac{1}{a^2}\int\frac{dx}{(a^2+x^2)^{n-1}} - \frac{1}{a^2}\int\frac{x^2}{(a^2+x^2)^n}dx = \frac{1}{a^2}R_{n-1} -$
$\frac{1}{a^2}\int x\frac{x}{(a^2+x^2)^n}dx = \frac{1}{a^2}R_{n-1} - \frac{1}{2}\frac{1}{a^2}\int x\frac{d(a^2+x^2)}{(a^2+x^2)^n} = \frac{1}{a^2}R_{n-1} - \frac{1}{2a^2}\int xd\frac{(a^2+x^2)^{1-n}}{1-n} =$
$\frac{1}{a^2}R_{n-1} - \frac{1}{2a^2}\frac{x}{(1-n)(a^2+x^2)^{n-1}} + \frac{1}{2a^2}\frac{1}{1-n}\int\frac{dx}{(a^2+x^2)^{n-1}} \Leftrightarrow R_n = \frac{1}{2(n-1)a^2}\frac{x}{(a^2+x^2)^{n-1}} +$
$\frac{2n-3}{2(n-1)}\frac{1}{a^2}R_{n-1}$, where $n = 2,3, \dots$
Similarly, we can show that
$T_n = \int\frac{dx}{(a^2-x^2)^n} \Leftrightarrow T_n = \frac{1}{a^2}\frac{2n-3}{2n-2}T_{n-1} + \frac{1}{a^2}\frac{1}{2n-2}\frac{x}{(a^2-x^2)^{n-1}}$, where $n = 2,3, \dots$

v. $L_n = \int(lnx)^n\,dx = \int(lnx)^n x'\,dx = x(lnx)^n - \int x[(lnx)^n]'\,dx = x(lnx)^n -$
$\int xn(lnx)^{n-1}\,(lnx)'dx = x(lnx)^n - n\int(lnx)^{n-1}\,dx = x(lnx)^n - nL_{n-1}$.

vi. For $n \in \mathbb{Q}$, $K_n = \int(a^2-x^2)^n\,dx = x(a^2-x^2)^n - \int xn(a^2-x^2)^{n-1}\,(-2x)dx =$
$\frac{1}{2n+1}x(a^2-x^2)^n + \frac{2na^2}{2n+1}K_{n-1}$.

2.14. INTEGRATION OF RATIONAL FUNCTIONS[508]

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[508] Ibid.

In order to integrate a rational function, namely, a function of the form $\int \frac{P(x)}{Q(x)} dx$, where $P(x)$ and $Q(x)$ are polynomials, we must apply the theory of partial fractions. When we split $\frac{P(x)}{Q(x)}$ in partial fractions, each non-repeated linear factor $ax + b$ of $Q(x)$ produces a term of the form $\frac{A}{ax+b}$. Thus, the term $(ax + b)^n$ of $Q(x)$ produces a sum of n terms of the form

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}.$$

Similarly, each irreducible and non-repeated quadratic factor $ax^2 + bx + c$ of $Q(x)$ produces a term of the form $\frac{Ax+B}{ax^2+bx+c}$. Thus, the term $(ax^2 + bx + c)^n$ produces a sum of n terms of the form

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}.$$

Case Studies:

i. $\int \frac{A}{(x-r)^n} dx = \begin{cases} Aln|x-r|, x > r or x < r and n = 1 \\ \frac{A}{1-n} \cdot \frac{1}{(x-r)^{n-1}}, x \neq r and n > 1 \end{cases}.$

ii. $\int \frac{dx}{ax^2+bx+c}$:

If r_1 and r_2 are the roots of the quadratic polynomial $ax^2 + bx + c$, then $ax^2 + bx + c = a(x - r_1)(x - r_2)$.
If $D = b^2 - 4ac > 0$, then $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$.
If $D = b^2 - 4ac < 0$, then $r_1, r_2 \in \mathbb{C}$ and $r_1 \neq r_2$.
If $D = b^2 - 4ac = 0$, then $r_1 = r_2 = r \in \mathbb{R}$.
In case $D = 0$, we work as follows: $\int \frac{dx}{ax^2+bx+c} = \int \frac{dx}{a(x-r)^2} = \frac{1}{a}\int \frac{dx}{(x-r)^2} = \frac{1}{a}\int \frac{d(x-r)}{(x-r)^2} = \frac{1}{a} \cdot \frac{(x-r)^{-1}}{-1} + c = -\frac{1}{a(x-r)} + c$.
In case $D > 0$, we work as follows: $\int \frac{dx}{ax^2+bx+c} = \int \frac{dx}{a(x-r_1)(x-r_2)} = I = \frac{1}{a}\int \frac{dx}{(x-r_1)(x-r_2)}$. Notice that
$\frac{1}{(x-r_1)(x-r_2)} = \frac{A}{x-r_1} + \frac{B}{x-r_2} = \frac{A(x-r_2)+B(x-r_1)}{(x-r_1)(x-r_2)} \Rightarrow A(x - r_2) + B(x - r_1) = 1 \ \forall x \in \mathbb{R}$.
Setting $x = r_2$, we obtain $(r_2 - r_1)B = 1 \Rightarrow B = \frac{1}{r_2-r_1}$. Setting $x = r_1$, we obtain $(r_1 - r_2)A = 1 \Rightarrow A = \frac{1}{r_1-r_2}$. Therefore,
$I = \frac{1}{a}\int (\frac{\frac{1}{r_1-r_2}}{x-r_1} + \frac{\frac{1}{r_2-r_1}}{x-r_2}) dx = \frac{1}{a(r_1-r_2)}\left(\int \frac{dx}{x-r_1} - \int \frac{dx}{x-r_2}\right) = \frac{1}{a(r_1-r_2)}(ln|x - r_1| - ln|x - r_2|) + c = \frac{1}{a(r_1-r_2)} ln\left|\frac{x-r_1}{x-r_2}\right| + c$.
In case $D < 0$, we work as follows: $ax^2 + bx + c$ can be expressed as the sum of two second powers. Thus, the integral reduces to the following form:
$\int \frac{du}{u^2+k^2} = \frac{1}{k} arctan\frac{u}{k} + c$.

2.15. INTEGRATION OF IRRATIONAL FUNCTIONS[509]

Case Studies:

i. In order to compute integrals of the form $\int f\left(x, \sqrt{ax^2+bx+c}\right) dx$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$, we use the Euler substitutions:
   (i) If $a > 0$, then we set $\sqrt{ax^2+bx+c} = y + \sqrt{a}x$ or $y - \sqrt{a}x$.
   (ii) If $a < 0$, then we set $\sqrt{ax^2+bx+c} = y|x - r_1|$, where r_1 is a root of $ax^2 + bx + c = 0$, given that $b^2 - 4ac > 0$.
   (iii) If $a < 0$ and $c > 0$, then we set $\sqrt{ax^2+bx+c} = yx + \sqrt{c}$ or $yx - \sqrt{c}$.

ii. Assume that we must compute an integral of the form $\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}} dx$, where $P_n(x)$ is a polynomial of degree n with respect to x. Then
   $\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}} dx = Q_{n-1}(x)\sqrt{ax^2+bx+c} + k \int \frac{dx}{\sqrt{ax^2+bx+c}}$,
   where $Q_{n-1}(x)$ is a polynomial of degree $n-1$ with respect to x, and k is a constant. Differentiating both sides of the aforementioned equation and multiplying by $\sqrt{ax^2+bx+c}$, we obtain
   $P_n(x) = Q'_{n-1}(x)(ax^2+bx+c) + \frac{1}{2} Q_{n-1}(x)(2ax+b) + k$.
   From the last equation, we obtain a system of $n+1$ linear equations, so that we can determine the coefficients of the polynomial $Q_{n-1}(x)$ and the constant k.

iii. If we have to compute an integral of the form $\int R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right) dx$, where R is a rational function, n is an even natural number, and $ad \neq cb$, then we set $\frac{ax+b}{cx+d} = y^n \Rightarrow x = \frac{b-dy^n}{cy^n-a}$, and $dx = \frac{n(ad-bc)y^{n-1}}{(a-cy^n)^2} dy$. Of course, if n is even, then $\frac{ax+b}{cx+d}$ must be positive. If, however, we have to compute $\int R\left(x, \sqrt[m]{\frac{ax+b}{cx+d}}, \sqrt[n]{\frac{ax+b}{cx+d}}, \ldots\right) dx$, where R is a rational function, $ad \neq cb$, and $m, n, \ldots \in \mathbb{N}$, then we set $\frac{ax+b}{cx+d} = y^t$, where t is the least common multiple of $m, n, \ldots$

iv. If we have to compute an integral of the form $\int x^m (a+bx^n)^k dx$, where $a, b \in \mathbb{R}^*$, and $m, n, k \in \mathbb{Q}$, then we apply Tchebychev's rule, according to which at least one of the numbers k, $\frac{m+1}{n}$, and $\frac{m+1}{n} + k$ must be an integer. Hence:
   (i) If $k \in \mathbb{Z}$, then we set $y^t = x$, where t is the least common multiple of the denominators of m and n.
   (ii) If $\frac{m+1}{n} \in \mathbb{Z}$, then we set $a + bx^n = y^d$, where d is the denominator of k.
   (iii) If $\frac{m+1}{n} + k \in \mathbb{Z}$, then we set $ax^{-n} + b = y^d$, where d is the denominator of k.

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[509] Ibid.

2.16. INTEGRATION OF TRIGONOMETRIC FUNCTIONS[510]

Case Studies:

i. If all the terms inside a trigonometric integral are raised to an even degree power, then we make the substitution $tanx = w \Rightarrow dx = \frac{dw}{1+w^2}$, since $tanx = w \Rightarrow (tanx)' = dw \Rightarrow \frac{1}{cos^2x}dx = dw \Rightarrow (1 + tan^2x)dx$ or $(1 + w^2)dx = dw \Rightarrow dx = \frac{dw}{1+w^2}$. Then, of course, $cosx = \frac{1}{\sqrt{1+w^2}}$, since $\frac{1}{\sqrt{1+w^2}} = \frac{1}{\sqrt{1+tan^2x}} = \frac{1}{\sqrt{\frac{1}{cos^2x}}} = \frac{1}{\frac{1}{cosx}} = cosx$. Moreover, then $sinx = \frac{w}{\sqrt{1+w^2}}$, since $\frac{w}{\sqrt{1+w^2}} = \frac{tanx}{\sqrt{1+tan^2x}} = \frac{\frac{sinx}{cosx}}{\frac{1}{cosx}} = sinx$.

ii. If at least one of the terms inside the trigonometric integral is raised to an odd degree power, then we make the substitution $tan\frac{x}{2} = w \Rightarrow dx = \frac{2dw}{1+w^2}$. Then $cosx = \frac{1-w^2}{1+w^2}$, and $sinx = \frac{2w}{1+w^2}$.

iii. Now, let us assume that we have to compute an integral of the form $\int sin^m x cos^n x dx$. Then we have to distinguish between the following three sub-cases:

(i) If $m, n \in \mathbb{Z}_+^*$, then we apply the method of integration by parts: $I_{m,n} = \int sin^m x cos^n x dx = \int sin^m x cos^{n-1} x d(sinx) = (sin^{m+1}x)(cos^{n-1}x) - \int (sinx)\,[msin^{m-1}xcos^n x - (n-1)sin^{m+1}xcos^{n-2}x]dx = sin^{m+1}xcos^{n-1}x - mI_{m,n} + (n-1)\int sin^m x(1 - cos^2x)\,cos^{n-2}xdx = sin^{m+1}xcos^{n-1}x - mI_{m,n} + (n-1)I_{m,n-2} - (n-1)I_{m,n}$. Thus,
$I_{m,n} = \frac{sin^{m+1}xcos^{n-1}x}{m+n} + \frac{n-1}{m+n}I_{m,n-2}$, where $n \geq 2$, and
$I_{m,n} = -\frac{sin^{m+1}xcos^{n+1}x}{m+n} + \frac{m-1}{m+n}I_{m-2,n}$, where $m \geq 2$.

(ii) If $m, n \in \mathbb{Z}$, then:
if m is odd and positive, then we set $cosx = w$;
if n is odd and positive, then we set $sinx = w$;
if $m + n$ is even and negative, then we set $tanx = w$;
if m and n are even and positive, then we apply the trigonometric formulas $sin^2x = \frac{1-cos2x}{2}$ and $cos^2x = \frac{1+cos2x}{2}$;
if $m + n = 0$, then we obtain $\int tan^m xdx = \int tan^{m-2}x(1 + tan^2x - 1)\,dx = \int tan^{m-2}xdtanx - \int tan^{m-2}xdx = \frac{tan^{m-1}x}{m-1} - \int tan^{m-2}xdx$.

(iii) If m and n are rational numbers, then the given integral reduces to a binomial integral, setting $sinx = w$; and then the given integral becomes
$\int w^m\,(1 - w^2)^{\frac{n-1}{2}}dw$.

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[510] Ibid.

2.17. INTEGRATION OF HYPERBOLIC FUNCTIONS[511]

Usually, we integrate hyperbolic functions by setting $tanh\frac{x}{2} = t$, so that $dx = \frac{2dt}{1-t^2}$, $sinhx = \frac{2t}{1-t^2}$, and $coshx = \frac{1+t^2}{1-t^2}$. For instance, setting $tanh\frac{x}{2} = t$, we compute the integral $\int \frac{dx}{sinhx} = \int \frac{1}{\frac{2t}{1-t^2}} \cdot \frac{2dt}{1-t^2} = \int \frac{dt}{t} = ln|t| + c = ln\left|tanh\frac{x}{2}\right| + c$. However, sometimes, the substitution $e^x = t$ may prove to be more helpful. For instance, $\int \frac{dx}{sinhx+coshx} = \int \frac{2dx}{e^x-e^{-x}+e^x+e^{-x}} = \int e^{-x}\,dx = -e^{-x} + c$.

2.18. THE THEORY OF RIEMANN INTEGRATION

Apart from the definition of the integral as the limit of a sum, which was studied in section 2.11, there is a more rigorous approach to integration developed by Bernhard Riemann. Riemann, one of the towering figures of modern mathematics, was the main instigator of setting up the theory of integration as a rigorous subfield of mathematical analysis independently of physics. In this section, we shall study the theory of Riemann integration, and we shall prove its equivalence to the aforementioned definition of the integral as the limit of a sum.

The Riemann Integral[512]

Assume that f is a bounded function defined on $[a,b]$, and that $P = \{a = x_0, x_1, x_2, \dots, x_{k-1}, x_k, \dots, x_n = b\}$ is a partition of $[a,b]$, so that $[x_0,x_1], [x_1,x_2], \dots, [x_{k-1},x_k], \dots, [x_{n-1},x_n]$ are the n subintervals of $[a,b]$. Hence, f is bounded on each subinterval. Let $\Delta x_k = x_k - x_{k-1}$ be the length of the kth subinterval (where $k = 1,2,\dots,n$), and let m_k and M_k be the infimum (greatest lower bound) and the supremum (least upper bound) of f in $[x_{k-1},x_k]$, respectively (see Figure 2.36). Then we define the sums $U(P,f)$ and $L(P,f)$ as follows:

$$U(P,f) = M_1\Delta x_1 + M_2\Delta x_2 + \cdots + M_k\Delta x_k + \ldots + M_n\Delta x_n = \sum_{k=1}^{n} M_k\Delta x_k,$$

and

$$L(P,f) = m_1\Delta x_1 + m_2\Delta x_2 + \cdots + m_k\Delta x_k + \ldots + m_n\Delta x_n = \sum_{k=1}^{n} m_k\Delta x_k.$$

These sums are called, respectively, the “upper sum” and the “lower sum” of f corresponding to the partition P of $[a,b]$.

If $m = inf([a,b])$ and $M = sup([a,b])$, then, $k \in \mathbb{N}$, it holds that

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[511] Ibid.
[512] Ibid.

$m \leq m_k \leq M_k \leq M.$

Because $\Delta x_k \geq 0$,

$m\Delta x_k \leq m_k\Delta x_k \leq M_k\Delta x_k \leq M\Delta x_k.$

Therefore, for the n sums, it holds that

$m\sum_{k=1}^{n}\Delta x_k \leq \sum_{k=1}^{n} m_k\Delta x_k \leq \sum_{k=1}^{n} M_k\Delta x_k \leq M\sum_{k=1}^{n}\Delta x_k \Rightarrow m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a).(*)$

Notice that a pair of lower and upper sums corresponds to each partition of $[a,b]$. Then let A be the set of all possible upper sums, and let B be the set of all possible lower sums, namely:

$A = \{U(P,f)\ \forall P\ of\ [a,b]\}$ and
$B = \{L(P,f)\ \forall P\ of\ [a,b]\}$.

Hence, due to $(*)$, the sets A and B are bounded, so that $\sup(A) = M(b-a)$, and $\inf(B) = m(b-a)$.
Let us set $J = \inf(A)$ and $I = \sup(B)$, namely:

$J = \inf(\{U(P,f)\ \forall P\ of\ [a,b]\}$ and
$I = \sup(\{L(P,f)\ \forall P\ of\ [a,b]\}$.

Then J and I are called, respectively, the "upper integral" and the "lower integral" of f on $[a,b]$, and they are denoted as follows:

$$J = \int_{a}^{\bar{b}} f(x)dx$$

and

$$I = \int_{\bar{a}}^{b} f(x)dx$$

(given the aforementioned notation). If $J = I$, that is, if $\int_{a}^{\bar{b}} f(x)dx = \int_{\bar{a}}^{b} f(x)dx$, then f is said to be "Riemann integrable," or simply "integrable," on (or over) $[a,b]$, and the common value of its upper and lower integrals is denoted by

$$\int_{a}^{b} f(x)dx$$

and is called the "integral" of the function f on (or over) $[a,b]$.

Notice that the existence of the integral $\int_a^b f(x)dx$ implies that the function f is bounded and integrable on $[a,b]$. However, the converse is not necessarily true, in the sense that a bounded function may not be Riemann integrable. For instance, the function $f:[a,b] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, x \in [a,b] \; and \; x \; is \; rational \\ 1, x \in [a,b] \; and \; x \; is \; irrational \end{cases}$$

is not Riemann integrable, because: in any subinterval $[x_{k-1}, x_k]$ of $[a,b]$, $m_k = 0$ and $M_k = 1$, so that $U(P,f) = \sum_{k=1}^{n} M_k \Delta x_k = (b-a)$ and $L(P,f) = \sum_{k=1}^{n} m_k \Delta x_k = 0$, and, therefore, $J = (b-a)$ and $I = 0$, that is, $J \neq I$.

If $f(x) = k \; \forall x \in [a,b]$ is an arbitrary constant function, then it is Riemann integrable, because, for any partition P of $[a,b]$, $U(P,f) = k(b-a)$ and $L(P,f) = k(b-a)$, so that $J = k(b-a) = I$.

Obviously, the manner in which we have just defined a Riemann integrable function implies that the corresponding function must be bounded, and that neither of the limits of integration is infinite. However, as I shall explain later, these constraints can be removed, so that Riemann's theory of integration can be generalized to apply to unbounded functions with one limit of integration or both the limits of integration being equal to infinity.

If P' and P are two partitions of $[a,b]$ such that $P \subset P'$, then P' is said to be a "refinement" of P, or P' is said to be "finer" than P. If $P' = P_1 \cup P_2$, then P' is a refinement of both P_1 and P_2, and, in particular, it is called a "common refinement" of P_1 and P_2.

Theorem[513]*:* If we increase the number of points in a partition P, then:

i. the new upper sum $U(P_1,f)$ will not exceed $U(P,f)$, that is, $U(P,f) \geq U(P_1,f)$, and
ii. the new lower sum $L(P_1,f)$ will not recede $L(P,f)$, that is, $L(P,f) \leq L(P_1,f)$.

Proof: (i) Let $P = \{a = x_0, x_1, \dots, x_{k-1}, x_k, \dots, x_n = b\}$ be a partition of $[a,b]$, and let P' be another partition of $[a,b]$ such that P' has one more point x' than P, and $x_{k-1} \leq x' \leq x_k$. If

$M_k = \sup\,(f(x))$ in $[x_{k-1}, x_k]$,
$M_k' = \sup\,(f(x))$ in $[x_{k-1}, x']$,
$M_k'' = \sup\,(f(x))$ in $[x', x_k]$,

then $M_k \geq M_k'$ and $M_k \geq M_k''$, so that

$$(M_k - M_k') \geq 0 \text{ and } (M_k - M_k'') \geq 0. \qquad (*)$$

Notice that $U(P,f) - U(P',f) = M_k(x_k - x_{k-1}) - [M_k'(x' - x_{k-1}) + M_k''(x_k - x')]$. Adding and subtracting $M_k x'$ in the right-hand side, we obtain $U(P,f) - U(P',f) = (M_k - M_k')(x' - x_{k-1}) + (M_k - M_k'')(x_k - x') \geq 0$, and, therefore, due to equation

513 Ibid.

$(*)$, $U(P,f) \geq U(P',f)$. Repeating the same process, we can prove the theorem when P' has n more points than P, where n is any non-zero natural number. (ii) The proof of this part of the theorem is analogous to the proof of part (i); and, in this case, we use $m'_k \geq m_k$ and $m''_k \geq m_k$.■

Theorem[514]*:* If P_1 and P_2 are two arbitrary partitions of $[a,b]$, then no lower sum can exceed any upper sum, symbolically:

i. $U(P_1,f) \geq L(P_2,f)$ and
ii. $U(P_2,f) \geq L(P_1,f)$.

Proof: Let $P' = P_1 \cup P_2$, so that $P' \supset P_1$ and $P' \supset P_2$. Then, due to the previous theorem on partitions, applied on P' and P_1 as well as on P' and P_2, we obtain

$$U(P_1,f) \geq U(P',f), \tag{$*$}$$

$$L(P_1,f) \leq L(P',f), \tag{$**$}$$

$$U(P_2,f) \geq U(P',f), \tag{$***$}$$

$$L(P_2,f) \leq L(P',f). \tag{$****$}$$

Moreover, by the definition of a partition,

$$U(P',f) \geq L(P',f), \tag{$*****$}$$

so that, due to $(*)$,$(****)$, and $(*****)$,

$$U(P_1,f) \geq U(P',f) \geq L(P',f) \geq L(P_2,f) \Rightarrow U(P_1,f) \geq L(P_2,f),$$

and, due to $(**)$, $(***)$, and $(*****)$,

$$U(P_2,f) \geq U(P',f) \geq L(P',f) \geq L(P_1,f) \Rightarrow U(P_2,f) \geq L(P_1,f).■$$

Corollary: The lower integral can never exceed the upper integral, symbolically:

$$\int_a^{\bar{b}} f(x)dx \geq \int_{\bar{a}}^{b} f(x)dx.$$

Proof: For the sake of contradiction, suppose that

$$\int_a^{\bar{b}} f(x)dx < \int_{\bar{a}}^{b} f(x)dx,$$

and that c is a value such that

[514] Ibid.

$\int_a^{\bar{b}} f(x)dx < c < \int_{\bar{a}}^{b} f(x)dx.$

Then the assumption that $\int_a^{\bar{b}} f(x)dx < c$ implies that there is a partition P_1 of $[a,b]$ such that $U(P_1,f) < c$, and the assumption that $c < \int_{\bar{a}}^{b} f(x)dx$ implies that there is a partition P_2 of $[a,b]$ such that $L(P_2,f) > c$. But then $U(P_1,f) < L(P_2,f)$, which is a contradiction. Therefore, $\int_a^{\bar{b}} f(x)dx \geq \int_{\bar{a}}^{b} f(x)dx.$■

For instance, we can show that the function $f: [0,1] \to \mathbb{R}$ such that $f(x) = x$ is Riemann integrable (and compute its integral) on $[0,1]$ as follows: We divide the interval $[0,1]$ into n equal subintervals, obtaining the partition $P = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n} = 1\right\}$, and, in each subinterval $\left[\frac{k-1}{n}, \frac{k}{n}\right]$, $m_k = \frac{k-1}{n}$ and $M_k = \frac{k}{n}$. Moreover, $\Delta x_k = x_k - x_{k-1} = \frac{1}{n}, k = 1,2, \ldots, n$. Hence,

$$L(P,f) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = \sum_{k=1}^{n} \frac{k-1}{n} \cdot \frac{1}{n} = \frac{1}{n^2}\big(0 + 1 + \cdots + (n-1)\big) = \frac{(n-1)n}{2n^2} = \frac{1}{2}\left(1 - \frac{1}{n}\right),$$ and

$$U(P,f) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) = \sum_{k=1}^{n} \frac{k}{n} \cdot \frac{1}{n} = \frac{1}{n^2}(1 + 2 + \cdots + n) = \frac{n(n+1)}{2n^2} = \frac{1}{2}\left(1 + \frac{1}{n}\right).$$

By definition,

$\int_{\bar{0}}^{1} f(x)dx = \sup\left(\{L(P,f), \forall P\ of\ [0,1]\}\right) = lim_{n\to\infty} \frac{1}{2}\left(1 - \frac{1}{n}\right) = \frac{1}{2}$, and
$\int_0^{\bar{1}} f(x)dx = \inf\left(\{U(P,f), \forall P\ of\ [0,1]\}\right) = lim_{n\to\infty} \frac{1}{2}\left(1 + \frac{1}{n}\right) = \frac{1}{2}.$

Because $\int_{\bar{0}}^{1} f(x)dx = \int_0^{\bar{1}} f(x)dx = \frac{1}{2}$, it follows that $\int_0^1 f(x) = \frac{1}{2}$.

The Construction of a Riemann Integral by Darboux

For every partition P of an interval $[a,b]$, the length of the greatest subinterval is called the "norm of the partition," and it is denoted by $\|P\|$, namely,

$$\|P\| = max\{x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}\} = max\{(x_k - x_{k-1}), k = 1,2, \ldots, n\}.$$

The French mathematician Jean-Gaston Darboux (1842–1917)studied integration by considering upper and lower integrals, which exist for any bounded real-valued function f on the interval $[a,b]$. Then the "Darboux integral" exists if and only if the upper and the lower integrals are equal, and, obviously, the definition of the "Darboux integral" is equivalent to the definition of the "Riemann integral" (i.e., a function is "Riemann integrable" if and only if it is "Darboux integrable"). In fact, Darboux obtained "lower bounds" for the area under the curve $y = f(x)$ via inscribed rectangles, and "upper bounds" for the area under the curve $y =$

$f(x)$ via rectangles that circumscribed the curve (see Figure 2.36): the sum of the areas of the inscribed rectangles (which underestimates the area under the curve) is said to be a lower sum, and the sum of the area of the circumscribed rectangles (which overestimates the area under the curve) is said to be an "upper sum." We can consider the aforementioned inscribed rectangles as being the largest rectangles that can be inscribed underneath the curve $y = f(x)$ using an arbitrary partition P, and, by analogy, we can consider the aforementioned circumscribed rectangles as being the largest rectangles that can be circumscribed above the curve using an arbitrary partition Q. The area of an inscribed rectangle whose base is $[x_{k-1}, x_k]$ and whose height is m_k is $m_k(x_k - x_{k-1})$, and the sum of the areas of the n inscribed rectangles is $s(P,f) = \sum_{k=1}^{n} m_k(x_k - x_{k-1})$. By analogy, the area of a circumscribed rectangle whose base is $[x_{k-1}, x_k]$ and whose height is M_k is $M_k(x_k - x_{k-1})$, and the sum of the areas of the n circumscribed rectangles is $S(Q,f) = \sum_{k=1}^{n} M_k(x_k - x_{k-1})$. Then it logically follows that

$$s(P,f) \leq area\ under\ the\ curve \leq S(Q,f). \qquad (*)$$

Given that $(*)$ holds for any choice of partition, it also holds if we take the supremums (least upper bounds) of the lower sums over all partitions P and the infimums (greatest lower bounds) of the upper sums over all partitions Q, symbolically:

$$\sup_{\mathrm{P}} s(P,f) \leq area\ under\ the\ curve \leq \inf_{\mathrm{Q}} S(Q,f).$$

If $\sup_{\mathrm{P}} s(P,f) = \inf_{\mathrm{Q}} S(Q,f)$, then the area $A(f,[a,b])$ under the curve $y = f(x)$ over the interval $[a,b]$ is defined as follows:

$$A(f,[a,b]) = \sup_{\mathrm{P}} s(P,f) = \inf_{\mathrm{Q}} S(Q,f);$$

and then $A(f,[a,b])$ is called the integral of $f(x)$ over the interval from a to b.

The aforementioned remarks were proved by Darboux in a rigorous way as follows:

Theorem[515]*:* Assume that $|f(x)| \leq c\ \forall x \in [a,b]$, $\delta > 0$, P_1 and P_2 are two partitions of $[a,b]$ such that $\|P_1\| \leq \delta$, and P_2 consists of P_1 and at most p additional points. Then

$$U(P_1,f) - U(P_2,f) \leq 2c\delta p.$$

Proof: Let $p = 1$, so that P_2 contains one more point than P_1, and let $x' \in (x_{k-1}, x_k)$ be the additional point of P_2, so that

$$P_2 = \{a = x_0, x_1, \dots, x_{k-1}, x', x_k, \dots, x_n = b\}.$$

Moreover, let M_k, M_k', and M_k'' be the supremums of f in $[x_{k-1}, x_k]$, $[x_{k-1}, x']$, and $[x', x_k]$, respectively. Then

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[515] Ibid.

$U(P_1, f) - U(P_2, f) = M_k(x_k - x_{k-1}) - [M'_k(x' - x_{k-1}) + M''_k(x_k - x')]$
$= (M_k - M'_k)(x' - x_{k-1}) + (M_k - M''_k)(x_k - x')$. $(*)$

Notice that $|f(x)| \leq c \Rightarrow -c \leq M'_k \leq M_k \leq c \Rightarrow M_k - M'_k \leq 2c$, and that $M_k - M'_k \geq 0$. Hence, $0 \leq M_k - M'_k \leq 2c$. Similarly, $0 \leq M_k - M''_k \leq 2c$. Therefore, due to $(*)$, $U(P_1, f) - U(P_2, f) \leq 2c(x' - x_{k-1}) + 2c(x_k - x') \leq 2c(x_k - x_{k-1}) \leq 2c\delta$. Consequently, $U(P_1, f) - U(P_2, f) \leq 2c\delta$, and, if P_2 contains p more points than P_1, then $U(P_1, f) - U(P_2, f) \leq 2c\delta p$.■

Darboux's Theorem[516]*:* Assume that f is a bounded function on $[a, b]$. Then, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that:

i. $U(P, f) < \int_a^{\bar{b}} f(x)dx + \varepsilon$ and
ii. $L(P, f) > \int_{\bar{a}}^{b} f(x)dx - \varepsilon$

for all partitions P of $[a, b]$ with $\|P\| \leq \delta$. In other words,

$lim_{\|P\|\to 0} U(P, f) = \int_a^{\bar{b}} f(x)dx$ and
$lim_{\|P\|\to 0} L(P, f) = \int_{\bar{a}}^{b} f(x)dx$.

Proof: (i) Let P be a partition of $[a, b]$ such that $\|P\| \leq \delta$. By hypothesis, f is a bounded function on $[a, b]$, that is, $|f(x)| \leq c \ \forall x \in [a, b]$. Moreover, by definition, $J = \int_a^{\bar{b}} f(x)dx$ is the infimum of the set of all possible upper sums, and, therefore, there exists a partition $P_1 = \{a = x_0, x_1, x_2, \ldots, x_{p-1}, x_p = b\}$ of $[a, b]$ such that

$U(P_1, f) < J + \frac{\varepsilon}{2}$,

where $\varepsilon > 0$. Assume that P_2 is another partition of $[a, b]$ such that $P_2 = P_1 \cup P$; and P_2 is finer than P, and it contains at most $(p - 1)$ more points than P. Therefore (according to the immediately preceding theorem),

$U(P, f) - U(P_2, f) \leq 2c\delta(p - 1)$.

Because $P_2 \supset P_1 \Rightarrow U(P_2, f) \leq U(P_1, f)$, it holds that $U(P, f) - 2c\delta(p - 1) \leq U(P_2, f) \leq U(P_1, f) \Rightarrow U(P, f) \leq 2c\delta(p - 1) + U(P_1, f)$.

If we set $2c\delta(p - 1) = \frac{\varepsilon}{2}$, then $U(P, f) < \frac{\varepsilon}{2} + \left(J + \frac{\varepsilon}{2}\right) \Rightarrow U(P, f) < J + \varepsilon \Rightarrow U(P, f) < \int_a^{\bar{b}} f(x)dx + \varepsilon$. (ii) The proof of (ii) is analogous to the proof of (i).■

[516] Ibid.

Criteria of Integrability and Methods of Integration

Theorem (Riemann Condition)[517]*:* Let f be a function bounded on $[a,b]$. Then f is Riemann integrable on $[a,b]$ if and only if, $\forall\varepsilon > 0$, there is a partition P of $[a,b]$ such that $U(P,f) - L(P,f) < \varepsilon$.

Proof: Let us assume that f is Riemann integrable on$[a,b]$. Then

$$I = \int_{\bar{a}}^{b} f(x)dx = J = \int_{a}^{\bar{b}} f(x)dx.$$

Because $J = \int_{a}^{\bar{b}} f(x)dx = \inf(\{U(P,f)\ \forall P\ of\ [a,b]\})$, there is a partition, say P_1, of $[a,b]$ such that $U(P_1,f) < J + \frac{\varepsilon}{2}$ where $\varepsilon > 0$. Moreover, because $I = \int_{\bar{a}}^{b} f(x)dx = \sup(\{L(P,f)\ \forall P\ of\ [a,b]\})$, there is a partition, say P_2, of $[a,b]$ such that $L(P_2,f) > I - \frac{\varepsilon}{2}$. Let $P = P_1 \cup P_2$, so that $P \supseteq P_1$ and $P \supseteq P_2$. Therefore,

$$U(P,f) \leq U(P_1,f) < J + \frac{\varepsilon}{2} \Rightarrow U(P,f) < J + \frac{\varepsilon}{2}, \quad (*)$$

$$L(P,f) \geq L(P_2,f)) > I - \frac{\varepsilon}{2} \Rightarrow L(P,f) > I - \frac{\varepsilon}{2} \Rightarrow -L(P,f) < -I + \frac{\varepsilon}{2}, \quad (**)$$

and, by adding $(*)$ and $(**)$, we obtain
$U(P,f) - L(P,f) < \varepsilon$, that is, $I = J$.

Now, we shall prove the converse. Suppose that, $\forall\varepsilon > 0$, there is a partition, say P, of $[a,b]$ such that $U(P,f) - L(P,f) < \varepsilon$. Because $J = \inf(\{U(P,f)\ \forall P\ of\ [a,b]\}) \leq U(P,f)$ and $I = \sup(\{L(P,f)\ \forall P\ of\ [a,b]\}) \geq L(P,f)$, it follows that $J - I \leq U(P,f) - L(P,f) < \varepsilon \Rightarrow J < I + \varepsilon$. Since ε is arbitrary, $J \leq I$. Moreover, it is known that $I \leq J$. Therefore, $I = J$, which proves that $f(x)$ is Riemann integrable.■

Theorem[518]*:* If $f: [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, then f is Riemann integrable on $[a,b]$.

Proof: Because f is continuous on a closed interval, it is also uniformly continuous on that interval, and, therefore,

$$\forall\varepsilon > 0, \exists\delta > 0 | \forall x, y \in [a,b], |x - y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{b-a}. \quad (*)$$

Moreover, because f is continuous on the closed interval$[a,b]$, it attains maximum and minimum values in every subinterval of $[a,b]$. If P is a partition of $[a,b]$ with $\|P\| < \delta$ (where $\|P\|$ is the norm of the partition, namely, the length of the greatest subinterval, i.e., $\|P\| = |x_1 - x_2|\ \forall x_1, x_2 \in [x_{k-1}, x_k]$), then there exist x_k' and x_k'' in $[x_{k-1}, x_k]$, where $k =$

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[517] Ibid.
[518] Ibid.

$1,2,\ldots,n$, such that $f(x_k') = M_k$ and $f(x_k'') = m_k$, where M_k and m_k are the supremum and the infimum of f in $[x_{k-1}, x_k]$, respectively. Moreover, because $|x_k' - x_k''| < \delta$, the condition $(*)$ implies that $|f(x_k') - f(x_k'')| < \frac{\varepsilon}{b-a}$. Consequently:

$U(P,f) - L(P,f) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) - \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = \sum_{k=1}^{n}(M_k - m_k)(x_k - x_{k-1}) = \sum_{k=1}^{n}[f(x_k') - f(x_k'')]\,(x_k - x_{k-1}) < \frac{\varepsilon}{b-a}\sum_{k=1}^{n}(x_k - x_{k-1}) = \frac{\varepsilon}{b-a}(b-a) = \varepsilon$, meaning that f is Riemann integrable on $[a,b]$, according to the aforementioned Riemann Condition.■

Using Riemann's definition of an integral, we can apply the following two methods of integration:

First Method: We divide the interval $[a,b]$ into n equal subintervals, and we apply the equality $\int_a^b f(x)dx = lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta x_k$, where $x_{k-1} \le c_k \le x_k$, and $\Delta x_k = x_k - x_{k-1}$. Usually, we choose the point c_k to be the right-hand or the left-hand endpoint of each subinterval. However, often, it is preferable to make an arithmetic or a geometric partition, that is, to choose $x_0, x_1, x_2, \ldots, x_n$ in such a way that they are successive terms of an arithmetic progression whose common difference is $d = \frac{b-a}{n}$ or successive terms of a geometric progression whose common ratio is $r = \sqrt[n]{\frac{b}{a}}$ where $\frac{b}{a} > 0$. Arithmetic partition is preferable when we deal with functions of the following forms: $f(x) = ax$, $f(x) = sinx$, $f(x) = cosx$, $f(x) = tanx$, and $f(x) = a^x$. Geometric partition is preferable when we deal with functions of the following forms: $f(x) = log_a x$, $f(x) = ax^k$ with $k \in \mathbb{R} - \{1\}$, and $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials.

Example: We can compute $\int_a^b sinxdx$, $a < b$, as follows: Consider the points $x_0 = a, x_1 = a + d, x_2 = a + 2d, \ldots, x_n = a + nd = b$, where $d = \frac{b-a}{n}$. We choose the points c_k to be the left-hand endpoints of the subintervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$, so that $c_k = a + (k-1)d$. Hence,

$\int_a^b f(x)dx = lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\,\Delta x_k \Rightarrow \int_a^b sinxdx = lim_{n\to\infty}\sum_{k=1}^{n}(sin[a + (k-1)d])[(a+kd) - (a+(k-1)d)] = lim_{n\to\infty}\sum_{k=1}^{n}(sin[a+(k-1)d])\,d = lim_{n\to\infty} d[sina + sin(a+d) + sin(a+2d) + \cdots + sin(a+(n-1)d)]$.

Setting

$$S = sina + sin(a+d) + sin(a+2d) + \cdots + sin(a+(n-1)d),$$

and multiplying by $sin\frac{d}{2}$, we obtain

$$Ssin\frac{d}{2} = sinasin\frac{d}{2} + sin(a+d)sin\frac{d}{2} + \cdots + sin(a+(n-1)d)sin\frac{d}{2}.$$

But, because $sinasinb = \frac{cos(a-b)}{2} - \frac{cos(a+b)}{2}$, we have:

$$sinasin\frac{d}{2} = \frac{1}{2}\left[cos\left(a-\frac{d}{2}\right) - cos\left(a+\frac{d}{2}\right)\right],$$
$$sin(a+d)sin\frac{d}{2} = \frac{1}{2}\left[cos\left(a+\frac{d}{2}\right) - cos\left(a+\frac{3d}{2}\right)\right],$$
$$\vdots$$
$$sin(a+(n-1)d)sin\frac{d}{2} = \frac{1}{2}\left[cos\left(a+(n-1)d-\frac{d}{2}\right) - cos\left(a+(n-1)d+\frac{d}{2}\right)\right].$$

Adding the above equations by parts, we obtain

$$sinasin\frac{d}{2} + \cdots + sin(a+(n-1)d)sin\frac{d}{2} = \frac{1}{2}\left[cos\left(a-\frac{d}{2}\right) - cos\left(a+(n-1)d+\frac{d}{2}\right)\right].$$

Because $x_n = a + nd = b$, we have

$$sinasin\frac{d}{2} + \cdots + sin(a+(n-1)d)sin\frac{d}{2} = \frac{1}{2}\left[cos\left(a-\frac{d}{2}\right) - cos\left(b-\frac{d}{2}\right)\right].$$

Therefore,

$$Ssin\frac{d}{2} = sinasin\frac{d}{2} + sin(a+d)sin\frac{d}{2} + \cdots + sin(a+(n-1)d)sin\frac{d}{2} \Rightarrow 2Ssin\frac{d}{2} = cos\left(a-\frac{d}{2}\right) - cos\left(b-\frac{d}{2}\right) \Rightarrow S = \frac{cos\left(a-\frac{d}{2}\right)-cos\left(b-\frac{d}{2}\right)}{2sin\frac{d}{2}}.$$

Consequently, we obtain

$$\int_a^b sinxdx = lim_{n\to\infty} d\frac{cos\left(a-\frac{d}{2}\right)-cos\left(b-\frac{d}{2}\right)}{2sin\frac{d}{2}} = lim_{n\to\infty}\frac{cos\left(a-\frac{d}{2}\right)}{\frac{sin\frac{d}{2}}{\frac{d}{2}}} - lim_{n\to\infty}\frac{cos\left(b-\frac{d}{2}\right)}{\frac{sin\frac{d}{2}}{\frac{d}{2}}} =$$

$\frac{cosa}{1} - \frac{cosb}{1} = cosa - cosb$, since we have $lim_{n\to\infty}\frac{d}{2} = lim_{n\to\infty}\frac{b-a}{2n} = 0$, and $lim_{\frac{d}{2}\to 0}\frac{sin(d/2)}{d/2} = 1$ (since, in general, $lim_{x\to 0}\frac{sinx}{x} = 1$).

Second method: We divide the interval $[a,b]$ into n equal subintervals of length $\frac{b-a}{n}$. For the right-hand endpoint of $[x_{k-1}, x_k]$, it holds that $x_k = a + k\frac{b-a}{n}$, where $k = 1,2,\dots,n$. In the equation $\int_a^b f(x)dx = lim_{n\to\infty}\sum_{k=1}^n f(c_k)\Delta x_k$, we set $\Delta x_k = \frac{b-a}{n}$ and $c_k = x_k = a + k\frac{b-a}{n}$, and, thus, we obtain

$$\int_a^b f(x)dx = lim_{n\to\infty}\sum_{k=1}^n f\left(a+k\frac{b-a}{n}\right)\frac{b-a}{n} \Rightarrow \int_a^b f(x)dx = lim_{n\to\infty}\frac{b-a}{n}\sum_{k=1}^n f\left(a+k\frac{b-a}{n}\right).$$

Example: We can compute $\int_1^5 x^2\,dx$ as follows: We divide the interval $[a,b]$ into n equal subintervals of length $\frac{5-1}{n}=\frac{4}{n}$, so that we obtain

$$\sum_{k=1}^{n} f\left(1+k\frac{4}{n}\right)=\sum_{k=1}^{n}\left(1+k\frac{4}{n}\right)^2=\left(1+\frac{4}{n}\right)^2+\left(1+2\frac{4}{n}\right)^2+\left(1+3\frac{4}{n}\right)^2+\cdots+\left(1+n\frac{4}{n}\right)^2=\left(1+2\frac{4}{n}+\frac{4^2}{n^2}\right)+\left(1+4\frac{4}{n}+2^2\frac{4^2}{n^2}\right)+\left(1+6\frac{4}{n}+3^2\frac{4^2}{n^2}\right)+\cdots+\left(1+2n\frac{4}{n}+n^2\frac{4^2}{n^2}\right)=n+2\frac{4}{n}(1+2+3+\cdots+n)+\frac{4^2}{n^2}(1+2^2+3^2+\cdots+n^2)=n+2\frac{4}{n}\frac{n(n+1)}{2}+\frac{4^2}{n^2}\frac{n(n+1)(2n+1)}{6}=\frac{62n^3+72n^2+16n}{6n^2}.$$

Therefore, the fact that $\int_a^b f(x)dx=lim_{n\to\infty}\frac{b-a}{n}\sum_{k=1}^{n} f\left(a+k\frac{b-a}{n}\right)$ implies that

$$\int_1^5 x^2\,dx=lim_{n\to\infty}\frac{4}{n}\left(\frac{62n^3+72n^2+16n}{6n^2}\right)=\frac{124}{3}.$$

Properties of Riemann Integrable Functions

Theorem[519]*:* If a function f is integrable on $[a,b]$ and $[c,d]\subset[a,b]$, then: (i) f is integrable on $[c,d]$, and (ii) $\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^d f(x)dx+\int_d^b f(x)dx.$

Proof: Because f is integrable on $[a,b]$, there is a partition P of $[a,b]$ such that $U(P,f)-L(P,f)<\varepsilon$ for $\varepsilon>0$. Let

$P'=P\cup\{c,d\}$,
$P_1=P'\cap[a,c]$,
$P_2=P'\cap[c,d]$, and
$P_3=P'\cap[,b]$.

Then $U(P_2,f)-L(P_2,f)\leq U(P,f)-L(P,f)<\varepsilon$, which implies that f is integrable on $[c,d]$. Similarly, f is integrable on $[a,c]$ and on $[d,b]$. Moreover,

$$U(P,f)=U(P_1,f)+U(P_2,f)+U(P_3,f),$$

and, therefore, (ii) holds.■

Theorem[520]*:* If $f:[a,b]\to\mathbb{R}$ is integrable and $c\in\mathbb{R}$, then cf is also integrable on $[a,b]$, and $\int_a^b cf(x)dx=c\int_a^b f(x)dx.$

Proof: Suppose that $c\geq 0$. Then, for any set $A\subset[a,b]$,

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[519] Ibid.
[520] Ibid.

$\sup_{\mathrm{A}}(cf) = c\sup_{\mathrm{A}}(f)$ and $\inf_{\mathrm{A}}(cf) = c\inf_{\mathrm{A}}(f)$, so that

$U(P, cf) = cU(P, f)$ for every partition P of A. Computing the infimum over the set P^* of all partitions of $[a, b]$, we obtain

$U(cf) = \inf_{\mathrm{P}\in P^*} U(P, cf) = \inf_{\mathrm{P}\in P^*} cU(P, f) = c\inf_{\mathrm{P}\in P^*} U(P, f) = cU(f)$.

Similarly, $L(P, cf) = cL(P, f)$ and $L(cf) = cL(f)$. If f is integrable, then

$U(cf) = cU(f) = cL(f) = L(cf)$,
which implies that cf is integrable and $\int_a^b cf(x)dx = c\int_a^b f(x)dx$.

Now, let us consider the case of $-f$. Because

$\sup_{\mathrm{A}}(-f) = -\inf_{\mathrm{A}}(f)$ and $\inf_{\mathrm{A}}(-f) = -\sup_{\mathrm{A}}(f)$, we have
$U(P, -f) = -L(P, f)$ and $L(P, -f) = -U(P, f)$. Hence,
$U(-f) = \inf_{\mathrm{P}\in P^*} U(P, -f) = \inf_{\mathrm{P}\in P^*}[-L(P, f)] = -\sup_{\mathrm{P}\in P^*} L(P, f) = -L(f)$ and
$L(-f) = \sup_{\mathrm{P}\in P^*} L(P, -f) = \sup_{\mathrm{P}\in P^*}[-U(P, f)] = -\inf_{\mathrm{P}\in P^*} U(P, f) = -U(f)$.

Therefore, $-f$ if is integrable if f is integrable, and

$\int_a^b [-f(x)]\,dx = -\int_a^b f(x)dx$.

Finally, if $c < 0$, then $c = -|c|$, and, by successively applying the aforementioned results, we can show that the theorem holds.■

Theorem[521]*:* If $f, g: [a, b] \to \mathbb{R}$ are integrable functions, then: (i) $f + g$ is also integrable on $[a, b]$, and $\int_a^b (f + g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$; (ii) $f - g$ is also integrable on $[a, b]$, and $\int_a^b (f - g)(x)dx = \int_a^b f(x)dx - \int_a^b g(x)dx$.

Proof: (i) Let $h(x) = f(x) + g(x)\ \forall x \in [a, b]$. If we assume that

M_k and m_k are, respectively, the supremum and the infimum of f in $[x_{k-1}, x_k]$,
M'_k and m'_k are, respectively, the supremum and the infimum of g in $[x_{k-1}, x_k]$, and
M''_k and m''_k are, respectively, the supremum and the infimum of h in $[x_{k-1}, x_k]$, then

$$m_k + m'_k \leq m''_k \leq M''_k \leq M_k + M'_k$$

$$\Rightarrow \sum_{k=1}^n (m_k + m'_k)\,\Delta x_k \leq \sum_{k=1}^n m''_k\,\Delta x_k \leq \sum_{k=1}^n M''_k\,\Delta x_k \leq \sum_{k=1}^n (M_k + M'_k)\,\Delta x_k,$$

so that $L(P, f) + L(P, g) \leq L(P, h) \leq U(P, h) \leq U(P, f) + U(P, g)$.

Therefore,

[521] Ibid.

$U(P,h) \leq U(P,f) + U(P,g)$ and $(*)$
$L(P,h) \geq L(P,f) + L(P,g) \Leftrightarrow -L(P,h) \leq -L(P,f) - L(P,g)$. $(**)$

By $(*)$ and $(**)$,
$U(P,h) - L(P,h) \leq [U(P,f) - L(P,f)] + [U(P,g) - L(P,g)] < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$,

since f and g are integrable. Consequently, h is integrable on $[a,b]$.
Notice that

$U(P,f) + U(P,g) \geq U(P,h) \geq \int_a^b h(x)dx \geq L(P,h) \geq L(P,f) + L(P,g)$ and
$U(P,f) + U(P,g) \geq \int_a^b f(x)dx + \int_a^b g(x)dx \geq L(P,f) + L(P,g)$ imply that
$\left|\int_a^b f(x)dx + \int_a^b g(x)dx - \int_a^b h(x)dx\right| \leq [U(P,f) + U(P,g)] - [L(P,f) + L(P,g)] =$
$[U(P,f) - L(P,f)] + [U(P,g) - L(P,g)]$. Consequently,
$\int_a^b h(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.

(ii) The proof of (ii) is analogous to the proof of (i).■

Theorem[522]*:* If $f: [a,b] \to \mathbb{R}$ is integrable on $[a,b]$, then $|f|$ is also integrable on $[a,b]$, and $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|\, dx$.

Proof: Let us set

$F_1 = \frac{|f|+f}{2}$ and $F_2 = \frac{|f|-f}{2}$,

so that, $\forall x \in [a,b]$, $F_1(x) \geq 0$ and $F_2(x) \geq 0$, and

$f = F_1 - F_2$ and $|f| = F_1 + F_2$.

Given that f is integrable on $[a,b]$, so are F_1 and F_2, and, therefore, so is $|f|$.
Moreover,

$\left|\int_a^b f(x)dx\right| = \left|\int_a^b F_1(x)dx - \int_a^b F_2(x)dx\right| \leq \left|\int_a^b F_1(x)dx\right| + \left|\int_a^b F_2(x)dx\right| \leq$
$\int_a^b F_1(x)dx + \int_a^b F_2(x)dx \leq \int_a^b |f(x)|\, dx$.■

Remark: The converse may not hold. In other words, if $|f(x)|$ is integrable on $[a,b]$, then it does not necessarily hold that f is integrable on $[a,b]$. For instance, the function $f: [a,b] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 \; if \; x \; is \; rational \\ -1 \; if \; x \; is \; irrational \end{cases}$$

[522] Ibid.

is not integrable, since, for a partition P of $[a,b]$,

$$U(P,f)=\sum_{k=1}^{n} M_k\,\Delta x_k=(b-a)\neq L(P,f)=\sum_{k=1}^{n} m_k\,\Delta x_k=-(b-a).$$

Nevertheless, $|f(x)|$ is integrable on $[a,b]$, since $|f(x)|=1\ \forall x\in[a,b]$, and, therefore, $U(P,f)=L(P,f)=b-a$.

Theorem[523]*:* (i) If $f,g:[a,b]\to\mathbb{R}$ are integrable functions, then $fg:[a,b]\to\mathbb{R}$ is also integrable on $[a,b]$. (ii) If, in addition, $g\neq 0$ and $1/g$ is bounded, then $f/g:[a,b]\to\mathbb{R}$ is also integrable on $[a,b]$.

Proof: (i) First, we shall prove that the square of an integrable function is an integrable function. Assume that $f:[a,b]\to\mathbb{R}$ is integrable. If

m_k and M_k are, respectively, the infimum and the supremum of f in $[x_{k-1},x_k]$,
m'_k and M'_k are, respectively, the infimum and the supremum of $|f|$ in $[x_{k-1},x_k]$, and
m''_k and M''_k are, respectively, the infimum and the supremum of f^2 in $[x_{k-1},x_k]$, then
$M''_k-m''_k=(M'_k)^2-(m'_k)^2=(M'_k-m'_k)(M'_k+m'_k)$,

so that $M''_k-m''_k\leq 2M(M'_k-m'_k)$ where $M=\sup\,(|f|)$ in $[a,b]$.

Consequently, $U(P,f^2)-L(P,f^2)\leq 2M[U(P,|f|)-L(P,|f|)]$.

Because $|f|$ is integrable on $[a,b]$, f^2 is also integrable on $[a,b]$. Moreover, because $f,g:[a,b]\to\mathbb{R}$ are integrable on $[a,b]$, it holds that $f+g$, f^2, g^2, and $(f+g)^2$ are also integrable on $[a,b]$, and, therefore, $\frac{(f+g)^2-f^2-g^2}{2}$ is integrable on $[a,b]$.

In a similar way, we can prove that, if $g\neq 0$ and $1/g$ is bounded, then $f/g:[a,b]\to\mathbb{R}$ is also integrable on $[a,b]$.■

Theorem[524]*:* The Cauchy–Schwarz–Buniakowski Inequality states that

$$\sum_{k=1}^{n} a_k^2\sum_{k=1}^{n} b_k^2\geq\left(\sum_{k=1}^{n} a_k b_k\right)^2,$$

where $a_k,b_k\in\mathbb{R}$, $k=1,2,\dots,n$. For functions, this inequality can be reformulated as follows:

$$\int_a^b[f(x)]^2\,dx\int_a^b[g(x)]^2\,dx\geq\left[\int_a^b f(x)g(x)\right]^2,$$

where $f,g:[a,b]\to\mathbb{R}$ are integrable functions.

Proof: For any $\lambda\in\mathbb{R}$,

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[523] Ibid.
[524] Ibid.

$$0 \leq [\lambda f(x) + g(x)]^2 \Rightarrow 0 \leq \int_a^b [\lambda f(x) + g(x)]^2 dx = \lambda^2 \int_a^b [f(x)]^2 dx + 2\lambda \int_a^b f(x)g(x)dx + \int_a^b [g(x)]^2 dx,$$

so that we obtain the following polynomial with respect to λ:

$A\lambda^2 + 2B\lambda + C$ where
$A = \int_a^b [f(x)]^2 dx$,
$B = \int_a^b f(x)g(x)dx$, and
$C = \int_a^b [g(x)]^2 dx$.

Hence, $A\lambda^2 + 2B\lambda + C \geq 0$ holds $\forall \lambda \in \mathbb{R}$. Because the polynomial $A\lambda^2 + 2B\lambda + C$ is non-negative for any $\lambda \in \mathbb{R}$, and $A > 0$, the discriminant $(2B)^2 - 4AC$ of this polynomial must be non-positive, and, therefore, $B^2 \leq AC$, which proves the theorem.■

The Equivalence of the Definitions of the Integral of a Function

As I explained in section 2.11, the definite integral of a function can be defined as the limit of a sum, namely,$lim_{\|P\| \to 0} S(P, f, c_k) = \int_a^b f(x)dx$, and, in fact, this is the way in which Newton and Leibniz studied integral calculus in the seventeenth century. However, as I have already explained, Riemann's definition of an integral is based on the concept of boundedness rather than on the concept of a limit, so that, if J and I are, respectively, the upper and the lower integrals of f on $[a, b]$, that is, if $J = \int_a^{\bar{b}} f(x)dx$ and $I = \int_{\bar{a}}^b f(x)dx$, then f is said to be Riemann integrable on$[a, b]$ if $J = I$.

Theorem[525]*:* The two aforementioned definitions of integration (namely, Riemann's definition of an integral and the definition of an integral as the limit of a sum) are equivalent to each other, symbolically (according to the aforementioned notation):

$$J = I = l \Leftrightarrow lim_{\|P\| \to 0} S(P, f, c_k) = l.$$

Proof: First, we shall assume that $J = I = l$, and we shall prove that then $lim_{\|P\| \to 0} S(P, f, c_k) = l$. Hence, let

$$\int_a^{\bar{b}} f(x)dx = \int_{\bar{a}}^b f(x)dx = l,$$

which implies that f is bounded on $[a, b]$. Due to Darboux's Theorem, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|U(P, f) - l| < \varepsilon \text{ and } |L(P, f) - l| < \varepsilon \qquad (1)$$

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[525] Ibid.

for all partitions P of $[a,b]$ for which $\|P\| < \delta$.

By definition, $S(P,f,c_k) = \sum_{k=1}^{n} f(c_k)\Delta x_k$, where $c_k \in [x_{k-1}, x_k]$, and, thus,

$$U(P,f) \geq S(P,f,c_k) \geq L(P,f),$$

which, due to (1), implies that

$$l + \varepsilon > S(P,f,c_k) > l - \varepsilon \ \forall \|P\| < \delta,$$

so that $lim_{\|P\|\to 0} S(P,f,c_k) = l$.

Now, we shall assume that

$$lim_{\|P\|\to 0} S(P,f,c_k) = l, \tag{2}$$

and we shall prove that then $J = I = l$. Hence, let

$lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k = l$, so that, by (2), for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$l - \frac{\varepsilon}{4} < S(P,f,c_k) < l + \frac{\varepsilon}{4}. \tag{3}$$

We shall prove that, given (2), f is bounded on $[a,b]$ by applying the method of *reductio ad absurdum*. Thus, for the sake of contradiction, assume that f is not bounded on $[a,b]$, so that f is not bounded on $[x_{k-1}, x_k]$ for at least one value of k, say for $k = w$. Then let

$$\left|\sum_{k\neq w} f(c_k)\Delta x_k\right| = l_1. \tag{4}$$

Because f is not bounded on $[x_{k-1}, x_k]$, we can choose a $c_w \in [x_{w-1}, x_w]$ such that

$$|f(c_w)\Delta x_w| > |l| + l_1 + \varepsilon, \tag{5}$$

so that $\left|\sum_k f(c_k)\Delta x_k\right| \geq |f(c_w)\Delta x_w| - \left|\sum_{k\neq w} f(c_k)\Delta x_k\right|$. Because of (4) and (5),

$$\left|\sum_k f(\quad_k)\Delta x_k\right| > |l| + \varepsilon \Rightarrow lim_{\|P\|\to 0} S(P,f,c_k) \neq l,$$

which contradicts the assumption that $lim_{\|P\|\to 0} S(P,f,c_k) = l$. This contradiction implies that the assumption that f is not bounded on $[a,b]$ is wrong. In other words, f is bounded on $[a,b]$.

If m_k and M_k are, respectively, the infimum and the supremum of f in $[x_{k-1}, x_k]$, then there exist points p_k and q_k in $[x_{k-1}, x_k]$ such that

$f(p_k) > M_k - \frac{\varepsilon}{4(b-a)}$ and $f(q_k) < m_k + \frac{\varepsilon}{4(b-a)}$, so that

$$f(p_k)\Delta x_k > M_k \Delta x_k - \frac{\varepsilon}{4(b-a)}\Delta x_k \Rightarrow \sum_k f(p_k)\Delta x_k > U(P,f) - \frac{\varepsilon}{4}, \tag{6}$$

and $\sum_k f(q_k)\Delta x_k < L(P,f) + \frac{\varepsilon}{4}$. (7)

If $p_k = c_k$, then (6) becomes

$\sum_k f(c_k)\Delta x_k > U(P,f) - \frac{\varepsilon}{4}$,

implying, by (3), that $U(P,f) - \frac{\varepsilon}{4} < \sum_k f(c_k)\Delta x_k < l + \frac{\varepsilon}{4}$, which implies that

$$U(P,f) < l + \frac{\varepsilon}{2}. \tag{8}$$

Similarly, if $q_k = c_k$, then (7) becomes

$$L(P,f) > l - \frac{\varepsilon}{2}. \tag{9}$$

The inequalities (8) and (9) imply that

$$l - \frac{\varepsilon}{2} < L(P,f) \leq U(P,f) < l + \frac{\varepsilon}{2},$$

which implies that $J = I = l$.■

Monotonicity

Theorem[526]*:* If f is monotonic on $[a,b]$, then f is integrable on $[a,b]$.

Proof: Notice that, by definition, monotonic functions on $[a,b]$ are bounded functions. First, let us assume that f is increasing on $[a,b]$, that is, $f(b) \geq f(a)$ whenever $b > a$. Let P be a partition of $[a,b]$ such that

$\|P\| < \frac{\varepsilon}{f(b)-f(a)+1}$, where $\varepsilon > 0$.

Then $U(P,f) - L(P,f) = \sum_{k=1}^{n}(M_k - m_k)\,\Delta x_k$
$= \sum_{k=1}^{n}[f(x_k) - f(x_{k-1})]\,\Delta x_k < \frac{\varepsilon}{f(b)-f(a)+1}\sum_{k=1}^{n}[f(x_k) - f(x_{k-1})] <$
$\frac{\varepsilon}{f(b)-f(a)+1}[f(b) - f(a)] < \varepsilon$,

and, therefore, f is integrable on $[a,b]$, that is, the theorem holds in case f is increasing on $[a,b]$. By analogy, we can prove the theorem in case f is decreasing on $[a,b]$.■

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[526] Ibid.

Generalized Integrals

A "generalized integral" (also known as an "improper integral") is an integral with one or more infinite limits of integration and/or discontinuous integrands.

First Case: f is discontinuous at some points or at one point in the closed interval of integration $[a, b] \subset \mathbb{R}$.

i. If $f(x)$ is discontinuous at $x = x_0$, that is, if $f(x_0) \to \infty$, and $a < x_0 < b$, then
$\int_a^b f(x) = lim_{\varepsilon\to 0} \int_a^{x_0-\varepsilon} f(x)dx + lim_{\varepsilon\to 0} \int_{x_0+\varepsilon}^{b} f(x)dx.$

ii. If $f(x)$ is discontinuous at $x = x_0 = b$, that is, if $f(b) \to \infty$, then
$\int_a^b f(x) = lim_{\varepsilon\to 0} \int_a^{b-\varepsilon} f(x)dx$, or, equivalently,
$\int_a^b f(x) = lim_{k\to b^-} \int_a^k f(x)dx.$

iii. If $f(x)$ is discontinuous at $x = x_0 = a$, that is, if $f(a) \to \infty$, then
$\int_a^b f(x) = lim_{\varepsilon\to 0} \int_{a+\varepsilon}^{b} f(x)dx$, or, equivalently,
$\int_a^b f(x) = lim_{k\to a^+} \int_k^b f(x)dx.$

For instance, in $\int_0^n \frac{dx}{\sqrt{n^2-x^2}}$, the integrand is discontinuous at $x = n$, and, therefore, $\int_0^n \frac{dx}{\sqrt{n^2-x^2}} = lim_{\varepsilon\to 0} \int_0^{n-\varepsilon} \frac{dx}{\sqrt{n^2-x^2}} = lim_{\varepsilon\to 0} arcsin\frac{x}{n}|_0^{n-\varepsilon} = lim_{\varepsilon\to 0}\left(arcsin\frac{n-\varepsilon}{n} - arcsin\frac{0}{n}\right) = lim_{\varepsilon\to 0} arcsin1 = arcsin1 = \frac{\pi}{2}.$

Second Case: the interval of integration is infinite, that is, $(-\infty, b]$, $[a, +\infty)$, or $(-\infty, -\infty)$.

i. If $f(x)$ is continuous on (a, b) where $b = +\infty$, then
$\int_a^\infty f(x)dx = lim_{k\to\infty} \int_a^k f(x)dx.$

ii. If $f(x)$ is continuous on (a, b) where $a = -\infty$, then
$\int_{-\infty}^b f(x)dx = lim_{k\to-\infty} \int_k^b f(x)dx.$

iii. If $f(x)$ is continuous on (a, b) where $a = -\infty$ and $b = +\infty$, then
$\int_{-\infty}^\infty f(x)dx = lim_{k_1\to-\infty} \int_{k_1}^l f(x)dx + lim_{k_2\to\infty} \int_l^{k_2} f(x)dx.$

For instance, $\int_0^\infty \frac{dx}{e^x} = lim_{k\to\infty} \int_0^k \frac{dx}{e^x} = lim_{k\to\infty}(-e^{-x})|_0^k = 1.$

Riemann Integrability and Sets of Measure Zero

Theorem[527]*:* If f is bounded on $[a, b]$ and continuous except at finitely many points, say $c_1, c_2, \dots, c_n$, in $[a, b]$, then f is Riemann integrable.

Proof: Let m and M be, respectively, the infimum and the supremum of f on $[a, b]$, and let f be discontinuous at a finite number of points, say $c_1, c_2, \dots, c_n$, in $[a, b]$. Then we

[527] Ibid.

enclose these n points in n non-overlapping intervals $[a_1, b_1], [a_2, b_2], \dots, [a_n, b_n]$ such that the sum of their length is less than $\frac{\varepsilon}{2(M-m)}$, symbolically,

$$\sum_{i=1}^{n}(b_i - a_i) < \frac{\varepsilon}{2(M-m)}.$$

Hence, f is continuous on

$$[a, a_1], [b_1, a_2], [b_2, a_3], \dots, [b_n, b].$$

Let $P' = [a_1, b_1] \cup [a_2, b_2] \cup \dots \cup [a_n, b_n]$.

Because $\sum_{i=1}^{n}(b_i - a_i) < \frac{\varepsilon}{2(M-m)}$ and $\sum_{i=1}^{n}(M_i - m_i) < M - m$,

it holds that

$$U(P', f) - L(P', f) < \frac{\varepsilon}{2(M-m)}(M-m) = \frac{\varepsilon}{2}.$$

Notice that f is continuous on each of the $(n+1)$ intervals

$$[a, a_1], [b_1, a_2], [b_2, a_3], \dots, [b_n, b],$$

and, therefore, f is integrable on each of these intervals. Assume that P_1 is a partition of $[a, a_1]$, so that

$$U(P_1, f) - L(P_1, f) < \frac{\varepsilon}{2(n+1)}.$$

Similarly, assume that P_2 is a partition of $[b_1, a_2]$, so that

$$U(P_2, f) - L(P_2, f) < \frac{\varepsilon}{2(n+1)}.$$

We repeat the same process on each of the $(n+1)$ intervals, until we obtain partition P_{n+1}, which is a partition of $[b_n, b]$, so that

$$U(P_{n+1}, f) - L(P_{n+1}, f) < \frac{\varepsilon}{2(n+1)}.$$

Let $P = P' \cup P_1 \cup P_2 \cup \dots \cup P_{n+1}$. Then P is as partition of $[a, b]$, and

$U(P, f) - L(P, f) = U(P', f) - L(P', f) + \sum_{k=1}^{n+1}[U(P_k, f) - L(P_k, f)] < \frac{\varepsilon}{2} + \frac{\varepsilon}{2(n+1)}(n+1) = \varepsilon$, meaning that f is integrable on $[a, b]$.■

Theorem[528]*:* Assume that f is bounded on $[a, b]$ and discontinuous at infinitely many points of $[a, b]$. Then f is Riemann integrable if the set of its points of discontinuity has a finite number of accumulation points.

Proof: The proof of this theorem is similar to the proof of the immediately preceding theorem: Let $\{c_1, c_2, \dots, c_n\}$ be the finite set of the accumulation points of the set of discontinuous points of f in $[a, b]$. Then we enclose the points $c_1, c_2, \dots, c_n$ in n non-overlapping intervals $[a_1, b_1], [a_2, b_2], \dots, [a_n, b_n]$, and, therefore, as in the immediately preceding theorem,

$$U(P', f) - L(P', f) < \frac{\varepsilon}{2},$$

where $P' = [a_1, b_1] \cup [a_2, b_2] \cup \dots \cup [a_n, b_n]$.

Notice that only a finite number of points of discontinuity of f can lie in each of the $(n + 1)$ intervals $[a, a_1], [b_1, a_2], [b_2, a_3], \dots, [b_n, b]$, and the aforementioned method of proof can be applied in each of these subintervals in order to prove the theorem.■

The last two theorems help us to understand the mathematical concept of a measure. “Measure” is a rigorous generalization of the intuitive notion of size (e.g., of length, area, and volume).[529] In particular, the “Lebesgue measure” is a function that assigns real values (measures), including ∞, to subsets of an n-dimensional Euclidean space; and, for $n = 1,2, or 3$, the Lebesgue measure coincides with the standard measure of a length, an area, or a volume, respectively. In $\mathbb{R}$, the measure of an interval $[a, b]$ is its length, and it is denoted by $\mu([a, b])$, that is, $\mu([a, b]) = b - a$. In $\mathbb{R}^2$, the measure of a rectangle X whose dimensions are a and b is $\mu(B) = ab$ (i.e., its area), the measure of a circle C with radius r is $\mu(C) = \pi r^2$ (i.e., its area), etc. The measure of the empty set is zero, symbolically, $\mu(\emptyset) = 0$. The measure of a single point $x \in \mathbb{R}$ is zero, symbolically, $\mu(\{x\}) = 0$. Moreover, $\mu(A \cup B) = \mu(A) + \mu(B)$, provided that $A \cap B \neq \emptyset$; and $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$. The aforementioned principles determine the concept of a measure.

If we think of the integral of a positive continuous function as a measure of the area between the axis of abscissas and the graph of the function, then it is easy to understand the connection between the measure of planar sets and the integrals of real functions of one real variable.

A set $S \subset \mathbb{R}$ is said to be of “measure zero” if, for any $\varepsilon > 0$, there exists a countable family of intervals $I = \{[a_n, b_n]\}_{n \in \mathbb{N}}$ such that S is contained in the union of these intervals (i.e., I covers S), and the total length of this family of intervals is less than ε, symbolically, $\sum_n (b_n - a_n) < \varepsilon$. In other words, a set S in $\mathbb{R}$ is said to have measure zero if the sum of the lengths of the intervals enclosing the entire S can be made arbitrarily small. Obviously, any set of finitely many points is a set of measure zero; the set $\mathbb{Z}$ of all integers is an infinite set of measure zero in $\mathbb{R}$; and the set $\mathbb{Q}$ of all rational numbers is another example of an infinite set of measure zero. Moreover, any subset of a set of measure zero has measure zero. Therefore, the French mathematician Henri Lebesgue (1875–1941) has reformulated the last two theorems in the following equivalent way, using the concept of a set of measure zero:

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528 Ibid
529 Ibid.

Lebesgue's Theorem[530]*:* Assume that f is bounded on the interval $[a,b]$. Then f is Riemann integrable on $[a,b]$ if and only if the set of points of $[a,b]$ at which f is discontinuous has measure zero; symbolically: if $f:[a,b] \to \mathbb{R}$ is continuous outside a set $A \subset [a,b]$, then

$$f \; is \; Riemann \; integrable \Leftrightarrow \mu(A) = 0.$$

For instance, we can calculate $\int_0^2 x[x]\,dx$, where $[x]$ denotes the greatest integer not greater than x, as follows: Because $[x]$ is discontinuous for every integral value of x, the given function is discontinuous at $x = 1$ and $x = 2$ in $[0,2]$, and, hence, it has finitely many points of discontinuity. Then

$$\int_0^2 x[x]\,dx = lim_{x_0 \to 0}\left(\int_0^{1-x_0} x[x]\,dx + \int_{1+x_0}^{2-x_0} x[x]\,dx\right) = lim_{x_0 \to 0}\left(\int_0^{1-x_0} x \cdot 0\,dx + \int_{1+x_0}^{2-x_0} x \cdot 1\,dx\right) = lim_{x_0 \to 0}\left(\frac{x^2}{2}\right)\Big|_{1+x_0}^{2-x_0} = \frac{3}{2}.$$

The Mean Value Theorems of Integral Calculus and the Fundamental Theorem of Infinitesimal Calculus

The First Mean Value Theorem of Integral Calculus[531]*:* If f is Riemann integrable on the interval $[a,b]$, and if m and M are, respectively, the infimum and the supremum of f on $[a,b]$, then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

Proof: If f is Riemann integrable on $[a,b]$, and if $f(x) \geq 0$, then (using the previous notation), $\forall P$ of $[a,b]$,

$$U(P,f) \geq 0 \Rightarrow J \geq 0,$$

and, therefore, $\int_a^b f(x)dx \geq 0$. Similarly, if f is Riemann integrable on $[a,b]$, and if $f(x) \leq 0$, then $\int_a^b f(x)dx \leq 0$. Moreover, $\forall x \in [a,b]$,

$$\begin{aligned} & m \leq f(x) \leq M \Rightarrow [m - f(x) \leq 0 \;\&\; f(x) - M \leq 0] \\ & \Rightarrow \left[\int_a^b \big(m - f(x)\big)\,dx \leq 0 \;\&\; \int_a^b (f(x) - M)dx \leq 0\right] \\ & \Rightarrow \left[m(b-a) - \int_a^b f(x)dx \leq 0 \;\&\; \int_a^b f(x)dx - M(b-a) \leq 0\right]. \end{aligned}$$

Therefore, $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$.■

[530] Ibid.
[531] Ibid.

Remarks: (i) If k is a number between m and M, then $\int_a^b f(x)dx = k(b-a)$. (ii) If, in addition, f is continuous on $[a,b]$, then there exists a point $t \in [a,b]$ such that $\int_a^b f(x)dx = (b-a)f(t)$.

Generalized First Mean Value Theorem of Integral Calculus[532]*:* If $f(x)$ and $g(x)$ are two functions such that $f \cdot g$ and g are Riemann integrable on $[a,b]$; if $m \leq f(x) \leq M\ \forall x \in [a,b]$; and if $g(x)$ does not change sign in $[a,b]$, namely, either $g(x) \geq 0$ or $g(x) \leq 0$ whenever $x \in [a,b]$, then:

there exists a $u \in [m,M]$ such that

$$\int_a^b f(x)g(x)\,dx = u\int_a^b g(x)dx; \tag{1}$$

and, if, in addition, f is continuous over $[a,b]$, then there exists an $x_0 \in [a,b]$ such that

$$\int_a^b f(x)g(x)\,dx = f(x_0)\int_a^b g(x)dx. \tag{2}$$

Proof: Let us consider the case in which $g(x) \leq 0\ \forall x \in [a,b]$, since we can work similarly in case $g(x) \geq 0\ \forall x \in [a,b]$. The fact that $m \leq f(x) \leq M\ \forall x \in [a,b]$ implies that $Mg(x) \leq f(x)g(x) \leq mg(x) \Rightarrow$

$$M\int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)dx \leq m\int_a^b g(x)\,dx. \tag{$*$}$$

If $\int_a^b g(x)\,dx = 0$, then $(*)$ implies that $\int_a^b f(x)g(x)dx = 0$, which proves (1). If $\int_a^b g(x)\,dx < 0$, then $(*)$ implies that$m \leq \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} < M$, and, thus, setting $\frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} = u$, we obtain (1).

If $f(x)$ is continuous on $[a,b]$, then, setting $m = min(f)$ and $M = max(f)$, f takes on any given value in the interval $[min(f), max(f)]$, and, in particular, it takes on the value u, meaning that $\exists x_0 \in [a,b] | f(x_0) = u$, so that

$$\int_a^b f(x)g(x)\,dx = f(x_0)\int_a^b g(x)dx,$$

proving (2).■

Remark: In the aforementioned theorem, the number u is called the "average value" of function f over the given interval.

Corollary: If $f\colon [a,b] \to \mathbb{R}$ is integrable and $m \leq f(x) \leq M\ \forall x \in [a,b]$, then there exists a $u \in [m,M]$ such that $\int_a^b f(x)\,dx = u(b-a)$; and, if, in addition, f is continuous over $[a,b]$, then there exists an $x_0 \in [a,b]$ such that $\int_a^b f(x)\,dx = f(x_0)(b-a)$.

[532] Ibid.

Equivalently, we can say that, if $f(x)$ is continuous over $[a,b]$, then there exists at least one point $x_0 \in [a,b]$ such that $f(x_0) = \frac{1}{b-a}\int_a^b f(x)dx$, where $\frac{1}{b-a}\int_a^b f(x)dx$ is the average value of $f(x)$ over $[a,b]$.

The Mean Value Theorem for integrals means that, for every definite integral, there exists a rectangle with the same area and width, and that, if we superimpose this rectangle on the definite integral, the top of the rectangle intersects the function under consideration. This rectangle is said to be the "mean rectangle" for the corresponding definite integral. Moreover, the existence of this rectangle allows us to calculate the "average value" of the definite integral. For instance, let us draw a rectangle such that one of its sides has length h, and the other side has length b. Let us draw the y-axis through side h, and let us draw the x-axis through side b. Then the average height of the rectangle is h. The same result can be obtained using calculus: we can write the function for the height of the rectangle at any point x, namely, $f(x) = h$, and then we can apply the Mean Value Theorem in order to find the average value of $f(x)$ from 0 to b, namely, $\frac{1}{b}\int_0^b f(x)\,dx = \frac{1}{b}\int_0^b hdx = \frac{1}{b}(h \cdot b - h \cdot 0) = h$.

In general, the average value of f in $[a,b]$ corresponds to the "average height" of f across $[a,b]$, the integral $\int_a^b f(x)dx$ corresponds to the area under the curve of f from a to b, and the factor $b - a$ corresponds to the width of the given interval, so that

$$Average\ Value \equiv Average\ Height = \frac{Area}{Width} = \frac{\int_a^b f(x)dx}{b-a}.$$

Example: The average value of the function $f(x) = 8 - 2x$ over the interval $[0,4]$ is $\frac{1}{4-0}\int_0^4 (8-2x)dx = 4$. The point x_0 at which $f(x_0)$ is equal to the average value of f over $[0,4]$ can be found as follows: $8 - 2x_0 = 4 \Rightarrow x_0 = 2$.

Assume that $f: I = [a,b] \to \mathbb{R}$ is an integrable function. Then, $\forall x \in [a,b]$, the restriction of f in the interval $[a,x]$ is an integrable function, given that, as we have already shown, this is one of the properties of integrable functions (if a function f is integrable on $[a,b]$ and $[c,d] \subseteq [a,b]$, then f is integrable on $[c,d]$). Hence, we can define a function $g: I \to \mathbb{R}$ such that

$$g(x) = \int_a^x f(t)dt$$

(notice that $g(a) = 0$): this integral function g is the formal definition of the "indefinite integral" of f, and, according to the following theorem, $g(x)$ is always continuous on I.

Theorem[533]*:* If $f: I = [a,b] \to \mathbb{R}$ is Riemann integrable on I, then the function $g: I \to \mathbb{R}$ defined by $g(x) = \int_a^x f(t)dt$ is continuous on I.

Proof: Let $x, y \in I$ with $x < y$. Then
$g(y) - g(x) = \int_a^y f(t)dt - \int_a^x f(t)dt = \int_a^y f(t)dt + \int_x^a f(t)dt = \int_x^y f(t)\,dt$.

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[533] Ibid.

Therefore,

$$lim_{y\to x}|g(y)-g(x)|=0 \Rightarrow lim_{y\to x}g(y)=g(x),$$

meaning that f is continuous on I.■

Theorem[534]*:* Let $f: I=[a,b]\to\mathbb{R}$ be Riemann integrable on I, and let $g: I\to\mathbb{R}$ be a function defined by $g(x)=\int_a^x f(t)dt$. Then g is differentiable at any point x_0 at which f is continuous, and, in particular, $g'(x_0)=f(x_0)$.

Proof: We suppose that f is continuous at $x_0\in I$. Then, $\forall\varepsilon>0$, there exists a $\delta>0$ such that $|f(x_0+h)-f(x_0)|<\varepsilon$ whenever $|h|<\delta$ and $x_0+h\in I$. For any such h, given that $\frac{1}{h}\int_{x_0}^{x_0+h}1\cdot dx=1$, we obtain:

$$\left|\frac{g(x_0+h)-g(x_0)}{h}-f(x_0)\right|=\left|\frac{1}{h}\int_{x_0}^{x_0+h}f(x)dx-f(x_0)\frac{1}{h}\int_{x_0}^{x_0+h}1dt\right|=\frac{1}{|h|}\left|\int_{x_0}^{x_0+h}(f(x)-f(x_0))dx\right|\leq\frac{1}{|h|}\left|\int_{x_0}^{x_0+h}|f(x)-f(x_0)|dx\right|\leq\frac{1}{|h|}\cdot\varepsilon|h|=\varepsilon,$$

meaning that

$$g'(x_0)=lim_{h\to 0}\frac{g(x_0+h)-g(x_0)}{h}=f(x_0).■$$

Corollary: If $f: I=[a,b]\to\mathbb{R}$ is continuous on I, and if $g: I\to\mathbb{R}$ is defined by $g(x)=\int_a^x f(t)dt$, then $g'(x)$ exists, and, in particular, $g'(x)=f(x)\ \forall x\in I$.

As I have already mentioned, if there exists a differentiable function $g(x)$ such that its derivative $g'(x)=f(x)$, then the function $g(x)$ is said to be an “antiderivative,” or a “primitive,” of $f(x)$. Hence, the last theorem provides a *sufficient* condition for the existence of a primitive of a function, namely, the condition of continuity. As a conclusion, every continuous function has at least one primitive, and, in particular, the integral function $g(x)=\int_a^x f(t)dt$ is a primitive of f. The fact that the continuity of a function is not a necessary condition for the existence of a primitive can be realized by considering the following example: given the function $f: [0,1]\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}2xsin\frac{1}{x}-cos\frac{1}{x}\ ifx\neq 0\\ 0\ ifx=0\end{cases},$$

its primitive is $g: [0,1]\to\mathbb{R}$ defined by

$$g(x)=\begin{cases}x^2sin\frac{1}{x}\ ifx\neq 0\\ 0\ ifx=0\end{cases},$$

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[534] Ibid.

since $g'(x) = f(x)\ \forall x \in [0,1]$, but $f(x)$ is not continuous at 0.

The Fundamental Theorem of Infinitesimal Calculus[535]*:* Let f be a continuous function on the interval $I = [a, b]$. Then a function $g: I \to \mathbb{R}$ satisfies the relation

$$\int_a^x f(t)dt = g(x) - g(a)\ (*)$$

if and only if $g'(x) = f(x)\ \forall x \in I$.

Proof: Suppose that $(*)$ holds. Then, setting $h(x) = \int_a^x f(t)dt$, we obtain $h(x) = g(x) - g(a)\ \forall x \in I$. Then, by the corollary of the immediately preceding theorem, the fact that f is continuous implies that

$$h'(x) = f(x) = g'(x)\ \forall x \in I.$$

Conversely, if $g: I \to \mathbb{R}$ is such that $g'(x) = f(x)\ \forall x \in I$, then it holds that $g'(x) = h'(x)\ \forall x \in I$. Hence, given that two functions having equal derivatives differ from each other by a constant, there exists a number c such that

$$g(x) = h() + c.$$

Because $h(a) = 0$, it holds that $c = g(a)$, and, therefore,

$$g(x) - g(a) = h(x) = \int_a^x f(t)dt. \blacksquare$$

Corollary: If $f: I \to \mathbb{R}$ is continuous on $I = [a, b]$, and if $g'(x) = f(x)\ \forall x \in I$, then

$$\int_a^b f(x)dx = g(b) - g(a).$$

Remarks: (i) Integration and differentiation are inverse operations. (ii) The indefinite integral of a function is a function, whereas the definite integral of a function is a number.

Weierstrass's Second Mean Value Theorem (Generalized Second Mean Value Theorem of Integral Calculus)[536]*:* If $f(x)$ and $g(x)$ are two functions such that $f(x)$ is monotonic over $[a, b]$, and $g(x)$ is integrable on $[a, b]$ and does not change sign in $[a, b]$, then there exists an $x_0 \in [a, b]$ such that

$$\int_a^b f(x)g(x)dx = f(a)\int_a^{x_0} g(x)dx + f(b)\int_{x_0}^b g(x)dx.$$

Proof: Let $F: [a, b] \to \mathbb{R}$ be defined by

$$F(t) = \int_a^b f(x)g(x)dx - f(a)\int_a^t g(x)dx - f(b)\int_t^b g(x)dx.\ (*)$$

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[535] Ibid.
[536] Ibid.

Given that $g(x)$ is integrable, the functions $\int_a^t g(x)dx$ and $\int_t^b g(x)dx$ are continuous (we have already proved that, if $f: I = [a, b] \to \mathbb{R}$ is Riemann integrable on I, then the function $g: I \to \mathbb{R}$ defined by $g(x) = \int_a^x f(t)dt$ is continuous on I).

Let us consider the case in which f is increasing on $[a, b]$ and $g(x) \geq 0\ \forall x \in [a, b]$. Then

$$f(a) \leq f(x) \leq f(b)\ \forall x \in [a, b],$$

and

$$f(a)g(x) \leq f(x)g(x) \leq f(b)g(x) \Rightarrow$$
$$f(a)\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq f(b)\int_a^b g(x)dx. \qquad (**)$$

Hence, due to $(**)$, $(*)$ implies that

$F(a) = \int_a^b f(x)g(x)dx - f(b)\int_a^b g(x)dx \leq 0$, and
$F(b) = \int_a^b f(x)g(x)dx - f(a)\int_a^b g(x)dx \geq 0$.

Consequently, $F(a) \cdot F(b) \leq 0$, so that:

$F(a) = 0 \Rightarrow x_0 = a$,
$F(b) = 0 \Rightarrow x_0 = b$, and
$F(a) \cdot F(b) < 0 \Rightarrow [\exists x_0 \in (a, b) | F(x_0) = 0]$, due to Bolzano's Theorem.■

The French mathematician Pierre Ossian Bonnet (1819–1892) has formulated and proved the following version of the Mean Value Theorem of Integral Calculus, which is, in fact, a corollary of Weierstrass's version of the Mean Value Theorem of Integral Calculus:

Bonnet's Form of the Second Mean Value Theorem[537]*:* Assume that a function $g: [a, b] \to \mathbb{R}$ is integrable on $[a, b]$, and that it does not change sign in $[a, b]$. Then:

if f is decreasing and positive, then there exists an $x_0 \in [a, b]$ such that $\int_a^b f(x)g(x)dx = f(a)\int_a^{x_0} g(x)dx$;
if f is increasing and positive, then there exists an $x_0 \in [a, b]$ such that $\int_a^b f(x)g(x)dx = f(b)\int_{x_0}^b g(x)dx$.

[537] Ibid.

2.19. NUMERICAL INTEGRATION[538]

When the integrand $f(x)$ is known only at certain points (e.g., those obtained by sampling), or when a formula for the integrand is known, but it is difficult or impossible to find an antiderivative that is an elementary function, we may use numerical methods of integration, that is, approximate formulas for definite integrals. The simplest approximate formula for definite integrals is

$$\int_a^b f(x)dx \approx \frac{1}{2}(b-a)[f(a)+f(b)],$$

which is exact when $f(x)$ is linear. However, a much better approximate formula for definite integrals is

$$\int_a^b f(x)dx \approx \frac{1}{6}(b-a)\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right], \qquad (*)$$

which is known as "Simpson's Rule" (named after the eighteenth-century British mathematician Thomas Simpson, who formulated it, but, before him, Johannes Kepler had already used similar formulas, and, for this reason, "Simpson's Rule" is sometimes called "Kepler's Rule"). Formula $(*)$ derives from the observation that, if $p(x)=Ax^2+Bx+C$, then $\int_a^b p(x)\,dx=\frac{b-a}{6}\left[p(a)+4p\left(\frac{a+b}{2}\right)+p(b)\right]$, and it is used in order to approximate any integral $\int_a^b f(x)dx$, where f is an arbitrary function, and not necessarily a quadratic polynomial.

2.20. APPLICATIONS OF INTEGRATION AND BASIC PRINCIPLES OF DIFFERENTIAL EQUATIONS[539]

In this section, we shall study some of the most important applications of the definite integral.

2.20.1. The Calculation of Areas Using Integrals

Let f be a non-negative, continuous function defined on the interval $[a,b]$, and let A denote the set of the points (x,y) such that

$0 \le y \le f(x)$ and $a \le x \le b$,

that is, A is the plane region that is bounded by the straight lines $x=a$ and $x=b$, the x-axis ($y=0$), and the curve whose equation is $y=f(x)$. As we have already shown in sections 2.11 and 2.18, the area of A is

[538] Ibid.
[539] Ibid.

$$A = \int_a^b f(x)dx.$$

Moreover, it can be easily verified that, if y is a continuous function of x on the interval $[a, b]$ where $x = g(t)$ and $y = f(t)$, then the area of the plane region A that is determined by a function defined in a parametric form is

$$A = \int_a^b y dx = \int_{t_1}^{t_2} f(t) g'(t) dt,$$

provided that $g(t_1) = a$ and $g(t_2) = b$ and that g' and f are continuous on $[t_1, t_2]$.

2.20.2. The Calculation of the Area between two Arbitrary Curves

In the first case, we want to determine the area A between the equations $y = f(x)$ and $y = g(x)$ over the interval $[a, b]$ under the assumption that $f(x) \geq g(x)$, meaning that the graph of $f(x)$ is above the graph of $g(x)$. Then

$$A = \int_a^b [(upper\ function) - (lower\ function)] dx = \int_a^b [f(x) - g(x)]\, dx,$$

where $a \leq x \leq b$.

In the second case, we want to determine the area A between the equations $x = f(y)$ and $x = g(y)$ over the interval $[c, d]$ under the assumption that $f(y) \geq g(y)$, namely, $x = f(y)$ is on the right-hand side of $x = g(y)$. Then

$$A = \int_c^d [(right\ function) - (left\ function)] dy = \int_c^d [f(y) - g(y)]\, dy,$$

where $c \leq y \leq d$.

Examples:

i. *The area of a triangle:* A triangle consists of tree lines connecting the three vertices. In order to find the area bounded by these three lines, we must find the equations of these three lines and integrate their differences. For instance, in order to find the area of the triangle with vertices (0,0), (1,1), and (2,0), we notice that it consists of the following three lines: $y = 0$, $y = x$, and $y = 2 - x$, as shown in Figure 2.38.

   For the left half of the triangle (i.e., between the points $x = 0$ and $x = 1$), we need to find the area between $y = x$ and $y = 0$. For the right half of the triangle (i.e., between the points $x = 1$ and $x = 2$), we need to find the area between $y = 2 - x$ and $y = 0$. Hence, finally, we compute

   $A(triangle) = A(left\ half) + A(right\ half)$, namely:

   $\int_0^1 (x - 0)\, dx + \int_1^2 [(2 - x) - 0]\, dx = 1\ square\ unit.$

ii. *The area of a square:* If a is the length of the side of the square, then the area of the square is given by $A(square) = \int_0^a a dx = ax|_0^a = a^2$.

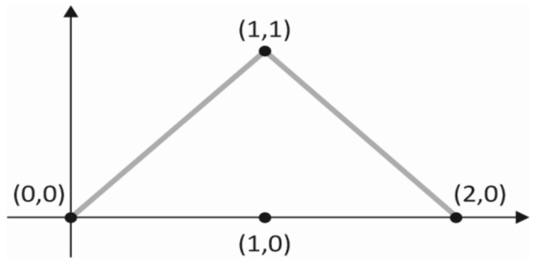

Figure 2.38. The Area of a Triangle.

iii. *The area of a circle:* The equation of a circle centered at the origin $(0,0)$ with radius r is $x^2+y^2=r^2$. Hence, $y=\sqrt{r^2-x^2}$, which is the equation of the upper half of the circle, and, therefore, we must multiply it by 2, and then integrate it to obtain

$A=2\int_{-r}^{r}\sqrt{r^2-x^2}dx.$

If we factor out an $r^2$from the square root, then we obtain

$A=2r\int_{-r}^{r}\sqrt{1-\frac{x^2}{r^2}}dx.$

Let us set $\frac{x}{r}=sinu$ (which is plausible, because, inside the circle, $x\leq r\Leftrightarrow\frac{x}{r}\leq 1$), so that

$\frac{1}{r}dx=cosudu$ and $dx=rcosudu$. Then

$A=2r\int_{x=-r}^{x=r}\left(\sqrt{1-sin^2u}\right)(rcosu)\,du=2r^2\int_{x=-r}^{x=r}\left(\sqrt{cos^2u}\right)(cosu)\,du=2r^2\int_{x=-r}^{x=r}cos^2u\,du.$

When $x=-r$, it holds that $\frac{x}{r}=-1=sinu\Rightarrow u=-\frac{\pi}{2}$.

When $x=r$, it holds that $\frac{x}{r}=1=sinu\Rightarrow u=\frac{\pi}{2}$.

Hence, $A=2r^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}cos^2udu=2r^2\left(\frac{u}{2}+\frac{sin2u}{4}\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=2r^2\left[\left(\frac{\pi}{4}+sin\frac{\pi}{4}\right)-\left(-\frac{\pi}{4}+\frac{sin3\pi}{4}\right)\right]$. Because $sin\pi=0=sin3\pi$, we obtain the area of the circle: $A=2r^2\left(\frac{\pi}{2}\right)=\pi r^2$.

2.20.3. The Calculation of the Volume of a Solid of Revolution

In order to obtain a solid of revolution, we start out with a curve $y=f(x)$on an interval $[a,b]$, as shown, for instance, in Figure 2.39, and then we rotate this curve (360º) about a given axis, so that a volume is generated, as shown, for instance, in Figure 2.40.

In order to determine the volume of a solid of revolution on the interval $[a,b]$, we work as follows: We divide the interval $[a,b]$ into n subintervals, each of which has width $\Delta x=\frac{b-a}{n}$, and then we choose a point x_k^* (where $k=1,2,\dots,n$) from each subinterval. When we want to determine the area between two curves, we approximate the area by using rectangles on each subinterval, but, understandably, when we want to compute the volume of a solid of revolution, we use disks on each subinterval to approximate the area. The area of the face of each disk is given by $A(x_k^*)$, and the volume of each disk is given by $V_k=A(x_k^*)\Delta x$. Hence, the volume of the corresponding solid of revolution on the interval $[a, b]$ can be approximated by $V \approx \sum_{k=1}^{n} A(x_k^*)\Delta x$, and, then, its exact volume is

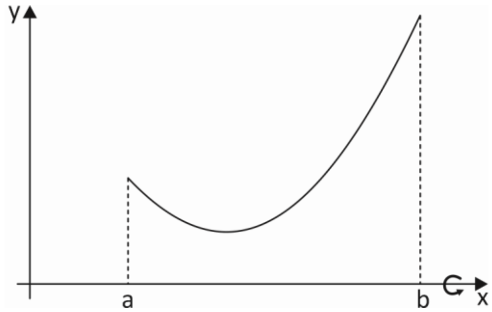

Figure 2.39. A Curve.

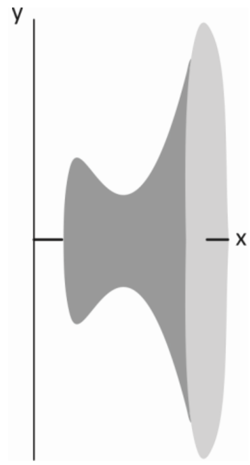

Figure 2.40. A Solid of Revolution.

$$V = lim_{n\to\infty} \sum_{k=1}^{n} A(x_k^*)\Delta x = \int_a^b A(x)dx,$$

where $a \leq x \leq b$. In other words, in this case, the volume is the integral of the cross-sectional area $A(x)$ at any x, and $x \in [a, b]$. Given that $A = \pi r^2$, $r = f(x)$, and $f(x)$ is a non-negative continuous function from $[a, b]$ to $\mathbb{R}$, the volume of the solid generated by a region under $y = f(x)$ bounded by the x-axis and the vertical lines $x = a$ and $x = b$ via revolution about the x-axis is

$$V = \pi \int_a^b [f(x)]^2 \, dx;$$

we take disks with respect to x, and $r = y = f(x)$; dx indicates that the area is rotated about the x-axis.

If we rotate a curve about the y-axis, thus obtaining a cross-sectional area that is a function of y instead of x, then the aforementioned formula becomes

$V = \int_c^d A(y)dy,$

where $c \leq y \leq d$. Given that, in this case, $A = \pi r^2$, and $r = f(y)$, the volume of the solid generated by a region under $x = f(y)$ bounded by the y-axis and the horizontal lines $y = c$ and $y = d$, via revolution about the y-axis is

$V = \pi \int_c^d [f(y)]^2 \, dy;$

we take disks with respect to y, and $r = x = f(y)$; dy indicates that the area is rotated about the y-axis.

If we have two curves y_1 and y_2 that enclose some area, and we rotate that area about the x-axis, then the volume of the solid formed is given by

$V = \pi \int_a^b [(y_2)^2 - (y_1)^2] \, dx.$

Examples:

i. *The volume of a sphere:* A sphere of radius r centered at the origin $(0,0,0)$ can be generated by revolving the upper semicircular disk enclosed between the x-axis and $x^2 + y^2 = r^2$ about the x-axis. Given that the upper half of this circle is the graph of $y = f(x) = \sqrt{r^2 - x^2}$, the volume of the corresponding sphere is $V = \pi \int_a^b [f(x)]^2 \, dx = \pi \int_{-r}^{r} (r^2 - x^2) dx = \pi \left(r^2 x - \frac{x^3}{3} \right) |_{-r}^{r} = \frac{4}{3} \pi r^3$.

ii. *The volume of a cone:* A cone with base radius r and height h can be formed by rotating a straight line through the origin $(0,0,0)$ about the x-axis. The slope of the straight line is $tan\theta = \frac{r}{h}$, so that the equation of the line is $y = \frac{r}{h} x$, and the limits of integration are $x = 0$ and $x = h$. Therefore, $V = \pi \int_0^h \left(\frac{r}{h} x \right)^2 dx = \frac{\pi r^2}{h^2} \left(\frac{x^3}{3} \right) |_0^h = \frac{1}{3} \pi r^2 h.$

iii. *The volume of a cylinder* with base radius r and height h (assuming that the plane xOy is the cylinder's base plane) is $V = \pi \int_0^h r^2 dx = \pi r^2 h$.

2.20.4. The Arc Length of a Curve[540]

Let us consider a curve γ defined by the parametric equations

$x = g(t)$ and $y = f(t)$, where $t \in [a, b]$,

as shown, for instance, in Figure 2.41, and let $P = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[a, b]$.

[540] Ibid.

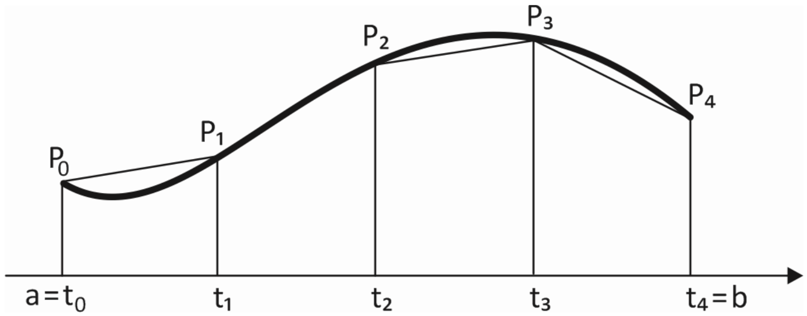

Figure 2.41. The Arc Length of a Curve.

Let $A_k = [g(t_k), f(t_k)]$, where $k = 1,2, \dots, n$, be the corresponding points of γ, as shown in Figure 2.41. Then these points define a polygonal line. The sum

$$L_P = \sum_{i=1}^{n} \sqrt{[g(t_i) - g(t_{i-1})]^2 + [f(t_i) - f(t_{i-1})]^2}$$

is the length of the polygonal line that is defined by the points A_k (corresponding to a partition P); and the finer the partition P, the more the corresponding polygonal line tends to be identified with the curve γ. Now, let us consider the set L of all numbers L_P, which correspond to all possible partitions P of $[a, b]$, symbolically, $L = \{L_P | P\ is\ a\ partition\ of\ [a, b]\}$. If this set L is bounded, then the curve is said to be "alignable," and the supremum $S = L(\gamma)$ of this set is said to be the length of the curve γ. Moreover, we write $S = L_a^b(\gamma)$ in order to denote the length of the arc of the curve that is defined on the interval $[a, b]$.

Notice that, if γ is an alignable curve on $[a, b]$, and if $a < c < b$, then

$$L_a^b(\gamma) = L_a^c(\gamma) + L_c^b(\gamma).$$

If the derivatives g' and f' are continuous on $[a, b]$, then the curve γ is alignable on $[a, b]$, and its length is given by

$$S = L(\gamma) = \int_a^b \sqrt{[g'(t)]^2 + [f'(t)]^2}\, dt,$$

where $t \in [a, b]$. If γ is defined by $y = f(x)$, where $x \in [a, b]$, and if the derivative $f'(x)$ exists and is continuous on $[a, b]$, then, setting $x = t$ and $y = f(t)$ in the aforementioned equation, we obtain the following formula:

$$S = \int_a^b \sqrt{1 + [f'(x)]^2 dx},$$

where $x \in [a, b]$. If γ is defined in polar coordinates, that is, by $r = r(\theta)$, then

$$S = \int_{\theta_1}^{\theta_2} \sqrt{r^2(\theta) + [r'(\theta)]^2}\, d\theta,$$

where $\theta \in [\theta_1, \theta_2]$.

Examples:

i. Let us calculate the length of a curve γ whose parametric equations are $x = acos\omega t$, $y = asin\omega t$, and $z = bt$, where $a > 0$, on the interval $[0,2\pi]$. Notice that this curve is known as the circular helix (i.e., one with constant radius), and it represents the orbit of a particle that performs a uniform circular motion with an angular speed ω and radius a, while the circle performs a uniform linear motion with a speed b. Then $S = L(\gamma) = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt =$ $\int_0^{2\pi} \sqrt{[a\omega(-sin\omega t)]^2 + (a\omega cos\omega t)^2 + b^2}\, dt = \int_0^{2\pi} \sqrt{a^2\omega^2 + b^2}\, dt =$ $\sqrt{a^2\omega^2 + b^2} \int_0^{2\pi} dt = 2\pi \sqrt{a^2\omega^2 + b^2}$.

ii. We can calculate the length of a circle c defined by $x^2 + y^2 = a^2$, where $a > 0$ is the radius, using the formula $L(c) = \int_0^{2\pi} \sqrt{\left(\frac{dX}{d\theta}\right)^2 + \left(\frac{dY}{d\theta}\right)^2}\, d\theta$, where $X = acos\theta$ and $Y = asin\theta$, so that $\frac{dX}{d\theta} = -asin\theta$, and $\frac{dY}{d\theta} = acos\theta$. Hence, $L(c) = \int_0^{2\pi} \sqrt{a^2cos^2\theta + a^2sin^2\theta}\, d\theta$, and, because $cos^2\theta + sin^2\theta = 1$, we obtain $L(c) = \int_0^{2\pi} ad\theta = a\theta|_0^{2\pi} = 2\pi a$.

2.20.5. Work

The work that a constant force, F, does when moving an object over a distance equal to Δx along a straight line is $W = F\Delta x$. However, the force may vary in both magnitude and direction. In case of a variable force, $F(x)$, in the direction of motion, the formula of work is $W = \int_a^b F(x)dx$ (assuming a displacement from point a to point b), and, in case of a variable force in a variable direction, the formula of work is $= \int_a^b F(x)cos\theta dx$.

2.20.6. Some Basic Applications of Integral Calculus to Economics[541]

In economics, "marginal revenue" (MR) is the additional revenue gained by producing one more unit of an economic good or a service. Therefore, it can de defined as the derivative of "total revenue" (TR) with respect to the quantity of sales (Q), symbolically, $MR = \frac{dTR}{dQ}$. Hence, given a marginal revenue function $MR(Q)$, the total revenue is given by $TR(Q) = \int MR(Q)dQ$, where integration is carried out over a certain interval of sales quantity Q. Moreover, "marginal cost" (MC) denotes the additional cost of producing one extra unit of output, and, therefore, it can be defined as the derivative of "total cost" (TC) with respect to

[541] See: Lovell, *Economics with Calculus*.

output quantity Q, symbolically, $MC = \frac{dTC}{dQ}$. Hence, given a marginal cost function $MC(Q)$, the total cost is given by $TC(Q) = \int MC(Q)dQ$, where integration is carried out over a certain interval of output quantity Q.

Given that "total profit" (TP) is defined as $TP = TR - TC$, the equation of "marginal profit" (MP) is $MP = MR - MC \Rightarrow \frac{dTP}{dQ} = \frac{dTR}{dQ} - \frac{dTC}{dQ}$.

Let $I(t)$ denote the rate of investment flow (as a function of time, t). If $K(t)$ denotes the "capital stock" at time t, then

$$\frac{dK(t)}{dt} = I(t).$$

Moreover, notice that an increase in $I(t)$ raises the rate of income flow $Y(t)$ as follows:

$$\frac{dY(t)}{dt} = \frac{1}{s} \cdot \frac{dI(t)}{dt},$$

where s is a constant standing for the marginal propensity to save.

The "total capital stock" $K(t)$ during the time interval $[a, b]$ is given by

$$K(t) = \int_a^b I(t)dt.$$

2.20.7. A Social Utility Model and Optimal Control

The "social utility" to nation X at any point in time $t|(U_{Xt})$ is assumed to depend on both consumption C_X and on infrastructure spending I_X, so that

$$U_{Xt} = u(C_X, I_X).$$

The objective for nation X is to maximize a welfare function (integral) that gives the discounted present value of all future utility levels:

$$U_X^* = \int_0^\infty e^{-rt} u(C_X, I_X)dt.$$

Hence, in this case, the welfare of nation X, namely, U_X^*, is obtained by adding (integrating) the contributions to welfare at each instant of time over all the time periods from the present, $t = 0$.[542]

The welfare integral is maximized by the choice of an additional investment in infrastructure, Z_X, and the level of consumption, C_X, under the constraints

$$\begin{Bmatrix} M_X' = Z_X - \lambda_X M_X \\ P_X = C_X + Z_X \end{Bmatrix},$$

[542] See: Athans and Falb, *Optimal Control*.

where: M_X' represents the rate of change in the infrastructure of nation X (i.e., we take the first derivative); the first constraint says that the increase in X's infrastructure is equal to its gross investment in new infrastructure, denoted by Z_X, minus the cost of operating its old infrastructure, denoted by $\lambda_X M_X$; and the second constraint says that the gross national product, denoted by P_X, is divided between consumption, C_X, and the additional investment in infrastructure, Z_X.

2.20.8. Integration and Ordinary Differential Equations[543]

The Fundamental Theorem of Infinitesimal Calculus is a rigorous explanation of the dialectical relationship between integration and differentiation, and, thus, it is a major underpinning of the theory of differential equations.

By the term "ordinary differential equation," we refer to any equation that contains an unknown function, some of its derivatives, and an independent variable. The "order" of a differential equation is the order of the highest ordered derivative occurring in the given differential equation. The fundamental problem of the theory of differential equations is to find all of the functions $y = f(x)$ that satisfy some differential equation. Every function $y = f(x)$ that satisfies some differential equation is said to be a "solution" of the given differential equation.

A family of functions

$$y = f(x, c) \qquad (*)$$

where c is a constant belonging to $A \subseteq \mathbb{R}$, is said to be a "general solution" of a differential equation

$$y' = F(x, y) \qquad (**)$$

if, for every $c \in A$, $(*)$ is a solution of $(**)$. The solution that we obtain for each particular value of c is said to be a "partial solution" of the differential equation $(**)$.

The theory of differential equations is a branch of mathematics in which the study of theoretical problems can hardly be distinguished from the study of practical problems, and dynamicity, which is a major characteristic of modern mathematics, is clearly manifested. Moreover, the theory of differential equations has played an important role in the transition from the eighteenth-century infinitesimal calculus to advanced mathematical analysis and modern geometry. One of the major advantages of differential equations is that they constitute one of the major underpinnings and instruments of the "mathematization" (i.e., of the "mathematical modelling") of many problems both in the context of the natural sciences and in the context of the social sciences.

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[543] See: Ayres, *Theory and Problems of Differential Equations*; Borzì, *Modelling with Ordinary Differential Equations*; Derrick and Grossman, *Introduction to Differential Equations with Boundary Value Problems*; Hochstadt, *Differential Equations*; Imhoff, *Differential Equations in 24 Hours*; Simmons, *Differential Equations with Applications and Historical Notes*.

Example: If $s(t)$ gives the position of a moving particle as a function of time, then velocity, $v(t)$, is given by the formula

$$s'(t) = \frac{ds(t)}{dt} = v(t),$$

and acceleration, $a(t)$, is given by the formula

$$v'(t) = \frac{dv(t)}{dt} = a(t).$$

Suppose that we wish to study the motion of an object of mass m that is in free fall in vacuum (this hypothesis is a simplification of physical reality in order to facilitate the mathematization of this problem, which can easily lead to generalizations that, under certain conditions, provide satisfactory approximations of the actual state of affairs). Moreover, for reasons of simplicity, we shall assume that the orbit of the object's fall is so small in comparison with the radius of the Earth that (without a significant error in the conclusion) we can suppose that the object's weight (i.e., the force acting on it due to gravity) is constant. Then the object's acceleration is constant, too, and it is denoted by g (g is said to be the "gravitational acceleration," namely, the free fall acceleration of an object in vacuum; $g \approx 9.80\, m/sec^2$). In view of the foregoing, we have:

$v'(t) = -g$ (the negative sign indicates that the object's motion is accelerating downward), and
$\int v'(t)dt = \int(-g)\, dt \Rightarrow v(t) = -gt + c_1$, where c_1 is a constant.

The last equation gives the value of velocity if we know the constant c_1. Furthermore, we obtain:

$s'(t) = v(t) \Rightarrow \int s'(t)dt = \int(-gt + c_1)\, dt \Rightarrow s(t) = -\frac{1}{2}gt^2 + c_1 t + c_2$, where c_2 is a constant.

Hence, we can determine displacement, too, provided that we know c_2. In general, constants are determinable quantities.

In physics, constants are functions of the initial conditions of the phenomenon under investigation. For instance, in the aforementioned phenomenon of free fall in vacuum, we must take into consideration whether the object was left to fall, in which case its initial velocity is $v_0 = 0$, or whether it was given a non-zero initial velocity $v_0 = v(t_0)$. In any case, applying the formula of velocity for $t_0 = 0$, we obtain $v(t_0) = -g0 + c_1 \Rightarrow c_1 = v_0$, and, therefore, $v(t) = -gt + v_0$. By analogy, regarding displacement, we have: $s(t_0) = -\frac{1}{2}g0^2 + c_1 0 + c_2$, and, therefore, setting $s(t_0) = s_0$, we obtain $s(t) = -\frac{1}{2}gt^2 + v_0 t + s_0$, which is the formula of "uniformly accelerated motion."

The aforementioned results are based on the hypothesis that we study motion in vacuum. If, however, we decide to take account of the resistance of the Earth's atmosphere during the object's fall, then we must modify the aforementioned model as follows: we assume that a force due to the resistance of the Earth's atmosphere is applied on the moving object in the direction opposite to the object's motion (for which reason this force has a negative sign), and that this force is proportional to the moving object's speed. In other words, for a suitable $k > 0$, this force is equal to $-kv(t)$. Then we assume that the total force that is applied on the moving body is $-mg - kv(t)$, that is, weight and air resistance. Consequently, according to Newton's Second Law of Motion (i.e., $Force = Mass \times Acceleration$), we obtain the linear differential equation

$$-mg - kv(t) = mv'(t) \Rightarrow v'(t) + \frac{k}{m}v(t) + g = 0,$$

so that, in this case, we must solve the given linear differential equation in order to find v.

Separation of Variables[544]*:* If a differential equation may be written in the form

$$\frac{dy}{dx} = f(x)g(y),$$

then it is said to be solvable by "separation of variables" as follows:

$$\int \frac{dy}{g(y)} = \int f(x)\,dx.$$

Remark: In case we have a differential equation of the form

$$y^{(n)} = f(x) \Leftrightarrow \frac{d^n y}{dx^n} = f(x), \tag{1}$$

then, by integrating (1), we obtain

$$\frac{d^{n-1}y}{dx^{n-1}} = \int f(x)dx + c_1. \tag{2}$$

By setting $\int f(x)dx = f_1\,(x)$ and then integrating (2), we obtain

$$\frac{d^{n-2}y}{dx^{n-2}} = \int f_1(x)dx + c_1\,x + c_2. \tag{3}$$

Repeating the same process, we obtain the general solution of (1), which is of the form

$$y = w(x) + \frac{c_1}{(n-1)!}x^{n-1} + \frac{c_2}{(n-2)!}x^{n-2} + \cdots + c_n,$$

meaning that the general solution of $y^{(n)} = f(x)$ can be obtained through n successive integrations.

For instance, let us find the general solution of the differential equation $x^2dy - ydx = 0$ as well as its partial solution that satisfies the condition $y(2) = 4$ (i.e., the integral curve that passes through the point $P(2,4)$). We shall apply the method of separation of variables:

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[544] Ibid.

$x^2dy - ydx = 0 \Rightarrow \frac{dy}{y} = \frac{dx}{x^2} \Rightarrow \frac{dy}{y} = x^{-2}dx \Rightarrow \int \frac{dy}{y} = \int x^{-2}dx \Rightarrow lny = \frac{x^{-1}}{-1} + c \Rightarrow$ $lny = -\frac{1}{x} + c \Rightarrow y = e^{-\frac{1}{x}+c} \Rightarrow y = e^c e^{-\frac{1}{x}} \Rightarrow y = ke^{-\frac{1}{x}}$, which is the general solution of the given differential equation. In order to find the partial solution for which $x = 2 \Rightarrow y = 4$ (i.e., the integral curve that passes through the point $P(2,4)$), we must determine the constant k. If we substitute $x = 2$ and $y = 4$ into the general solution, then we obtain $4 = ke^{-\frac{1}{2}} \Rightarrow k = 4e^{\frac{1}{2}} = 4\sqrt{e}$. Hence, if we substitute this value of k into the general solution, then we shall obtain the required partial solution, namely, $y = 4\sqrt{e}e^{-\frac{1}{x}}$.

Homogeneous Differential Equations[545]*:* A differential equation is said to be “homogeneous” if it may be written in the form

$$f(x,y)dx + g(x,y)dy = 0, \tag{1}$$

where the functions $f(x,y)$ and $g(x,y)$ are homogeneous with respect to x and y of the same degree of homogeneity, meaning that

$f(x,y)$ may be written in the form $x^m A\left(\frac{y}{x}\right)$ and (2)

$g(x,y)$ may be written in the form $x^m B\left(\frac{y}{x}\right)$. (3)

Thus, due to (2) and (3), (1) becomes (for $x^m \neq 0$):

$$A\left(\frac{y}{x}\right)dx + B\left(\frac{y}{x}\right)dy = 0 \Rightarrow \frac{dy}{dx} = -\frac{A\left(\frac{y}{x}\right)}{B\left(\frac{y}{x}\right)},$$

which ultimately reduces to the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \Leftrightarrow y' = f\left(\frac{y}{x}\right), \tag{4}$$

where $f\left(\frac{y}{x}\right)$ is a homogeneous function whose degree of homogeneity is equal to zero. In order to find the general solution of (4), we set

$$\frac{y}{x} = w \Leftrightarrow y = xw \tag{5}$$

where w is a function of the independent variable x, that is, $w = w(x)$.

By differentiating (5), we obtain

$$dy = wdx + xdw,$$

and, after dividing by dx, we obtain

[545] Ibid.

$$\frac{dy}{dx} = w + x\frac{dw}{dx}. \tag{6}$$

Therefore, due to (5) and (6), the original differential equation becomes

$$w + x\frac{dw}{dx} = f(w) \Rightarrow x\frac{dw}{dx} = f(w) - w \Rightarrow \frac{dw}{f(w)-w} = \frac{dx}{x}. \tag{7}$$

The differential equation (7), which is equivalent to (1), can be solved by the method of separation of variables. In particular, (7) gives

$$\int \frac{dw}{f(w)-w} = lnx + lnc, \text{ or } \int \frac{dw}{f(w)-w} = lncx, \text{ or } cx = e^{\int \frac{dw}{f(w)-w}}. \tag{8}$$

In (8), we have to compute the integral $\int \frac{dw}{f(w)-w}$ and then to make the substitution $w = \frac{y}{x}$ in order to ultimately find the general solution of (1).

For instance, let us solve the differential equation $(x^2 - y^2)dx + 2xydy = 0$. This differential equation is homogeneous, because the expressions $f(x, y) = x^2 - y^2$ and $g(x, y) = 2xy$ are homogeneous with respect to x and y, and their degree of homogeneity is 2. We set

$$\frac{y}{x} = w \Leftrightarrow y = xw \tag{$*$}$$

where $w = w(x)$. By differentiating ($*$) with respect to x, we obtain

$$y' = w + xw'. \tag{$**$}$$

Due to ($*$) and ($**$), the given differential equation becomes

$(x^2 - x^2w^2) + 2x^2w(w + xw') = 0 \Rightarrow x^2(1 - w^2) + 2x^2w(w + xw') = 0,$ and, because, by ($*$), $x \neq 0$, we divide the last expression by x^2 to obtain

$$(1 - w^2) + 2w(w + xw') = 0 \Rightarrow 1 - w^2 + 2w^2 + 2xw\frac{dw}{dx} = 0 \Rightarrow 1 + w^2 + 2xw\frac{dw}{dx} = 0 \Rightarrow \frac{2wd}{w^2+1} = -\frac{dx}{x} \Rightarrow \int \frac{2wdw}{w^2+1} = -\int \frac{dx}{x} \Rightarrow ln(w^2 + 1) = -lnx + lnc \Rightarrow ln(w^2 + 1) = ln\left(\frac{c}{x}\right) \Rightarrow w^2 + 1 = \frac{c}{x}.$$

By the substitution $w = \frac{y}{x}$, we realize that the general solution of the given differential equation is $y^2 + x^2 = cx$.

Differential Equations Reducible to Homogeneous Differential Equations[546]*:* The differential equations of the form

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[546] Ibid.

$$\frac{dy}{dx} = f\left(\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}\right), \tag{$*$}$$

where $a_1, b_1, c_1, a_2, b_2, c_2$ are real constants, are reducible to homogeneous differential equations. In order to solve $(*)$ by reducing it to a homogeneous differential equation, we distinguish the following two cases:

Case I: If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \Leftrightarrow a_1b_2 - a_2b_1 \neq 0$, then we can find the general solution of $(*)$ as follows: We solve the system of equations

$$\begin{Bmatrix} a_1x + b_1y + c_1 = 0 \\ a_2x + b_2y + c_2 = 0 \end{Bmatrix}. \tag{1}$$

Let $(x, y) = (x_0, y_0)$ be the solution of (1). Then we set

$$\begin{Bmatrix} x = x_0 + w \\ y = y_0 + v \end{Bmatrix}, \tag{2}$$

and, by differentiating (2), we obtain

$$\begin{Bmatrix} dx = dw \\ dy = dv \end{Bmatrix}, \tag{3}$$

so that, by (2) and (3), the differential equation $(*)$ becomes

$$\frac{dv}{dw} = f\left(\frac{a_1(x_0+w)+b_1(y_0+v)+c_1}{a_2(x_0+w)+b_2(y_0+v)+c_2}\right) \Rightarrow \frac{dv}{dw} = f\left(\frac{a_1x_0+b_1y_0+c_1+a_1w+b_1v}{a_2x_0+b_2y_0+c_2+a_2w+b_2v}\right).$$

But $a_1x_0 + b_1y_0 + c_1 = 0$ and $a_2x_0 + b_2y_0 + c_2 = 0$, because (x_0, y_0) is the solution of (1), and, therefore,

$$\frac{dv}{dw} = f\left(\frac{a_1w+b_1v}{a_2w+b_2v}\right). \tag{4}$$

The differential equation (4) is homogeneous with respect to v and w, and, in order to find its general solution, we set $\frac{v}{w} = z \Leftrightarrow v = wz$, where $z = z(w)$, and we work according to the method of solving homogeneous differential equations, which I have already explained. When we find the general solution of (4), we set $z = \frac{v}{w}$, and then, by (2), $w = x - x_0$ and $v = y - y_0$ in order to ultimately find the general solution of $(*)$.

Case II: If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \lambda \Rightarrow a_1b_2 - a_2b_1 = 0$, then we can find the general solution of $(*)$ as follows: Because $a_1 = \lambda a_2$ and $b_1 = \lambda b_2$, $(*)$ becomes

$$\frac{dy}{dx} = f\left(\frac{\lambda(a_2x+b_2y)+c_1}{a_2x+b_2y+c_2}\right). \tag{5}$$

We set $a_2x + b_2y = w$, where $w = w(x)$, and, by differentiating (5) with respect to x, we obtain $a_2 + b_2y' = w' \Leftrightarrow y' = \frac{1}{b_2}(w' - a_2)$, so that (5) becomes

$$\frac{1}{b_2}\left(\frac{dw}{dx} - a_2\right) = f\left(\frac{\lambda w + c_1}{w + c_1}\right). \tag{6}$$

The differential equation (6) can be solved by the method of separation of variables. When we find the general solution of (6), we set $w = a_2x + b_2y$.

First-Order Linear Differential Equations[547]*:* The general form of these equations is

$$\frac{dy}{dx} + Ay = B, \tag{$*$}$$

where A and B are functions of x, that is, $A = A(x)$ and $B = B(x)$. The general solution of $(*)$ is

$$y = e^{-\int Adx}\left(c + \int Be^{\int Adx}\, dx\right),$$

where c is an arbitrary constant.

Proof: If $B = 0$, then $(*)$ becomes $\frac{dy}{dx} + Ay = 0$, and it is said to be a homogeneous linear differential equation, which can be solved by separation of variables: $\frac{dy}{y} = -Adx \Rightarrow \int \frac{dy}{y} = -\int Adx \Rightarrow lny = -\int Adx + c \Rightarrow y = e^{-\int Adx + c} = e^c e^{-\int Adx} = ce^{-\int Adx}$, which is the general solution of the aforementioned homogeneous linear differential equation; and, if $c = 1$, then we obtain its partial solution $y_1 = e^{-\int Adx}$.

In order to find the general solution of $(*)$, we consider a new unknown function z of x such that

$$y = y_1 z. \tag{1}$$

By differentiating (1) with respect to x, we obtain

$$y' = y_1' z + y_1 z'. \tag{2}$$

Hence, by (1) and (2), the differential equation $(*)$ becomes

$$y_1' z + y_1 z' + Ay_1' z = B \Leftrightarrow (y_1' + Ay_1)z + y_1 z' = B.$$

But $y_1' + Ay_1 = 0$, since y_1 is a partial solution of $\frac{dy}{dx} + Ay = 0$, and, therefore,

$$y_1 z' = B \Rightarrow e^{-\int Adx} z' = B \Rightarrow z' = Be^{\int Adx} \Rightarrow z = \int Be^{\int Adx} dx + c.$$

[547] Ibid.

Because $y_1 = e^{-\int A dx}$ and $z = \int B e^{\int A dx} dx + c$, equation (1) gives the general solution of $(*)$, which is $y = e^{-\int A dx}\left(c + \int B e^{\int A dx}\, dx\right)$.■

Linearization of Nonlinear Differential Equations[548]*:* Problems of nonlinear analysis started to exist ever since the creation of the universe. Some of them were solved by ancient Greek mathematicians, but many new nonlinear problems were created, both in pure mathematics and in other sciences, such as biology, physics, astronomy, economics, etc. The distinction between linear and nonlinear analysis is not quite clear, because a considerable part of information about a nonlinear system can be extracted from a linear approximation of the corresponding nonlinear problem. Moreover, it is often possible to extract information about the solution of a linear system from a relevant nonlinear one.[549] The term "linearization" of a nonlinear differential equation refers to the reduction of a nonlinear differential equation to a linear differential equation that is either equivalent or almost equivalent to the given nonlinear differential equation, that is, the solution of the linear differential equation may give the solution of the nonlinear differential equation either exactly or approximately within an acceptable error. Two well-known examples of linearization of nonlinear differential equations are the following:

i. The Bernoulli equation:
$$\frac{dy}{dx} + Ay = By^n, \qquad (*)$$
where A and B are functions of x, and $n \in \mathbb{R} - \{0,1\}$ (if $n = 0$, then the equation is linear; if $n = 1$, then the equation can be solved by separation of variables). Multiplying both sides of $(*)$ by y^{-n}, we obtain
$$y^{-n}\frac{dy}{dx} + Ay^{1-n} = B. \qquad (1)$$
Let $y^{1-n} = w$, (2)
where $w = w(x)$. By differentiating (2) with respect to x, we obtain
$$(1-n)y^{-n}\frac{dy}{dx} = \frac{dw}{dx} \Leftrightarrow y^{-n}y' = \frac{w'}{1-n}. \ (3)$$
Hence, (1), due to (2) and (3), yields
$$\frac{w'}{1-n} + Aw = B \Rightarrow \frac{dw}{dx} + (1-n)Aw = (1-n)B, \ (4)$$
which is a linear differential equation (whose dependent variable is w), and it can be solved according to the aforementioned method of solving linear differential equations. When we find the general solution of (4), we set $w = y^{1-n}$, according to (2), and, thus, we obtain the general solution of $(*)$.

ii. The Riccati equation:
$$\frac{dy}{dx} = A + By + Cy^2, \qquad (*)$$
where A, B, and C are functions of x. We can find the general solution of the Riccati equation only if we know one of its partial solutions. Suppose that $y = y_1$ is a partial solution of $(*)$, so that
$$\frac{dy_1}{dx} + A + By_1 + Cy_1^2 = 0. \qquad (1)$$

[548] Ibid.
[549] Lomonosov, "Invariant Subspaces for the Family of Operators which Commute with a Completely Continuous Operator."

Then, using the transformation

$$y = y_1 + w, \tag{2}$$

where $w = w(x)$, and differentiating (2) with respect to x, we obtain

$$\frac{dy}{dx} = \frac{dy_1}{dx_1} + \frac{dw}{dx}. \tag{3}$$

Hence, $(*)$, due to (2) and (3), yields

$$\frac{dy_1}{dx} + A + By_1 + Cy_1^2 + \frac{dw}{dx} + (B + 2Cy_1)w + Cw^2 = 0.$$

But, due to (1), $\frac{dy_1}{dx} + A + By_1 + Cy_1^2 = 0$, so that we obtain

$$\frac{dw}{dx} + (B + 2Cy_1)w = -Cw^2, \tag{4}$$

which is a Bernoulli equation (where w is the dependent variable), and it can be solved according to the aforementioned method of solving the Bernoulli equation: multiplying both sides of (4) by w^{-2} we obtain

$$w^{-2}\frac{dw}{dx} + (B + 2Cy_1)w^{-1} = -C. \tag{5}$$

If we set

$$w^{-1} = z, \tag{6}$$

where $z = z(x)$, and we differentiate (5) with respect to x, then we obtain

$$-w^{-2}w' = z'. \tag{7}$$

Therefore, (5), due to (6) and (7), yields

$$-z' + (B + 2Cy_1)z = -C \Leftrightarrow z' - (B + 2Cy_1)z = -C, \tag{8}$$

which is a linear differential equation (where z is the dependent variable), and its general solution is

$$z = e^{-\int -(B+2Cy_1)dx}\left(c + \int Ce^{\int -(B+2Cy_1)dx}\, dx\right).$$

By substituting this value of z into (6), we find w, and, by substituting the so found value of w into (2), we find the general solution of the Riccati equation.

2.21. INTEGRATION OF MULTIVARIABLE FUNCTIONS

In this section, we shall apply the concept of integration to functions of several variables $x, y, \ldots$ As we have already realized, integration can be used in order to find the area under a curve given by some function of one variable. By adding a dimension, we obtain a double integral, by which we can find the volume under a surface given by some function of two variables.

The concept of a Riemann integral can be extended to the computation of volumes under the graph of bivariate functions.[550] Assume that the domain of a bivariate function is the Cartesian product of two closed intervals, that is, a rectangle, say

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 | a \leq x \leq b, c \leq x \leq d\}.$$

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[550] See: Apostol, *Calculus*; Courant and John, *Introduction to Calculus and Analysis*; Edwards, *A Treatise on the Integral Calculus*; Fraleigh, *Calculus with Analytic Geometry*; Haaser and Sullivan, *Real Analysis*; McLeod, *The Generalized Riemann Integral*; Nikolski, *A Course of Mathematical Analysis*; Piskunov, *Differential and Integral Calculus*; Rudin, *Principles of Mathematical Analysis*; Spivak, *Calculus*; Taylor, *General Theory of Functions and Integration*.

Analogously to section 2.18, we partition R into rectangular subregions

$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, where $1 \leq i \leq n$ and $1 \leq j \leq m$,

so that we obtain the partition P defined by

$a = x_0 < x_1 < \cdots < x_n = b$, and $c = y_0 < y_1 < \cdots < y_m = d$.

Such a partition can be constructed by drawing straight lines $x = x_i$ and $y = y_j$ ($i = 1,2, \ldots, n$ and $j = 1,2, \ldots, m$) that are parallel to the y-axis and the x-axis, respectively. The area of each of the aforementioned subregions R_{ij} is

$A_{ij} = (x_i - x_{i-1})(y_j - y_{j-1}) = \Delta x_i \Delta y_j$,

so that the area of the rectangle R is

$A = (b - a)(d - c) = \sum_{i=1}^{n} \sum_{j=1}^{m} A_{ij}$.

Assume that f is a bounded function defined on the aforementioned rectangle R. Let m_{ij} and M_{ij} be the infimum (greatest lower bound) and the supremum (least upper bound) of f in R_{ij}, respectively:

$m_{ij} = inf_{(x,y) \in R_{ij}}\big(f(x,y)\big)$, and
$M_{ij} = sup_{(x,y) \in R_{ij}}\big(f(x,y)\big)$.

Then the real number $U(P,f) = \sum_{i=1}^{n} \sum_{j=1}^{m} M_{ij} A_{ij}$ is the "upper sum" of f corresponding to the partition P of R, and the real number $L(P,f) = \sum_{i=1}^{n} \sum_{j=1}^{m} m_{ij} A_{ij}$ is the "lower sum" of f corresponding to the partition P of R. Analogously to section 2.18, a pair of lower and upper sums corresponds to each partition of R, and, for every partition P of R, it holds that $mA \leq L(P,f) \leq U(P,f) \leq MA$, where m and M are, respectively, the infimum and the supremum of f in R, and A is the area of R. If X is the set of all possible upper sums, if Y is the set of all possible lower sums, if $J = inf(X) = inf(\{U(P,f)\ \forall P\ of\ R\})$, and if $I = sup(Y) = sup(\{L(P,f)\ \forall P\ of\ R\})$, then J and I are called, respectively, the "upper double integral" and the "lower double integral" of f on R, and they are denoted as follows:

$$J = \overline{\iint_R} f(x,y)dxdy$$

and

$$I = \iint_{\underline{R}} f(x,y)dxdy$$

(given the aforementioned notation). If $I = J$, that is, if $\iint_{R}^{-} f(x,y)dxdy = \iint_{\bar{R}} f(x,y)dxdy$, then f is said to be "Riemann integrable," or simply "integrable," on the rectangle R, and the common value of its upper and lower integrals is denoted by

$$\iint_R f(x,y)dxdy$$

and is called the "double integral" of f on the region R.

In other words, for a bivariate function $f: R \subset \mathbb{R}^2 \to \mathbb{R}$, we compute $f(p_i, q_j)$ for evaluation points (p_i, q_j) arbitrarily chosen in R_{ij}, and, thus, we approximate the volume V under the graph of f by the following Riemann sum:

$$V \approx \sum_{i=1}^{n} \sum_{j=1}^{m} f(p_i, q_j)(x_i - x_{i-1})(y_j - y_{j-1}),$$

where $(x_i - x_{i-1})(y_j - y_{j-1})$ is the area of rectangle R_{ij}, and each term $f(p_i, q_j)(x_i - x_{i-1})(y_j - y_{j-1})$ in the aforementioned sum is the volume of the bar $[x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [0, f(p_i, q_j)]$. Now, let us refine the aforementioned partition by assuming that the diameter of the largest rectangle (i.e., the "norm" $\|R_{ij}\| = max(R_{ij})$) tends to 0, symbolically, $\|R_{ij}\| \to 0$. If the Riemann sum converges for every such sequence of partitions for arbitrarily chosen points (p_i, q_j), then this limit is said to be the double integral of f over R, symbolically,

$$\iint_R f(x,y)\, dxdy = lim_{\|R_{ij}\| \to 0} \sum_{i=1}^{n} \sum_{j=1}^{m} f(p_i, q_j)(x_i - x_{i-1})(y_j - y_{j-1}). \qquad (*)$$

In short, if $f(x,y)$ is continuous on a region R in the plane $z = 0$, then:

i. the area of the region R is
$A = \iint_R dA \equiv \iint_R dxdy$;
and
ii. the volume of the solid that lies below the surface $z = f(x,y)$ and above the given region (assuming that this integral exists) is
$V = \iint_R f(x,y)\, dA \equiv \iint_R f(x,y)\, dxdy$.

In view of the foregoing, if $R = [a,b] \times [c,d]$, whenever the integrand is $f(x,y)$, we have to integrate over two variables, x and y, so that, for each variable, we have an integration sign, and, in order to indicate the variables involved, we have dx and dy, symbolically,

$$\iint_R f(x,y)\, dxdy \equiv \int_c^d \int_a^b f(x,y)dxdy,$$

where $f(x,y)$ is an integrable function of two real variables.[551] In this case, we compute the innermost integral first, and then we work our way outward. In particular, we compute the dx integral inside first, while treating y as a constant, and then we integrate the result over y as we would do with any variable. However, the order in which we do the integrations does not matter, provided that we keep track of the limits of integration of each variable. For instance, in the double integral $\int_c^d \int_a^b f(x,y)dxdy$, dx is associated with the x integrand, which runs from a to b, while dy is associated with the y integrand, which runs from c to d, and, therefore,

$$\int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx,$$

so that the limits of integration of each integrand remain the same (this result is known as Fubini's Theorem).[552]

Let X be a set in $\mathbb{R}^2$ such that, $\forall \varepsilon > 0$, there is a finite number of rectangles R_i, where $i = 1, \dots, n$, such that X is contained in the union of these rectangles, and, if A_i is the area of rectangle R_i, then $\sum_{i=1}^n A_i < \varepsilon$. If this is the case (namely, if X can be covered by finitely many rectangles of arbitrarily small area), then X is called a set with "Jordan content zero" (named after the French mathematician Camille Jordan). Obviously, singletons have content zero, and any finite set of points has content zero (e.g., given a set of n points, we can surround each of them with an interval of $\frac{\varepsilon}{n}$). If a set contains some rectangle of positive area, then it is said to be a set with "positive (Jordan) content." Moreover, a region U in $\mathbb{R}^2$ is a set with area if the boundary of U has content zero (and then U is said to be a "domain"); and, as I mentioned earlier, if $R = \{(x,y) \in \mathbb{R}^2 | a \leq x \leq b, c \leq x \leq d\}$, then the area of R is $(b-a)(d-c)$.

The concept of the measure of a set is more general than the concept of the (Jordan) content of a set. In $\mathbb{R}^2$, a set of (Lebesgue) "measure zero" is a set that can be covered by a countable collection of rectangles of arbitrarily small total area. Although there are sets of measure zero whose algebraic sum is non-measurable, and, hence, their content is undefined, compact sets have measure zero if and only if they have content zero. In the theory of Riemann integration, we deal with compact sets, and, therefore, the notion of content is sufficient.

For the definition of the Riemann integral of a bivariate function, the partition P need not consist of rectangles. In other words, the same concept of a Riemann integral also applies in case of non-rectangular regions R. The partition has to consist of subregions R_i of R whose areas can be determined, so that the rationale of $(*)$ is satisfied.

Analogously to section 2.18, if a bounded function $f(x,y)$ is defined on a region $R \subset \mathbb{R}^2$, and if the boundary of R and the set of discontinuities of f each have content zero in $\mathbb{R}^2$, then f is integrable on R.[553] This statement is an important criterion for the existence of a double integral. In view of the foregoing, we realize that the properties of double integrals are similar to those of definite single integrals [554]: If U is a set with area in $\mathbb{R}^2$ (i.e., the boundary

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[551] Ibid.
[552] Ibid.
[553] Ibid.
[554] Ibid.

of U has content zero); if both X and Y have area and $U = X \cup Y$ (i.e., every point of U is a point of X or Y or their boundary points, and X and Y have at most some of their boundary points in common); and if f and g are functions defined and continuous on U, then the double integrals of f and g over U exist and have the following properties[555]:

i. $\iint_U dxdy = theareaofU$.
ii. $\iint_U [af(x,y) + bg(x,y)]\, dxdy = a \iint_U f(x,y)\, dxdy + b \iint_U g(x,y)\, dxdy$ for any constants a and b.
iii. $\iint_U f(x,y)\, dxdy = \iint_X f(x,y)\, dxdy + \iint_Y f(x,y)\, dxdy$.
iv. If $f(x,y) \leq g(x,y)\ \forall (x,y) \in U$, then
v. $\iint_U f(x,y)\, dxdy \leq \iint_U g(x,y)\, dxdy$.
vi. $\left|\iint_U f(x,y)\, dxdy\right| \leq \iint_U |f(x,y)| dxdy$.
vii. $\left|\iint_U f(x,y)\, dxdy\right| \leq max_U(|f|)A(U)$, where $A(U)$ is the area of U.
viii. If m and M are, respectively, the infimum and the supremum of f on U, then there exists a value w with $m \leq w \leq M$ such that $mA(U) \leq \iint_U f(x,y)\, dxdy = w \leq MA(U)$, where $A(U)$ is the area of U (this is a version of the Mean Value Theorem for double integrals). Notice that, in view of the aforementioned analysis of the Riemann integral of a bivariate function, if $f(x,y)$ is an integrable function over a region R with area A, then the "average value" of f over R is equal to $\frac{1}{A}\iint_R f(x,y)\, dxdy$.

In general, increasing the number of integrals in the context of multiple integration is the same as increasing the number of dimensions: a single integral gives a two-dimensional area, a double integral gives a three-dimensional volume, a triple-integral gives a four-dimensional hypervolume, etc.[556] Those sets in $\mathbb{R}^2$ which are suitable domains of integration are said to be sets with area. Those sets in $\mathbb{R}^3$ which are suitable domains of integration are said to be sets with volume; and we define the volume V of a rectangular parallelepiped $R = \{(x,y,z) \in \mathbb{R}^3 | a_1 \leq x \leq b_1, a_2 \leq y \leq b_2, a_3 \leq z \leq b_3\}$, where $a_k < b_k$ for $k = 1,2,3$, as follows: $V = (b_1 - a_1)(b_2 - a_2)(b_3 - a_3)$. Using the notion of the volume of a rectangular parallelepiped, we can define a set U in $\mathbb{R}^3$ to have content zero if, $\forall \varepsilon > 0$, there exists a finite set of rectangular parallelepipeds $R_1, \dots, R_n$ such that U is contained in the union of these rectangular parallelepipeds, and $V(R_1) + \cdots + V(R_n) < \varepsilon$, where $V(R_i)$ denotes the volume of R_i for $i = 1,2,\dots,n$.

In case of triple integrals, the function f is a function of three variables, and integration takes place over a closed and bounded region R in $\mathbb{R}^3$. Let R_i (where $i = 1,2,\dots,n$) be a closed and bounded subregion of a closed and bounded region R in $\mathbb{R}^3$; let $V(R_i)$ be the volume of R_i; and let (p_i, q_i, r_i) be an arbitrary point of R_i. Then the "triple integral" is defined as

$$\iiint_R f(x,y,z)\, dxdydz = lim_{n\to\infty} \sum_{i=1}^{n} f(p_i, q_i, r_i) V(R_i).$$

[555] Ibid.
[556] Ibid.

If, in particular, R is the rectangular parallelepiped $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$, then

$$\iiint_R f(x, y, z)\, dxdydz = lim_{n\to\infty} \sum_{j=1}^{\lambda} \sum_{i=1}^{\mu} \sum_{k=1}^{\nu} f\left(p_j, q_i, r_k\right) \Delta x_j \Delta y_i \Delta z_k,$$

where $n = \lambda \cdot \mu \cdot \nu$ is the number of elementary rectangular parallelepipeds (subregions R_{jik}) each of which has volume $V\left(R_{jik}\right) = \Delta x_j \Delta y_i \Delta z_k$.

If $R = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ and the number of rectangular parallelepipeds is n^3, then

$$\iiint_R f(x, y, z)\, dxdydz = lim_{n\to\infty} \sum_{i=1}^{n} \sum_{i=1}^{n} \sum_{i=1}^{n} f(p_i, q_i, r_i)\, \Delta x_i \Delta y_i \Delta z_i.$$

The properties of triple integrals are analogous to the properties of double integrals. Similarly, we can develop integrals over $\mathbb{R}^n$ for any $n > 3$, which enable us to integrate functions of $n > 3$ variables.

In general, the multiple integral of a function $f(x_1, \dots, x_n)$ in n variables over a domain U is represented by n nested integral signs in the reverse order of computation (in the sense that the leftmost integral is computed last), followed by the function and the integrand arguments in such an order that indicates that the integral with respect to the rightmost argument is computed last; and the domain of integration is either represented symbolically for every argument over each integral sign or it is indicated by a characteristic letter (variable) at the rightmost integral sign:

$$\int \dots \int_U f(x_1, \dots, x_n) d\, x_1 \dots dx_n.$$

2.22. Vector-Valued Functions[557]

When a function takes a real number and sends it to a vector (whose meaning was clarified in sections 2.2.5 and 2.2.6), then it is said to be a vector-valued function. In the real plane, namely, in the xy-plane, the general form of a vector-valued function is the following:

$$\vec{r}(t) = f(t)\hat{\imath} + g(t)\hat{\jmath}; \tag{1}$$

and, in the real 3-dimensional space, namely, in the xyz-space, the general form of a vector-valued function is the following:

$$\vec{r}(t) = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}; \tag{2}$$

where the component functions f, g, and h are real-valued functions of the parameter t, and $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are the corresponding unit vectors on the x-axis, the y-axis, and the z-axis,

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[557] Ibid.

respectively. The standard unit vectors in the direction of the x, y, and z axes of a 3-dimensional Cartesian coordinate system are

$$\hat{\imath} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \hat{\jmath} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \text{ and } \hat{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Analogously to section 2.6, the "limit" of a vector-valued function $\vec{r}(t)$ is $\vec{L}$ as t tends to a, symbolically,

$$lim_{t\to a}\vec{r}(t) = \vec{L}$$

if and only if

$$lim_{t\to a}\left\|\vec{r}(t) - \vec{L}\right\| = 0.$$

Therefore, (1) implies that

$$lim_{t\to a}\vec{r}(t) = [lim_{t\to a}f(t)]\hat{\imath} + [lim_{t\to a}g(t)]\hat{\jmath},$$

and (2) implies that

$$lim_{t\to a}\vec{r}(t) = [lim_{t\to a}f(t)]\hat{\imath} + [lim_{t\to a}g(t)]\hat{\jmath} + [lim_{t\to a}h(t)]\hat{k},$$

provided that the limits of the component functions f, g, and h as $t \to a$ exist. Similarly, we can define the limit of a vector-valued function of n component functions for $n > 3$.

Analogously to section 2.7, a vector-valued function $\vec{r}(t)$, where $t \in [a, b]$, is said to be "continuous" at a point $t_0 \in [a, b]$ if $lim_{t\to t_0}\vec{r}(t) = \vec{r}(t_0)$; and $\vec{r}(t)$ is said to be continuous on $[a, b]$ if it is continuous at every point of $[a, b]$.

Analogously to section 2.10, the derivative of a vector-valued function $\vec{r}(t)$, where $t \in [a, b]$, is defined as follows:

$$\overrightarrow{r'}(t) = lim_{\Delta t\to 0}\frac{\vec{r}(t+\Delta t)-\vec{r}(t)}{\Delta t},$$

provided that the limit exists. If $\overrightarrow{r'}(t)$ exists, then $\vec{r}(t)$ is said to be differentiable at t. If $\overrightarrow{r'}(t)$ exists $\forall t \in (a, b)$, then $\vec{r}(t)$ is said to be differentiable on the interval (a, b). In order for $\vec{r}(t)$ to be differentiable on $[a, b]$, $\vec{r}(t)$ must be differentiable on the interval (a, b), and the following two limits must exist as well:

$$\overrightarrow{r'}(a) = lim_{\Delta t\to 0^+}\frac{\vec{r}(a+\Delta t)-\vec{r}(a)}{\Delta t} \text{ and}$$

$$\overrightarrow{r'}(b) = lim_{\Delta t\to 0^-}\frac{\vec{r}(b+\Delta t)-\vec{r}(b)}{\Delta t}.$$

Consequently, (1) implies that

$$\overrightarrow{r'}(t) = f'(t)\hat{\imath} + g'(t)\hat{\jmath},$$

and (2) implies that

$$\overrightarrow{r'}(t) = f'(t)\hat{\imath} + g'(t)\hat{\jmath} + h'(t)\hat{k}.$$

The properties of the derivative of a vector-valued function are analogous to those of the derivative of a scalar-valued function.

Let f, g, and h be integrable real-valued functions on $[a, b]$. Then (1) implies that the indefinite integral of a vector-valued function $\vec{r}(t) = f(t)\hat{\imath} + g(t)\hat{\jmath}$ is

$$\int [f(t)\hat{\imath} + g(t)\hat{\jmath}]\, dt = [\int f(t)dt]\hat{\imath} + [\int g(t)dt]\hat{\jmath},$$

and the definite integral of a vector-valued function $\vec{r}(t) = f(t)\hat{\imath} + g(t)\hat{\jmath}$ is

$$\int_a^b [f(t)\hat{\imath} + g(t)\hat{\jmath}]dt = \left[\int_a^b f(t)dt\right]\hat{\imath} + \left[\int_a^b g(t)dt\right]\hat{\jmath}.$$

By analogy, (2) implies that

$$\int \left[f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}\right] dt = [\int f(t)dt]\hat{\imath} + [\int g(t)dt]\hat{\jmath} + [\int h(t)dt]\hat{k},$$

and

$$\int_a^b \left[f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}\right]dt = \left[\int_a^b f(t)dt\right]\hat{\imath} + \left[\int_a^b g(t)dt\right]\hat{\jmath} + \left[\int_a^b h(t)dt\right]\hat{k}.$$

The properties of the integral of a vector-valued function are analogous to those of the integral of a scalar-valued function.

Let us consider a function $f(x, y)$, namely, f depends on both x and y, and its graph is a surface in space. Then, in order to interpret and compute the rate of change of $f(x, y)$, we find the rate of change of $f(x, y)$ in a specific direction independently: if we want the rate of change in the x-direction, we differentiate $f(x, y)$ with respect to x while treating y as a constant, namely, we compute the partial derivative $\frac{\partial f(x,y)}{\partial x}$; similarly, if we want the rate of change in the y-direction, we differentiate $f(x, y)$ with respect to y while treating x as a constant, namely, we compute the partial derivative $\frac{\partial f(x,y)}{\partial y}$. The "gradient" of $f(x, y)$ is denoted by ∇f, and it is a concept that combines the two aforementioned partial derivatives; specifically, the gradient of $f(x, y)$ is a vector consisting of both partial derivatives of f in their associated positions; symbolically:

$$grad f \equiv \nabla f = \frac{\partial f(x,y)}{\partial x}\hat{\imath} + \frac{\partial f(x,y)}{\partial y}\hat{\jmath},$$

where $\hat{\imath}$ is the unit vector in the x-direction, and $\hat{\jmath}$ is the unit vector in the y-direction. By analogy, we can define the gradient of a function $f(x, y, z)$, etc.

Let us consider a vector-valued function $\vec{r}(x, y, z) = f(x, y, z)\hat{\imath} + g(x, y, z)\hat{\jmath} + h(x, y, z)\hat{k}$ such that the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial g}{\partial y}$, and $\frac{\partial h}{\partial z}$ exist and are continuous on $U \subseteq \mathbb{R}^3$. Then the "divergence" of $\vec{r}(x, y, z)$ is a vector operator that operates on a vector field, producing a scalar field that gives the quantity of the vector field's source at each point; and it is defined as follows:

$$div\vec{r} \equiv \vec{\nabla}\vec{r} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)\left(f\hat{\imath} + g\hat{\jmath} + h\hat{k}\right) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}.$$

For instance, as I explained in Chapter 1, in quantum physics, everything is described in terms of wave functions. In quantum mechanics, a wave function is a vector in a Hilbert space, and the vector coefficients are complex numbers. According to Paul Dirac's notation, in quantum physics, vectors are symbolized as follows (the bra-ket notation):

$$|\Psi\rangle = a_1\begin{pmatrix}1\\0\\0\end{pmatrix} + a_2\begin{pmatrix}0\\1\\0\end{pmatrix} + a_3\begin{pmatrix}0\\0\\1\end{pmatrix}, \text{ where } a_1, a_2, a_3 \in \mathbb{C}.$$

The aforementioned type of brackets helps us to keep track of whether a vector is a row vector or a column vector: $|\Psi\rangle$ is a column vector, whereas $\langle\Psi|$ is a row vector. In quantum mechanics, if we convert a column vector to a row vector, then we have to take the complex conjugate of each coefficient. In other words, for instance,

$|\Psi\rangle = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ and $\langle\Psi| = (a_1^*, a_2^*, a_3^*)$, where a_1^*, a_2^*, a_3^* are, respectively, the complex conjugates of a_1, a_2, a_3.

In quantum mechanics, all vectors describe probabilities, and, usually, we choose the basis in the space under consideration in such a way that the basis vectors correspond to possible measurement outcomes; for instance:

$$|\Psi\rangle = a_1\begin{pmatrix}1\\0\\0\end{pmatrix} + a_2\begin{pmatrix}0\\1\\0\end{pmatrix} + a_3\begin{pmatrix}0\\0\\1\end{pmatrix} \text{ corresponds to } |\Psi\rangle = a_1|X\rangle + a_2|Y\rangle + a_3|Z\rangle.$$

Hence, the probability of a particular measurement outcome is the absolute square of the scalar product with the basis vector that corresponds to the outcome; so that, for instance, the probability of measuring X is $|\langle X|\Psi\rangle|^2 = a_1 a_1^*$ (Born's Rule). In quantum physics, the gradient operator works as follows:

$$\nabla|\Psi\rangle = \frac{\partial}{\partial x}|\Psi\rangle\hat{\imath} + \frac{\partial}{\partial y}|\Psi\rangle\hat{\jmath} + \frac{\partial}{\partial z}|\Psi\rangle\hat{k}.$$

Chapter 3

RATIONALITY, TRUTH, AND ETHICS

3.1. BASIC PRINCIPLES OF LOGIC

Logic, namely, the science of correct reasoning, is a necessary underpinning of every philosophical and scientific endeavor. The first systematization of logic is due to the ancient Greek philosopher and scientist Aristotle, and, for this reason, the phrase "Aristotelian logic" is still commonly used. Aristotle's works on logic were grouped together by ancient commentators under the title *Organon* ("Instrument"). In particular, the *Organon* comprises the following logical treatises of Aristotle: (i) *Categories*, (ii) *On Interpretation*, (iii) *Prior Analytics*, (iv) *Posterior Analytics*, (v) *Topics*, and (vi) *On Sophistical Refutations*. The title *Organon*, meaning instrument, implies that logic is an instrument and a method used by philosophy and science, and, in particular, according to both Aristotle and the later Peripatetics, the ultimate purpose of correct reasoning is to create correct social relationships and to enable people to correctly communicate the results of philosophical and scientific research to each other. In the third century B.C., the Greek Stoic philosopher and logician Chrysippus founded a propositional calculus, studying implication, conjunction, and disjunction. In the mid-nineteenth century, the (largely self-taught) English mathematician, philosopher, and logician George Boole put logic within a rigorous mathematical setting, thus giving rise to what has been known as "Boolean algebra."[558]

As I mentioned in Chapter 2, logic involves a special set of symbols, namely:

$\wedge$ or &: conjunction ("and"),
$\vee$: disjunction ("or"),
$\neg$: negation ("not"),
$\rightarrow$ or $\Rightarrow$: material implication ("if . . . then . . .")
$\leftrightarrow$ or $\Leftrightarrow$: biconditional ("if and only if"),
$\forall$: universal quantification ("for every"),
$\exists$: "there exists,"
$\exists!$: "there exists exactly one,"

[558] See: Arnold, *Logic and Boolean Algebra*; Bell and Machover, *A Course in Mathematical Logic*; Ebbinghaus, Flum, and Thomas, *Mathematical Logic*; Epp, *Discrete Mathematics with Applications*; Kolman, Busby, and Ross, *Discrete Mathematical Structures*; Rautenberg, *A Concise Introduction to Mathematical Logic*.

$\nexists$: "there does not exist,"
$P(x)$: predicate letter (meaning that x (an object) has property P),
$|$: "such that,"
$\vdash$: turnstile ($x \vdash y$ means that x "proves" (i.e., syntactically entails) y; a sentence φ is "deducible" from a set of sentences Σ, expressed $\Sigma \vdash \varphi$, if there exists a finite chain of sentences $\psi_0, \psi_1, \psi_2, \ldots, \psi_n$ where ψ_n is φ and each previous sentence in the chain either belongs to Σ, or follows from one of the logical axioms, or can be inferred from previous sentences; $\nvdash$ denotes the negation of $\vdash$),
$\vDash$: double turnstile ($x \vDash y$ means that x "models" (i.e., semantically entails) y; a sentence φ is a "consequence" of a set of sentences Σ, expressed $\Sigma \vDash \varphi$, if every model of Σ is a model of φ).

Using the aforementioned notation, we can depict the well-known "logical square" that Aristotle articulated in the context of his *Organon* in order to describe the basic kinds of propositions as follows:

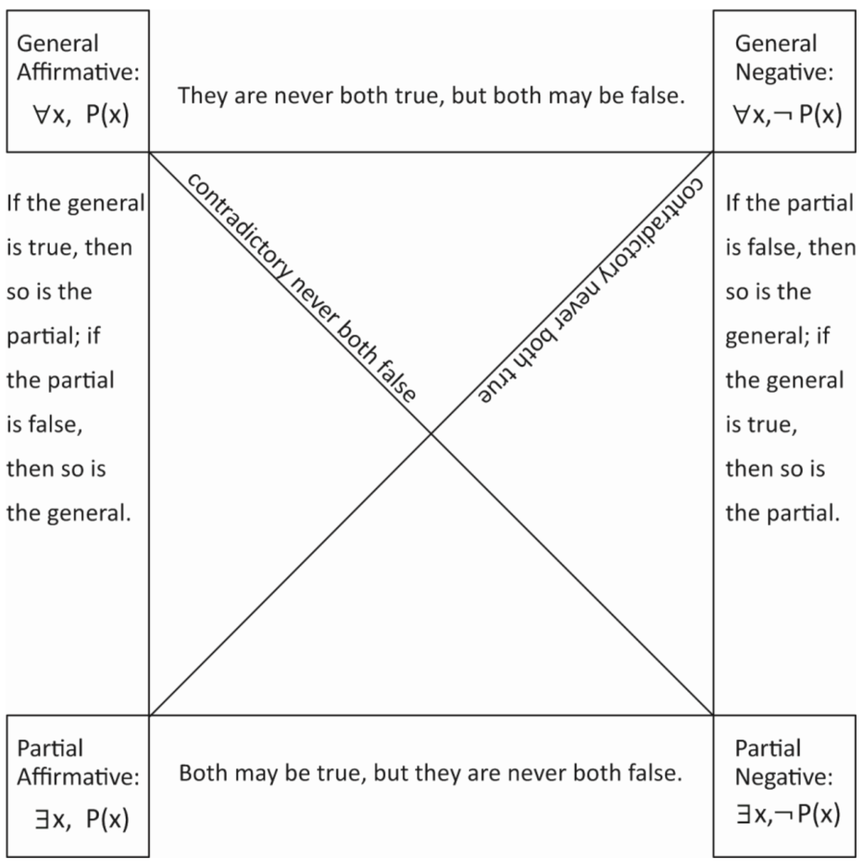

Figure 3.1. Aristotle's Logical Square.

George Boole realized that arguments expressed in an ordinary language (e.g., in ordinary English) can be expressed in the notation of mathematical logic and then studied in the context of "propositional calculus." For instance, consider the following argument:

- If you want to learn mathematics, then you must study methodically.
- If you must study methodically, then you must be taught an effective method of studying.

Therefore, if you want to learn mathematics, then you must be taught an effective method of studying.

The aforementioned argument involves various propositions, which we may present by letters as follows:

P: You want to learn mathematics.
Q: You must study methodically.
R: You must be taught an effective method of studying.

These propositions can be “true” or “false.” In section 3.2, we shall study propositions called “predicates,” which contain variables. The aforementioned argument can be formalized as follows:

$P \Rightarrow Q$
$Q \Rightarrow R$

$P \Rightarrow R$

where the two propositions above the dashed line are the “premises,” and the one below the dashed line is the “conclusion.” The reasoning process that leads from premises to a conclusion is called a “deductive process” or just a “deduction.”

It is important to distinguish between the terms “validity” and “truth” as they are used in logic. An argument, or a reasoning process, or a deduction is said to be valid (i.e., logically correct) if the truth of the conclusion follows from the truth of the premises. Notice that, if the premises are both true, then the conclusion is logically necessarily true, too, and, therefore, with one or more factually incorrect premises, an argument may still be valid, although its conclusion may be false. Furthermore, a valid argument based on false premises does not necessarily lead to a false conclusion. In other words, there is a substantial difference between *logical* (i.e., procedural) correctness, namely, “validity,” and *factual* correctness. If an argument is valid (i.e., logically correct), and if its premises are true (i.e., if the facts on which it is based are true), then it is said to be “sound.” In logic, we focus on the validity of arguments rather than on their soundness, and this fact explains the “instrumental” role of logic in philosophy and science.

A “Boolean algebra” is the 6-tuple

$$\langle A, \wedge, \vee, \neg, 0, 1\rangle,$$

consisting of a set A equipped with two binary operations, namely, $\wedge$(called “meet” or “and”) and $\vee$ (called “join” or “or”), a unary operation $\neg$ (called “complement” or “not”), and two elements 0 and 1 in A (called “bottom” and “top,” respectively, and denoted, respectively, by the symbols $\perp$ and $\top$), such that, the truth value of a true sentence is 1, the truth value of a false sentence is 0, and, for all elements a, b, and c of A, the following axioms hold:

i. Associativity:
$a\vee(b\vee c) = (a\vee b)\vee c$; $a\wedge(b\wedge c) = (a\wedge b)\wedge c$.
ii. Commutativity:
$a\vee b = b\vee a$; $a\wedge b = b\wedge a$.
iii. Absorption:
$a\vee(a\wedge b) = a$; $a\wedge(a\vee b) = a$.
iv. Identity:
$a\vee 0 = a$; $a\wedge 1 = a$.
v. Distributivity:
$a\vee(b\wedge c) = (a\vee b)\wedge(a\vee c)$; $a\wedge(b\vee c) = (a\wedge b)\vee(a\wedge c)$.
vi. Complements:
$a\vee\neg a = 1$ and $a\wedge\neg a = 0$.

For instance, the 2-element Boolean algebra has only two elements, namely, 0 and 1, and it is defined by the following rules:

Table 3.1. Truth Tables of a 2-Element Boolean Algebra

a	b	$a\wedge b$	$a\vee b$	a	$\neg a$
0	**0**	0	0	**0**	1
1	**0**	0	1	**1**	0
0	**1**	0	1		
1	**1**	1	1		

Those propositions whose truth value depends on the values of the variables in them are called "predicates."

3.2. PREDICATE CALCULUS

As I mentioned in section 3.1, George Boole developed a purely symbolic system for deduction in a rigorous language of predicates (or relations, or properties), and, thus, Predicate Calculus (henceforth, PC) emerged.[559] The formal system PC involves the following:

i. The alphabet of PC: a countable set of variables (or arguments): $v_1, v_2, v_3, \ldots$ and a two-place predicate letter P; two logical connectives: $\neg$and $\wedge$; one quantifier symbol:$\exists$; three improper symbols: the left parenthesis, the comma, and the right parenthesis, namely, (,), but quite often we may also use brackets [and] as well as the symbol | standing for "such that."

[559] Ibid.

ii. These symbols are used in order to build the (well-formed) formulas of PC, according to the following rules:
   a. If x, y are individual variables, then $P(x, y)$ is a formula of PC.
   b. If φ, ψ are formulas of PC, then so are $(\varphi \wedge \psi)$as well as $\neg\varphi$ and $\neg\psi$.
   c. If x is an individual variable and φ is a formula, then so is $\exists x\varphi$.
   d. Something is a formula of PC only by virtue of the aforementioned conditions (a), (b), and (c).

Remark: The alphabet contains only the logical symbols$\neg$, $\wedge$, and $\exists$, because the other usual symbols can be defined in terms of these three as follows:

$(\varphi \vee \psi)$ is defined as $\neg(\neg\varphi \wedge \neg\psi)$,
$(\varphi \rightarrow \psi)$ is defined as $\neg(\varphi \wedge \neg\psi)$,
$(\varphi \leftrightarrow \psi)$ is defined as $((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))$, and
$\forall x\varphi$ is defined as $\neg\exists x\neg\varphi$.

A variable is said to be "bounded" if it is determined by a quantifier; otherwise, it is said to be "free." For instance, in the formula $\exists xP(x, y)$, x is bounded, and y is free. If a formula of PC contains no free variables, then it is said to be a "sentence."

By an "interpretation," we mean the task of giving a certain meaning to the undefined terms of a formal system. Consider, for instance, the following sentences of PC:

i. $\forall x\forall y(P(x, y) \rightarrow P(x, y))$,
ii. $((P(x, y) \wedge P(y, z)) \rightarrow P(x, z))$, and
iii. $\forall y\exists xP(x, y)$.

If we interpret P as the ancestor relation over the domain of all (living and dead) people (and if we assume that such a relation is biologically determined in a definite way), then: (i) means that, "if x is an ancestor of y, then x is an ancestor of y, for every x and y," namely, it is a tautology; (ii) means that "if x is an ancestor of y, and if y is an ancestor of z, then x is an ancestor of z"; (iii) means that, "for every y, there exists an ancestor x." Thus, (i), (ii), and (iii) are true. However, if we interpret P as $<$ ("strictly less than") over the natural numbers, then (iii) is false. Moreover, if we interpret P as "the father of" over the domain of human beings, then (ii) is false. We can easily notice that (i) will remain true for any interpretation of P; such sentences of PC are said to be "universally valid" (and they are tautological in character).

A "formal system" is obtained by choosing a finite set of axioms (or schemes of axioms, i.e., selected formulas) and a finite set of rules of inference in a given language. In the case of PC, we have the following axioms and the following rule of inference (φ, ψ, χ are formulas, $x, y, y_1, \ldots, y_n, \ldots$ are variables, and $\varphi(y)$ is the result of substituting y for all free occurrences of x in $\varphi(x)$):

Axioms of Predicate Calculus[560]*:*

[560] Ibid.

i. $\forall y_1 \dots \forall y_n\ (\varphi \to (\psi \to \varphi))$.
ii. $\forall y_1 \dots \forall y_n\ ((\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi)))$.
iii. $\forall y_1 \dots \forall y_n\ ((\neg\varphi \to \neg\psi) \to ((\neg\varphi \to \psi) \to \varphi))$.
iv. $\forall y_1 \dots \forall y_n\ (\forall x(\varphi \to \psi) \to (\varphi \to \forall x\psi))$, provided that φ has no free occurrence of x.
v. $\forall y_1 \dots \forall y_n\ ((\varphi \to \psi) \to (\forall y_1 \dots \forall y_n \varphi \to \forall y_1 \dots \forall y_n \psi))$.
vi. $\forall y_1 \dots \forall y_n\ (\forall x \varphi(x) \to \varphi(y))$, provided that, as we substitute the free occurrences of x in $\varphi(x)$ with y, the y's are free in $\varphi(y)$, that is, they are not determined by quantifiers already occurring in φ.

Rule of Inference for Predicate Calculus[561]*:*

Modus Ponens: from φ and $(\varphi \to \psi)$, infer ψ. In other words, if a conditional statement ("if φ then ψ") is accepted, and the antecedent (φ) holds, then the consequent (ψ) may be inferred.

A "theorem" is a formula inferred by means of a rule of inference in a finite number of steps from axioms and previously inferred formulas. Hence, we are faced with the problem of determining that finite set of axioms (or schemes of axioms) from which the rule of inference will give only true sentences.

First, we must clearly determine the meaning of the term "true formula" in a given interpretation, regardless of whether it is a sentence. When we interpreted $P(x, y)$ as the ancestor relation, we realized that $P(x, y)$ may be true for some values of x and y and false for others. Therefore, we have to specify the values of free variables when we interpret them (such a problem does not exist in case of bounded variables).

An "interpretation of the formal system PC" is a structure

$$I = \langle U, R \rangle,$$

where U is a (non-empty) set whose members are $u, u_1, u_2, \dots$, and R is a relation on U. Then U is called the universe of the interpretation, and the predicate letter P is interpreted as R (for which reason R is two-place).

Assume that members of U are assigned to all the individual variables of PC in such a way that at most one member of U is assigned to each variable of PC (one member of U may be assigned to more than one variable). Then a formula φ is said to be satisfied in I by an assignment of u_1 to x, u_2 to y, etc. ($x, y, \dots$ are free variables in φ) if the relation over U corresponding to φ (i.e., if the replacement of each P in φ by R) holds between the elements assigned to the free variables of φ. In this case, we write

$$I \vDash \varphi[u_1, u_2, \dots],$$

where $u_1, u_2, \dots$ are all the assignments to the free variables of φ. Thus, for the interpretation $I = \langle U, R \rangle$ for PC, we have:

561 Ibid.

i. $I \vDash P(x_1, x_2)[u_1, u_2]$ if and only if $u_1 R u_2$.
ii. $I \vDash \neg\varphi[u_1, \dots]$ if and only if $I \vDash \varphi[u_1, \dots]$ does not hold.
iii. $I \vDash (\varphi \wedge \psi)[u_1, \dots]$ if and only if $I \vDash \varphi[u_1, \dots] \wedge I \vDash \psi[u_1, \dots]$.
iv. Let $\psi(x_1, \dots, x_n)$ be of the form $\exists y \varphi(x_1, \dots, x_n, y)$. Then
v. $I \vDash \psi[u_1, \dots, u_n]$ if and only if there exists a t in U such that $I \vDash \varphi[u_1, \dots u_n, t]$.

If $I \vDash \varphi[u_1, \dots]$ for every possible assignment to the free variables in φ, then we write $I \vDash \varphi$, and then φ is said to be "true" in I. If, for every interpretation I, $I \vDash \varphi$ holds, then φ is said to be "universally valid," and we write $\vDash \varphi$. If φ is a sentence (i.e., if it has no free variables), then $I \vDash \varphi$ if and only if $I \vDash \varphi[u_1, \dots, u_n]$ for some choice of $u_1, \dots, u_n$ (such a choice is, in fact, arbitrary).

For instance, consider the interpretation $I = \langle V, T \rangle$, where V is the universe of all (living or dead) human beings, and T is the ancestor relation over V, and the formula $\exists x P(x, y)$. If p and q are human beings, then

$I \vDash \exists x P(x, y)[p]$ if and only if $I \vDash P(x, y)[q, p]$

for some $q \in V$. This holds if and only if qTp for some $q \in V$. According to biology, this is always true (i.e., for each p, there is always someone who is an ancestor of p), and, therefore, there is always some $q \in V$ such that qTp, meaning that the formula $\exists x P(x, y)$ is always true in $I = \langle V, T \rangle$.

If Σ is a set of sentences of PC and I is an interpretation such that $I \vDash \varphi$ for every sentence φ in Σ, then I is said to be a "model" of Σ.

We are already familiar with the axioms and the rule of inference of PC. However, we have to prove that the axiomatic system PC is "complete" (that is, the theorems generated by the axiomatic system PC are all the universally valid formulas) and "consistent" (that is, the theorems generated by the axiomatic system PC are only universally valid formulas, or, in other words, the axiomatic system PC neither contains nor produces contradictions). Given the axiomatic system PC, a "proof" in PC is a finite sequence of formulas, each of which is either an axiom or inferred from formulas earlier in the sequence by a rule of inference. A "theorem" can be regarded as the last line of a proof, and we write $\vdash \varphi$ if φ is a theorem.

If Σ is a set of axioms, then we write $\Sigma \vdash \varphi$ if φ is provable in the axiomatic system composed of the initial axiomatic system PC and the sentences in Σ that have been accepted as additional axioms. A set of sentences Σ is "consistent" if there exists no formula φ for which both $\Sigma \vdash \varphi$ and $\Sigma \vdash \neg\varphi$ hold.

The Gödel–Henkin Completeness Theorem for PC[562]*:* The Austrian mathematician and logician Kurt Gödel (1906–78) and the American logician Leon Albert Henkin (1921–2006) have independently proved the following theorem, which is known as the "Gödel–Henkin Completeness Theorem for PC": If Σ is a consistent set of formulas, then there exists an interpretation I such that $I \vDash \psi$ for every ψ in Σ. Thus, if a formula is logically valid, then there is a finite deduction (i.e., a formal proof) of the formula.

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[562] Ibid.

Proof: In order to prove the aforementioned theorem, we need, first of all, to prove Lindenbaum's Lemma, according to which any consistent theory of predicate calculus can be extended to a complete consistent theory, or, in other words, every consistent set of formulas has a maximal consistent superset.

Notice that a consistent set of sentences Σ is said to be "complete" if, for each sentence φ of the language, either $\Sigma \vdash \varphi$ or $\Sigma \vdash \neg\varphi$. Then Lindenbaum's Lemma can be formulated as follows: If Σ is consistent, then there exists a complete consistent extension Σ^* of Σ, that is, for any sentence φ of the language, exactly one of the following holds: $\Sigma^* \vdash \varphi$ or $\Sigma^* \vdash \neg\varphi$. We can prove Lindenbaum's Lemma as follows: First, let us list all sentences $\varphi_1, \varphi_2, \varphi_3, \ldots$ of PC. Then let us define a sequence $\Sigma_0, \Sigma_1, \Sigma_2, \ldots$ of sets of sentences as follows: Set $\Sigma_0 = \Sigma$. Let

$$\Sigma_1 = \begin{cases} \Sigma_0 \; if \; \Sigma_0 \vdash \neg\varphi_1 \\ \Sigma_0 + \varphi_1 \; if \; \Sigma_0 \nvdash \neg\varphi_1 \end{cases},$$

that is, if φ_1 can be added to Σ_0 so that the new set is still consistent, then we add it to obtain Σ_1; otherwise, we leave $\Sigma_1 = \Sigma_0$. Under the same reasoning, we obtain

$$\Sigma_{n+1} = \begin{cases} \Sigma_n \; if \; \Sigma_n \vdash \neg\varphi_{n+1} \\ \Sigma_n + \varphi_{n+1} \; if \; \Sigma_n \nvdash \neg\varphi_{n+1} \end{cases},$$

so that each Σ_n is consistent. If Σ^* is what we get by adding or not adding, as the case may be, each formula in our list of sentences of PC, then Σ^* is consistent; because, given that the length of every proof is finite, a proof of inconsistency in Σ^* would imply that one of the Σ_n is inconsistent, but this is impossible because we have guaranteed by the construction that each of the Σ_n is consistent. Moreover, all sentences of the language are in the list $\varphi_1, \varphi_2, \varphi_3, \ldots$, and, at each stage n, we determine whether to add φ_n to the Σ_n. Thus, for each formula φ, exactly one of φ or $\neg\varphi$ is in Σ^* and is provable from Σ^*. In this way, we have proved Lindenbaum's Lemma, and, now, we shall return to the proof of the Gödel–Henkin Completeness Theorem for PC.

Step 1: We start with a theory Σ.

Step 2: Add the constants $c_1, c_2, c_3, \ldots$ to the language and revise the alphabet and the definition of a formula of PC to obtain Σ'. These individual constants were added in such a way that, whenever a property $\psi(v_1)$ is satisfied by some object in our universe, we can fix some constant c and assert $\psi(c)$, and, in this case, c is a definite "witness" that there exists an element satisfying the property ψ. Obviously, if Σ is consistent, then so is Σ' (since we simply add names of the objects in a prospective universe).

Step 3: List all the formulas whose unique free variable is v_1: $\psi_1(v_1), \ldots, \psi_n(v_1), \ldots$ Let ω_n be the formula

$\exists v_1 \psi_n(v_1) \rightarrow \psi_n(c)$,

where c is the first witness not previously used in a ψ or a ω.

Step 4: In order to add the ω_n as axioms, let

$\Sigma^0 = \Sigma'$,

$\Sigma^{n+1} = \Sigma^n + \omega_n$,
$\Sigma^\infty = \cup\, \Sigma^n$.

Thus, if we add all the new axioms to Σ, we obtain the system Σ^∞. Obviously, each Σ^n is consistent. Therefore, Σ^∞ is consistent (notice that a proof of Σ^∞'s inconsistency would be a proof of Σ^n's inconsistency for some n).

Step 5: By Lindenbaum's Lemma, Σ^∞ can be extended to a complete consistent extension Σ^*. Thus, for any sentences φ and ψ of Σ^*, we have:

a) $\Sigma^* \vdash \varphi or \Sigma^* \vdash \neg\varphi, i.e., \Sigma^* is complete.$
b) $\Sigma^* \vdash \neg\varphi if and only if \Sigma^* \nvdash \varphi.$
c) $\Sigma^* \vdash (\varphi \wedge \psi) if and only if \Sigma^* \vdash \varphi \wedge \Sigma^* \vdash \psi.$
d) $\Sigma^* \vdash \exists v_1 \psi(v_1) if and only if \Sigma^* \vdash \psi(c) for some c; since \exists v_1 \psi(v_1) \rightarrow \psi(c) is one of the \omega_n axioms.$

Step 6: Define a model $M = \langle U, R \rangle$ for Σ^*, so that we have the universe $U = \{c_{1,} c_2, \ldots\}$ and the relation R on U such that $c_i R c_j$ if and only if $\Sigma^* \vdash P(c_i, c_j)$.

Step 7: We conclude that (due to Steps 1, 2, 3, and 4, and due to the definition of an interpretation of PC)
$M \vDash \varphi if and only if \Sigma^* \vdash \varphi.$

Step 8: Since Σ is contained in Σ^*, it follows that $M \vDash \varphi$ for all φ in Σ. Thus, if Σ is consistent, then Σ has a model. In this way, we have proved the Gödel–Henkin Completeness Theorem for PC.■

Remark:Σ has a countable model, since the c_i form a countable set.

A set of sentences is said to be ω-complete if, whenever it deductively yields every instance of a universal generalization, it also yields the given generalization. In particular, an arithmetic theory T is said to be ω-complete if, for every formula $\varphi(x)$ in one free variable x, the fact that, in T, one can derive all the formulas of the form $\varphi(0), \varphi(1), \ldots$ implies that the formula $\forall x \varphi(x)$ is derivable in T. A set of sentences is said to be ω-consistent if, whenever it yields every instance of a universal generalization, it does not also yield that there exists an instance that contradicts it.

3.3. Axiomatic Model Theory

In the study of the foundations of science as a semiotic phenomenon (that is, as something strongly connected with the subject matter of the science of signs), language is a crucial factor; for, civilization, in general, can be regarded *sub specie semioticae*, on the grounds that it is a system of systems of meaning.[563] According to the Swiss linguist Ferdinand de Saussure, language is made of linguistic units, or "sings," which are composed of two parts, namely, a concept (or meaning) and a sound-image: respectively, "the signified" and the "signifier."[564] Since, according to Saussure, a language is a system of "signs" that express "ideas," the study of signs is of great significance. The American philosopher and mathematician Charles Sanders Peirce has placed particular emphasis on the notion of a

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[563] See: Ecco, *A Theory of Semiotics.*
[564] Saussure, *Course in General Linguistics*.

"sign" as something that substitutes for something else to somebody else from some point of view or relative to some properties.[565] As Charles Morris, has argued, something is a "sign" if it is interpreted as a sign of another thing by someone, namely, there are different "possible interpretations" and "possible interpreters."[566]

Consider the following syllogism:

(S) The Queen is the head of England, and
Elizabeth II is the Queen of England;
hence, Elizabeth II is the head of England.

In order to formalize (S), we shall use a predicate language $\mathbb{L}$ having, in addition to the usual system of PC and quantifiers, an one-place predicate P interpreted as "is the head of England," a two-place predicate letter T interpreted as "is the same as," and two constant letters e and q for "Elizabeth II" and "Queen," respectively. In addition, we shall use two of the usual PC connectives: $\wedge$ and $\Rightarrow$. Then (S) can be formalized as follows:

$(S')P(q) \wedge T(e,q) \Rightarrow P(e).$

The predicate letter T is called the "identity predicate," and PC equipped with T is called "predicate calculus with identity," denoted by PC(i).

Consider the structure $Q = \langle A, K, a_1, a_2, R \rangle$, where A is a non-empty set, K is a property on A (i.e., a subset of A), R is a relation on A, and a_1 and a_2 are members of A. It is easily seen that Q is a structure wherein we can interpret (S'). Thus, if e is interpreted as a_1, q as a_2, P as K, and T as R (i.e., R is the identity relation on A), and if $A = \{a_1, a_2\}$, we conclude that (S') asserts that, if a_2 has property K, and if a_1 and a_2 are the same, then a_1 has property K.

A "normal interpretation" for a language $\mathbb{L}$ is one for which the interpretation of the two-place predicate (T) is the identity relation. In every normal interpretation, the following are true:

Axioms of Equality:

i. $\forall x\, T(x,x)$, that is, T is reflexive;
ii. $\forall x \forall y\, (T(x,y) \Rightarrow T(y,x))$, that is, T is symmetric;
iii. $\forall x \forall y \forall z\, (T(x,y) \wedge T(y,z) \Rightarrow T(x,z))$, that is, T is transitive;
iv. *Leibniz's Law (the Identity of the Indiscernibles):* For any formula φ of $\mathbb{L}$,

$\forall x \forall y\, (T(x,y) \Rightarrow (\varphi(x,x) \Rightarrow \varphi(x,y)))$. In other words, according Leibniz, no two distinct things exactly resemble each other, or, equivalently, no two (distinct) objects have exactly the same properties.[567]

Two interpretations I_1 and I_2 of an axiomatic system Σ are said to be "isomorphic" if there exists a correspondence (i.e., an one-to-one mapping) from the elements of I_1 onto the

[565] Peirce, *Collected Papers*.
[566] See: Ecco, *A Theory of Semiotics*.
[567] Leibniz, "Discourse on Metaphysics."

elements of I_2 (namely, a bijection $I_1 \to I_2$) such that it is preserved by the relations and the operations of Σ. Notice that two isomorphic models differ only in the nature of their elements, but the structure in each is the same.

Example: Consider the axioms of an elementary part of geometry called “incidence geometry”:

i. For any two distinct points P and Q, there exists exactly one straight line l incident with P and Q.
ii. For every straight line l, there exist at least two distinct points incident with l.
iii. There exist three distinct points such that no straight line is incident with all three of them.

Now, consider a set $\{A, B, C\}$ of three letters that we call “points.” By “lines,” we mean those subsets that contain exactly two letters, such as $\{A, B\}$. A point is interpreted as “incident” with a “line” if it is a member of that subset; for instance, A lies on $\{A, B\}$ and $\{A, C\}$ but not on $\{B, C\}$. It is easily seen that the preceding interpretation is a model for incidence geometry. Furthermore, let us consider another set $\{a, b, c\}$ of three letters that we call “lines.” Now, by “points,” we mean those subsets that contain exactly two letters, such as $\{a, b\}$. Assume that incidence is set membership, namely, that “point” $\{a, b\}$ is “incident” with “lines” a and b but not with “line” c. This model has the same structure as the aforementioned three-point model, and these two models differ only in notation. An explicit isomorphism is given by the following correspondences: $A \leftrightarrow \{a, b\}$, $B \leftrightarrow \{b, c\}$, $C \leftrightarrow \{a, c\}$, $\{A, B\} \leftrightarrow b$, $\{B, C\} \leftrightarrow c$, and $\{A, C\} \leftrightarrow a$. Hence, A lies on $\{A, B\}$ and $\{A, C\}$ only, and its corresponding “point” $\{a, b\}$ lies on the corresponding “lines” b and a only; by analogy, incidence is preserved by the correspondence for B and C, too. Nevertheless, notice that, if we use the correspondence $\{A, B\} \leftrightarrow a, \{B, C\} \leftrightarrow b, \{A, C\} \leftrightarrow c$ for the “lines” and leave the correspondence for the “points” as it was before, then we do not obtain an isomorphism, because, for instance, A lies on $\{A, C\}$, but the corresponding “point” $\{a, b\}$ does not lie on the corresponding “line” c.

A set of axioms is called “categorical” if any two of its interpretations are isomorphic. Thus, a “categorical judgment” is one in which the attribute refers to the subject in an absolute, unconditional way (for instance, “Humans are mammals.”).

The Gödel–Henkin Completeness Theorem for PC is a very important result in model theory. Because of the fact that a formula that is satisfiable in some non-empty domain cannot be refutable, it follows that the aforementioned Completeness Theorem includes the Löwenheim–Skolem Theorem, which asserts that, if a formula is satisfiable, then it is satisfiable in a countably infinite domain (or, equivalently, that, if a countable theory has a model, then it has a countable model).[568] In particular the Norwegian mathematician and logician Thoralf Albert Skolem (1887–1963) not only showed that German mathematician Leopold Löwenheim’s argument was true once again, but also extended the theorem to the case of a countable infinity of formulas. Moreover, if they are simultaneously satisfiable, Kurt Gödel treated the case in the domain of natural numbers.

[568] See: Heijenoort, *From Frege to Gödel.*

If a set of sentences Σ is inconsistent, then some finite subset of Σ is inconsistent, too[569]; because, by hypothesis, for some formula φ, both $\Sigma \vdash \varphi$ and $\Sigma \vdash \neg\varphi$, namely, there exists a finite string of formulas $\varphi_1, \varphi_2, \dots, \varphi_n$ with $\varphi_n = \varphi \wedge \neg\varphi$ such that, for each $\varphi_i, i = 1,2, \dots, n$, one of the following holds: (a) φ_i is a logical or equality axiom, or (b) φ_i is a member of Σ, or (c) φ_i is inferred by the established rule of inference from two formulas in the list before φ_i; since the list is finite, it follows that the number of the formulas in it that are members of Σ is finite.

The Compactness Theorem (Kurt Gödel proved the countable compactness theorem in 1930, and Anatoly Maltsev proved the uncountable case in 1936)[570]*:* If every finite subset of Σ has a model, then Σ has a model (Σ is a countably infinite set of formulas).

Proof: This theorem can be easily proved by *reductio ad absurdum* as follows: Suppose, for contradiction, that every finite subset of Σ has a model, but Σ does not have a model. Then, by the contrapositive of the Gödel–Henkin Completeness Theorem, Σ is inconsistent. Therefore, there exists a sentence φ such that $\Sigma \vdash \varphi$ and $\Sigma \vdash \neg\varphi$, and each of these deductions consists of a finite chain of sentences. If Σ' is the set of sentences of Σ involved in the deduction of φ, and if $\Sigma''$is the set of sentences of Σ involved in the deduction of $\neg\varphi$, then we obtain two finite subsets of Σ such that $\Sigma' \vdash \varphi$ and $\Sigma'' \vdash \neg\varphi$. Obviously, $\Sigma' \cup \Sigma'' \vdash \varphi$ and $\Sigma' \cup \Sigma'' \vdash \neg\varphi$, since the sentences involved in the original deductions still belong to the union of the two finite subsets. Consequently, $\Sigma' \cup \Sigma''$ is inconsistent, and, hence, it does not have a model. By having thus produced a finite subset of Σ that does not have a model, we have reached the desired contradiction.■

Example[571]*:* Assume that Σ is a set of sentences with arbitrarily large finite normal models. In the language of Σ augmented by infinitely many new constant letters $c_1, c_2, \dots$, let Σ^* be the set of sentences consisting of Σ plus all the sentences $\neg T(c_i, c_j)$ for which $i \neq j$. We shall show that Σ^* has a model by showing that every finite subset of Σ^* has and then applying the Compactness Theorem. If Σ' is a finite subset of Σ^*, then Σ' contains some of the elements of Σ and finitely many sentences $\neg T(c_i, c_j)$ involving only finitely many of the constant letters c_i that will be among $c_1, c_2, \dots, c_n$ for an appropriate n. By hypothesis, Σ has a normal model $\langle A, \dots \rangle$ with at least n elements, so that, if $a_1, a_2, \dots$ are elements of A with $a_1, a_2, \dots, a_n$ distinct, then $\langle A, \dots, a_1, a_2, \dots \rangle$ is a model of Σ', where $a_1, a_2, \dots$ are the interpretations of $c_1, c_2, \dots$ Hence, Σ^* has a model, and so has a normal model $\langle B, \dots, b_1, b_2, \dots \rangle$, where $b_1, b_2, \dots$ are the interpretations of $c_1, c_2, \dots$ Consequently, $\langle B, \dots \rangle$ is a normal model of Σ, since Σ^* includes Σ, and $b_i \neq b_j$ whenever $i \neq j$, and, thus, B is infinite. We conclude that Σ has an infinite normal model. In general, we can easily reach the conclusion that the Gödel–Henkin Completeness Theorem implies that every consistent set of sentences in a countable language (i.e., a language with a countable number of formulas) with an equality predicate letter including the equality axioms has a countable normal model.

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569 See: Arnold, *Logic and Boolean Algebra*; Bell and Machover, *A Course in Mathematical Logic*; Ebbinghaus, Flum, and Thomas, *Mathematical Logic*; Epp, *Discrete Mathematics with Applications*; Kolman, Busby, and Ross, *Discrete Mathematical Structures*; Rautenberg, *A Concise Introduction to Mathematical Logic*.

570 Ibid.

571 Ibid.

3.4. Common Sense, Non-Monotonic Logic, and Many-Valued Logic

By now, it has become clear that the character of the system that is used in formalization is "monotonic," in the sense that, given the operator of logical consequence Con for two sets of formulas (databases) X and Y, it holds that, if $X \subseteq Y$, then $Con(X) \subseteq Con(Y)$. However, we must consider the case in which the addition of new data to the existing sets of formulas (databases) may contradict propositions that had been previously accepted as valid, and it may compel us to revise previous syllogisms.

For instance, consider the following logical proposition:

"Birds fly."

This proposition is not generally true, since there exist birds that do not fly. Let us add the following piece of information to our database:

"Hedwig is a bird."

Then common sense urges us to infer that Hedwig flies, since there is no clear evidence that Hedwig belongs to a non-typical case. Thus, the conclusion that "Hedwig flies" is due to the lack of data indicating that Hedwig is an exception to the established rule. Now, assume that we receive the following piece of information:

"Hedwig is a hen, or a penguin, or an ostrich, or a bird with broken wings."

Then the conclusion that "Hedwig flies" must be retracted.

The structure of the aforementioned syllogism is characteristic of what we call "non-monotonic reasoning."[572] The data management method and the proof procedure that characterize non-monotonic logic are different from those that characterize classical syllogisms. Thus, we should incorporate new hypotheses (which are regarded as "exceptions") into the proof procedure. In particular, the general form of formalization should be modified as follows:

"From the set X of hypotheses, we can infer y,
unless it happens to have z as a hypothesis."

In fact, non-monotonic systems have significant applications to medical diagnoses, database analysis, conflict analysis, intelligence analysis, etc.[573]

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[572] It is important to mention that reasoning may also prove to be defeasible when it is applied to an inconsistent stock of information obtained via different sources.

[573] See: Arieli and Avron, "General Patterns of Nonmonotonic Reasoning"; Elqayam and Over, "New Paradigm Psychology Reasoning"; Pearl, *Probabilistic Reasoning in Intelligent Systems*; Pollock, "Defeasible Reasoning."

Non-monotonic logic, originally systematized by the American computer scientists Jon Doyle and Drew McDermott in the early 1980s, implies that there are several possibilities. Hence, non-monotonic reasoning deals with the problem of deriving plausible yet not infallible conclusions from a given set of formulas, and, since the conclusions are not certain, it must be possible to retract some of them if new information shows that they are wrong. The concept of "possible worlds" underpins the development of many-valued logics.

Many-valued logics are similar to classical (Aristotelian) logic, because they accept the principle of truth functionality, according to which the truth of a compound sentence is determined by the truth values of its constituent sentences, as shown in section 3.1 (and, therefore, the truth value of a compound sentence remains unaffected when one of its component sentences is replaced by another sentence of the same truth value). However, in contrast to classical (Aristotelian) logic, multi-valued logics discard the principle of excluded middle, in the sense that they do not restrict the number of truth values to only two (denoted by "0" and "1"). In fact, multi-valued logics allow for a larger set K of truth degrees. In a k-valued logic, there are $k-2$ different ("intermediate") truth degrees between the extreme truth values "absolutely false" (denoted by "0") and "absolutely true" (denoted by "1").[574]

The Polish logician and mathematician Jan Łukasiewicz started studying multi-valued logic in the 1920s, developing a 3-valued logic, whose set of truth degrees consists of the values "true" (denoted by "1"), "false" (denoted by "0"), and "undetermined" or "possible" (denoted by "1/2"), in order to deal with Aristotle's "paradox of the sea-battle." In particular, Aristotle, in his treatise *On Interpretation* (the second treatise from Aristotle's *Organon*), formulated the famous "paradox of the sea-battle," which can be summarized as follows: Consider a statement, such as "a sea-battle will be fought tomorrow." According to Aristotle's principle of excluded middle, statements must be either true or false. But, if there are several possibilities out of which one is going to take place tomorrow, can there be a truth now about which one will take place? If the answer is "yes," then the following problem emerges: on what grounds could something that is still a mere possibility, nevertheless be true already now? If the answer is "no," then the following problem emerges: can we argue that all logically exclusive possibilities are necessarily untrue without denying that one of the possible outcomes must turn out to be the chosen one? This is a characteristic example of the so-called problem of future contingents: how can we assign truth values to contingent statements about the future? To solve the aforementioned problem (and resolve the corresponding paradox), Łukasiewicz developed his 3-valued logic, which accepts the values "true," "false," and "undetermined."

According to Łukasiewicz's principles governing implication and equivalence involving the third value ("possibly," or, symbolically, "1/2"), 3-valued logic gives rise to the following truth tables:

Table 3.2. Truth Tables of a 3-Valued Logic (Łukasiewicz Logic)

AND	True	Undetermined	False
True	T	U	F
Undetermined	U	U	F
False	F	F	F

[574] See: Ackerman, *An Introduction to Many-Valued Logics*; Bolc and Borowik, *Many-Valued Logics*; Yablonsky, *Introduction to Discrete Mathematics*.

OR	True	Undetermined	False
True	T	T	T
Undetermined	T	U	U
False	T	U	F

NOT	
True	F
Undetermined	U (i.e., $\neg U = U$)
False	T

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
T	U	U
F	T	T
F	F	T
F	U	T
U	T	T
U	F	U
U	U	T

In fact, in the 1920s, the American logician and mathematician Emil Leon Post formulated k additional truth degrees with $k \geq 2$, and, later, Łukasiewicz and the Polish-American logician and mathematician Alfred Tarski together formulated a logic of k truth values, where $k \geq 2$. In the 1930s, the German philosopher of science and logician Hans Reichenbach formulated an infinitely-valued logic.

In the 1960s, the American mathematician, logician, and computer scientist Lofti Zadeh of the University of California at Berkeley formulated the so-called "fuzzy logic," in which a proposition can possess a degree of truth anywhere between 0.0 and 1.0. In other words, in fuzzy logic, the truth values of variables may be any real number between 0.0 and 1.0 both inclusive. Fuzzy logic is based on the assumption that people make decisions based on imprecise and non-numerical information, and it has significant applications to control theory and artificial intelligence.[575]

As I have already explained, logic studies the concepts of syntactic consequence and semantic consequence, and, in this context, it is concerned with propositions, sets of propositions, and the relation of consequence among them. Hence, as Petr Hájek has pointed out, logic tries to represent all this "by means of well-defined logical calculi admitting exact investigation," but "various calculi differ in their definitions of sentences and notion(s) of consequence," and, in particular, "fuzziness is imprecision (vagueness); a fuzzy proposition may be true to some degree."[576] Hájek has explained the reasoning of fuzzy logic as follows:

> Standard examples of fuzzy propositions use a *linguistic variable* as, for example, *age* with possible values *young, medium, old* or similar. The sentence "The patient is young" is

[575] See: Hájek, *Metamathematics of Fuzzy Logic*; Cintula, Hájek, and Noguera, eds., *Handbook of Mathematical Fuzzy Logic*.

[576] Hájek, *Metamathematics of Fuzzy Logic*, p. 1.

> true to some degree—the lower the age of the patient (measured e.g., in years), the more the sentence is true. *Truth of a fuzzy proposition is a matter of degree* . . . In a narrow sense, fuzzy logic, FLn, is a logical system which aims at a formalization of approximate reasoning. In this sense, FLn is an extension of multivalued logic . . . In its wide sense, fuzzy logic, FLn, is fuzzily synonymous with the fuzzy set theory, FST, which is the theory of classes with unsharp boundaries.[577]

3.5. CRISES IN THE FOUNDATIONS OF MATHEMATICS AND MATHEMATICAL PHILOSOPHY

Mathematics is a large discipline, and it has many particular areas (such as number theory, algebra, geometry, mathematical analysis, topology, probability and statistics, combinatorics, etc.). In the domain that is known as the foundations of mathematics, one deals with the problem of the systematization of mathematics and with the problem of infinity (in fact, these two problems are interrelated). There is much to learn from the foundations of mathematics regarding both the structure of consciousness and the structure of the world. Moreover, the study of the foundations of mathematics refines and puts into practice critical-thinking skills. Ideally, "mathematical rigor" consists in the following set of attributes and tasks: (i) Formulating initial definitions in a careful and precise way. (ii) Making a minimum number of reasonable assumptions. (iii) Deriving new theorems in a step by step logical way. (iv) Constantly verifying and cross-checking the constituent components of one's research work in order to preclude mistakes and logical gaps from creeping into it. (v) Making careful computations. Hence, there is a substantial difference between giving one *confidence in* an argument and *formally proving* it. However, as I shall explain shortly, achieving rigor (in particular, logical continuity) is a very arduous task even within the context of mathematics.

3.5.1. The First Crisis in the Foundations of Mathematics

In the seventh century B.C., Thales of Miletus, a Greek mathematician, astronomer, and philosopher from Miletus, in Ionia, Asia Minor, officially initiated a new approach to mathematics. In contrast to the mathematics of other civilizations, such as the Babylonians and the Egyptians, Thales's approach to mathematics is based on the thesis that scientific propositions are not recipes for practical tasks, that is, techniques whose validity is determined by the method of trial and error, but they should be explained and proved.[578] In other words, Thales attempted to endow mathematics with rigor, which, in this case, means logical validity.

In the context of Thales's rigorous mathematics, by the term "line segment," we mean a part of a line that is bounded by two distinct endpoints, and contains every point on the line between the endpoints. Let us consider the line segments $a_1, a_2, a_3, \dots, a_n$ and the non-zero line segments $b_1, b_2, b_3, \dots, b_n$. The line segments $a_1, a_2, a_3, \dots, a_n$ are said to be "proportional" to $b_1, b_2, b_3, \dots, b_n$, respectively, if

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[577] Ibid, p. 2.

[578] Heilbron, *Geometry Civilized*; Holme, *Geometry*; Ostermann and Wanner, *Geometry and Its History*.

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \cdots = \frac{a_n}{b_n}.$$

Thus, two arbitrary line segments a and c are proportional to two other arbitrary line segments b and d, respectively, if b and d are non-zero, and it holds that

$$\frac{a}{b} = \frac{c}{d}. \quad (1)$$

Any equality between two ratios, such as (1), is said to be a "proportion" with terms a, b, c, and d, as shown above.

Assume that AB is a non-zero straight line segment, and that P is a point on AB. Then we say that the point P "divides internally" the straight line segment AB in a ratio λ, where $\lambda \geq 0$, if it holds that

$$\frac{PA}{PB} = \lambda.$$

If this is the case, then $\frac{PA}{PA+PB} = \frac{\lambda}{\lambda+1} \Leftrightarrow PA = \frac{\lambda}{\lambda+1} AB$, which implies the uniqueness of P. Similarly, we say that a point Q "divides externally" the straight line segment AB in a ratio λ, where $\lambda \geq 0$, if the points A, B, and Q are collinear, Q is external to AB, and it holds that

$$\frac{QA}{QB} = \lambda.$$

If this is the case, then $\frac{QA}{|QA-QB|} = \frac{\lambda}{|\lambda-1|}$ (given that $QA \neq QB$, it holds that $\lambda \neq 1$), so that $QA = \frac{\lambda}{|\lambda-1|}$ AB, which implies the uniqueness of Q.

Thales's Theorem[579]*:* If parallel straight lines intersect two straight lines, then they define proportional straight line segments on them. For instance, if parallel straight lines l_1, l_2, and l_3 intersect straight lines a and a' at points A, B, C and A', B', C', respectively, as shown in Figure 3.2, then

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}.$$

Corollary 1: Every straight line that is parallel to the bases of a trapezoid divides, both internally and externally, the non-parallel sides of the given trapezoid in equal ratios.

Corollary 2: Every straight line that is parallel to one side of a triangle divides, both internally and externally, the other two sides of the given triangle in equal ratios.

[579] Ibid.

Corollary 3: If two triangles have a common angle, and if they have parallel opposite sides, then they are said to be in Thales position, and then they are similar and have proportional sides.

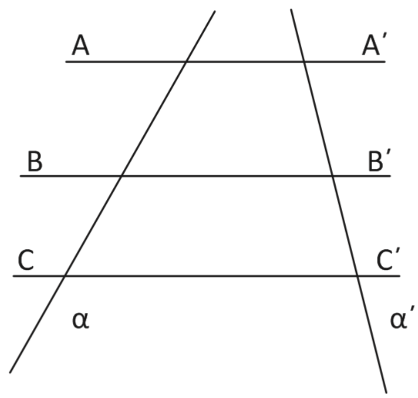

Figure 3.2. Thales's Theorem.

In the sixth century B.C., Pythagoras and his school (the so-called "Pythagoreans") endorsed Thales's approach to mathematics. From the Pythagorean perspective of mathematics, the relations between the objects of the world (e.g., magnitudes) correspond to the relations between natural (and, generally, integral) numbers. However, it was soon realized that things are not so simple, since it was realized that there exist magnitudes that do not have a common measure. According to the Pythagoreans, two objects (magnitudes) are "commensurable," that is, they have a common measure, if and only if there is a magnitude of the same kind contained an integral number of times in both of them. In other words, two magnitudes are "commensurable" if and only if their ratio is a rational number. However, the Pythagoreans encountered "incommensurable" magnitudes, namely, magnitudes whose ratio is an irrational number. For instance, the length of a diagonal of a unit square, namely, of a square whose sides have length 1, is, according to the Pythagorean Theorem, equal to $\sqrt{2}$, which is an irrational number; similarly, a circle's circumference and its diameter are incommensurable (see sections 2.1.2, 2.2.3, and 2.2.4). The awareness that there exist incommensurable magnitudes compelled ancient Greek mathematicians to inquire into the relations between incommensurable magnitudes. This event marked a major crisis in ancient mathematics.

According to ancient Greek mathematicians, quantities (magnitudes) are continuous and uniform objects, which are best represented by straight line segments, whereas their division into parts, namely, their measurement in terms of a "unit of measurement" (i.e., a definite magnitude of a quantity), represents the notion of discreteness. Ancient Greek mathematicians used the term "ratio of magnitudes" in order to refer to the relation between two magnitudes that can be measured in terms of a common unit of measurement, and, thus, the ancient Greek concept of a ratio is most similar to the more abstract modern concept of a number. In the context of ancient Greek mathematics, the objects of mathematics were quantities (represented by straight line segments), and the ratio between two quantities was a meta-object, namely, something that was used in order to study mathematical objects without being treated as a mathematical object itself. In other words, in the context of ancient Greek mathematics, a ratio (namely, a number) was construed as a measuring relationship between two quantities, and such a measuring relationship could be built up (and, hence, proved) in finitely many steps, using a common unit of measurement. Nevertheless, the discovery of incommensurable ratios demonstrated that a ratio could not be interpreted as a measuring relationship in the aforementioned way. In fact, as a result of the discovery of incommensurable ratios, the concept of a ratio (or a number) acquired its conceptual autonomy, and, instead of being treated as a meta-object, it started being treated as an object of mathematics. Therefore, ancient Greek mathematicians had to transcend the system of mathematics that was based on commensurable ratios (notice that a commensurable ratio could easily become an object of mathematical theory, since it is a rational number, and, therefore, it can be constructed in finitely many steps, whereas the decimal representation of an irrational number neither terminates nor infinitely repeats but extends forever without regular repetition).

In the fourth century A.D., Theon, one of the most important Greek mathematicians and commentators of Euclid's and Ptolemy's works, attempted to solve the problems that were generated as a result of the aforementioned crisis in the foundations of ancient Greek mathematics. In particular, Theon started from an extremely small (namely, infinitesimal) unit square such that the ratio between any of its sides and any of its diagonals is equal to 1 (given that it is infinitely small); symbolically, if a_1 is the length of each of the sides of the given infinitesimal unit square, and if δ_1 is the length of each of the diagonals of the given infinitesimal unit square, then $\frac{\delta_1}{a_1} = 1$. Subsequently, Theon formulated a recursive sequence of unit squares defined by

$$a_n = \delta_{n-1} + a_{n-1} \text{ and } \delta_n = 2a_{n-1} + \delta_{n-1},$$

so that the ratio between a diameter and a side of these unit squares approaches its real value (meaning the real relationship between a diameter and a side of these unit squares according to the Pythagorean Theorem), namely,

$$\frac{\delta_n}{a_n} \to \sqrt{2}.$$

He explained that he started from the case in which $\frac{\delta_1}{a_1} = 1$ because, just as the sperm of a living organism encompasses all the subsequent properties of the given organism, so any ratio (including the ratio between a diagonal and a side of a unit square) spermatically (namely, at the infinitesimal level) encompasses the unit.

Theon's aforementioned reasoning is underpinned by Aristotle's concept of a "potential infinity." The concept of modern mathematics that is semantically most similar to Aristotle's concept of a "potential infinity" is the convergence of a sequence of natural numbers (see section 2.4). Thus, from the perspective of ancient Greek mathematics, infinity is not a being (i.e., it is not an actual state), namely, it cannot be simultaneously considered in its whole extension, but it can only be considered as a becoming (i.e., a process). In this way, the concept of an infinite approach helps us to overcome the contradiction between incommensurable ratios and commensurable ratios, since we can think of an incommensurable ratio infinitely approaching a commensurable ratio and vice versa.

Similarly, the concept of an infinite approach helps us to overcome the contradiction between broken lines and curves as well as the contradiction between continuity and discreteness. This reasoning is endorsed by Euclid, and, therefore, in his *Elements*, he does not consider infinitely extended straight lines, but he always works with straight line segments, which, as he says, can be extended as much as one needs.

However, several intellectuals have used infinite processes in a way that is not rigorous. For instance, they have attempted to compute the length of the circumference of a circle by considering an inscribed polygon whose number of sides increases indefinitely, and, therefore, the length of each side of such a polygon decreases indefinitely, so that a triangle whose base is a side of the given polygon and whose vertex (i.e., the "top" corner opposite its base) is the center of the given circle could become such that its base coincides with the given circle's circumference. But then to what extent is such a shape a triangle, and beyond which point does a straight line segment (in this case, the base of a triangle) become a chord? One may argue that these changes happen when a straight line segment becomes infinitely small, but then one may counter-argue that, by becoming infinitely small, a straight line segment is not "something" any more, and it becomes "nothing." Hence, how is it possible that an infinite series of "nothing" (namely, of "no-things") gives "something," such as a circle? The aforementioned example indicates the problems that are generated as a result of the use of infinite processes in computations.

The aforementioned crisis in the foundations of mathematics was overcome by Eudoxus's theory of proportions and by the method of exhaustion, which derives from Eudoxus's theory of proportions, and it was used by Archimedes. As I mentioned in section 2.2.6, Archimedes was very careful in the use of infinite processes, and, therefore, he approximated π by using the fact that the circumference of a circle is bounded by the perimeter of an *inscribed* polygon and the perimeter of a *circumscribed* polygon. According to Eudoxus (an ancient Greek mathematician, astronomer, philosopher, and student of Archytas and Plato) and Archimedes, there is always a ratio between any two magnitudes, and we can always make any magnitude smaller or greater than a given magnitude, so that the ratio between two magnitudes a and b is the same as the ratio between two other magnitudes c and d if and only if, for any natural numbers m and n, it holds that

$ma \gtreqless nb \Rightarrow mc \gtreqless nd$, that is,
$\frac{m}{n} \gtreqless \frac{a}{b} \Rightarrow \frac{m}{n} \gtreqless \frac{c}{d}$, (1)

meaning that both of these ratios are characterized by the same placement property (i.e., ordering) with regard to other numbers. In (1), the equality sign (=) refers to commensurable ratios, whereas the inequality signs ($\gtrless$) refer to incommensurable ratios. These ideas of Eudoxus and Archimedes indicate that ancient Greek mathematicians discovered not only incommensurable magnitudes but also incommensurable numbers. Moreover, the ideas of Eudoxus and Archimedes are conceptually very similar to Dedekind's cuts (see section 2.2.4). Eudoxus's aforementioned theory of proportions underpins Archimedes's method of exhaustion for the solution of geometric problems, and Archimedes's method of exhaustion underpins modern infinitesimal calculus (see section 2.9).

It is important to notice that the way in which Eudoxus solved the problem of the existence of incommensurable ratios (i.e., his attempt to make rigorous the conundrum of irrationality that appears to exist in elementary geometry) marks a shift away from the traditional constructivist approach to mathematics toward formalism. In other words, Eudoxus does not explain what is a ratio (as a mathematical object), but he states only when two ratios are similar to each other. The constructivist approach to mathematics allows us to determine what is an object by being able to construct it, whereas the formalist approach to mathematics is not concerned with the substance of the mathematical object under consideration, and it is concerned only with the relations of the mathematical object under consideration to other mathematical objects. The difference between the constructivist approach to mathematics and the formalist approach to mathematics reappeared and gave rise to heated epistemological debates in the nineteenth and the twentieth centuries in the context of the controversy between the "school" of intuitionism and the "school" of formalism, to which I shall refer later in this section.

Eudoxus's decision to give primacy to the concept of a ratio as a meta-object over the concept of a quantity as an object underpins the dual way in which a "number" is construed in the context of modern mathematics: a "number" can be construed both as a multiplicative entity (in which case, for instance, "six" means a quantity six times bigger than a given quantity) and as an additive entity (in which case, for instance, "six" means a quantity whose magnitude is six units). Therefore, using the terminology of abstract algebra (studied in section 2.1.4), we may say that any real number can be considered both as an element of a multiplicative group and as an element of an additive group. Due to the fact that multiplication distributes over addition, the conception of real numbers as a multiplicative group can be considered as a group of automorphisms of the corresponding additive group, since, for every $a \in \mathbb{R}$, we may define the automorphism ψ_a of the additive group of real numbers by $\psi_a(x) = a \cdot x$ for every $x \in \mathbb{R}$ (the converse does not hold, because addition is not distributive over multiplication).

3.5.2. The Second Crisis in the Foundations of Mathematics

Another important crisis in the foundations of mathematics broke out in the seventeenth century. Whereas ancient Greek mathematics (as it is expounded and systematized by Euclid in his *Elements*) is based on a geometric way of thinking (arguably, with the exception of the research works of Archimedes and Diophantus), modern European mathematics is more inclined to an algebraic way of thinking (and, hence, it tends to give primacy to arithmetic over geometry). This shift was typified by the reduction of geometry to arithmetic in the context of the so-called analytic geometry, which is characterized by the use of coordinates and by the correspondence between curves and equations (see section 2.2.6). From the perspective of ancient Greek mathematics, the only curves that "exist" are those that can be constructed and, hence, be defined according to Euclid's *Elements*, whereas, from the perspective of analytic geometry, any curve that can be expressed through an algebraic equation is acceptable (see sections 2.2.6, 2.5, and 2.6). The use of coordinate systems implies that space itself is encoded by n-tuples (i.e., by sequences, or ordered lists, of n numbers), and, specifically, that the 2-dimensional space, the "plane," is encoded by pairs of numbers, so that the conception of space becomes subordinate to the conception of arithmetic.

In the seventeenth century, mathematicians (primarily the initiators and developers of calculus) were preoccupied with such geometric problems as the computation of areas and volumes of arbitrary geometric figures and the construction of tangents to curves (see section 2.9) as well as with such physical problems as the formulation of the law that determines the rate of change of velocity and of acceleration (with respect to time) when one knows the law that determines the rate of change of displacement of an object (i.e., its velocity) and vice versa. The tendency toward the study of the aforementioned types of geometric and physical problems was reinforced by Galileo's physical theory, which constrained Aristotle's theory of motion (according to which the term "motion" referred to any kind of change, development, and growth) to the study of change in the relative position of physical objects.

The most prominent seventeenth-century mathematicians realized that, when we treat geometric figures and the motions of physical bodies as "wholes," we cannot demonstrate significant apparent similarities between them, but, when we analyze them into (sufficiently) "small" pieces, they display great similarities to each other. Hence, the major problem of seventeenth-century mathematics consisted in determining the proper processes for dividing the "whole" into "small" parts, which would be more easily and more rigorously studied than the "whole," as well as in determining the proper processes for resynthesizing the behavior of the "whole" from the behavior of its "small" parts. However, the "small" parts into which an object of scientific research was divided were similar to the "small" parts into which another object of scientific research was divided, and, thus, they could give rise to generalizations (such as natural laws), only when the dimensions of such "small" parts tended to zero, and, thus, only when the number (i.e., the population size) of such "small" parts tended to infinity. Therefore, the need for the use of infinite processes, specifically, infinitesimals, became prominent again. Even though "infinitesimal methods" could lead to correct results and useful applications, they lacked the logical rigor that characterized ancient Greek mathematics, particularly, Euclid's *Elements*, and they were susceptible to contradictions. Some mathematicians argued that lengths consisted of (infinitely many) infinitesimal lengths, areas consisted of (infinitely many) infinitesimal areas, and volumes consisted of (infinitely many) infinitesimal volumes, while other mathematicians argued that lines consisted of an infinite number of points, surfaces consisted of an infinite number of lines, and solid bodies consisted of an infinite number of surfaces. In that era, namely, in the seventeenth century, the concept of a limit (which I studied in sections 2.3.4, 2.3.5, 2.4, 2.5, and 2.6) was not yet clarified. It is worth pointing out that the famous French Enlightenment scholar Voltaire described calculus as "the art of measuring exactly a thing whose existence cannot be conceived," thus expressing his bewilderment at the fact that the seventeenth-century infinitesimal calculus was a useful and powerful scientific instrument, but the actual things that it was talking about were almost beyond conception.[580]

As I mentioned in Chapter 2, Newton defined the derivative of a function as the "ultimate ratio" of "vanishing quantities," and Leibniz argued that the quantities dy and dx, which appear in the definition of the derivative of a function, are infinitely small yet non-zero quantities. These ambiguities ignited heated debates regarding the foundations of infinitesimal calculus. In fact, the major problem pertaining to the development of infinitesimal calculus in the seventeenth and the eighteenth centuries was the reduction of a continuous entity, namely, a "whole," to discrete entities, namely, infinitesimals (i.e.,

[580] Quoted in: Simoson, *Voltaire's Riddle*, p. 51.

infinitely small parts of the corresponding "whole"), by means of a non-well-defined concept, namely, the concept of infinity. However, the effectiveness of the application of infinitesimal methods to physics and astronomy played a significant role in the acceptance and the further development of infinitesimal calculus. In general, many eighteenth-century mathematicians drew their subject matter from many branches of physics, astronomy, navigation, cartography, commerce, and finance.

Before the invention of infinitesimal calculus, physics was mainly a qualitative science, in the sense that it described and attempted to interpret phenomena without using specific magnitudes and laws for the synthesis of magnitudes, whereas infinitesimal calculus allowed physics to become a clearly quantitative science. Regarding the significance of the transition from old, pre-Newtonian qualitative ("speculative") physics to quantitative physics, René Thom has pointedly argued as follows:

> . . . it is not the impossibility of giving a quantitative result that condemns the old qualitative theories to modern eyes, for what matters most for everyday use is almost always a qualitative result and not the precise value of some real number. When we drive our car from town A to town B a hundred miles away, we rarely calculate our route with precision. What matters is the qualitative result: that we will arrive at B after a finite and reasonable time without having hit any obstacles lying in our path . . . What condemns these speculative theories in our eyes is not their qualitative character but the relentlessly naïve form of, and the lack of precision in, the ideas that they use.[581]

Thus, what modern physics has been actually requesting from mathematics from the eighteenth century onward is not so much to provide physics with instruments and methods that ensure *computational* precision as to provide physics with instruments and methods that ensure *conceptual* precision and logical rigor. Having said that, I do not intend to underestimate the tendency of scientific consciousness toward *descriptive* accuracy, but, following René Thom's reasoning, I intend to highlight the significance of developing this tendency to its ultimate consequences, which involve *explanatory* accuracy. In this way, physics has played an important role in the further development of infinitesimal calculus, and, in the eighteenth and the nineteenth centuries, infinitesimal calculus developed into the broad discipline that is known as mathematical analysis (including differential equations, complex functions, and differential dynamics). Nevertheless, during the same period, almost every thorough mathematician was concerned with the epistemological problems of mathematics, and refused to content oneself with the production of theorems of advanced mathematical analysis while particular areas of mathematics (pertaining to basic mathematical analysis) were lacking in rigor and were dependent on geometric and physical perceptions.

In the eighteenth and the nineteenth centuries, neither the notion of evidentness (as a method of justification) nor the reduction to Euclid's geometry was deemed to be sufficient for the rigorous foundation of mathematical analysis. Furthermore, the creation of non-Euclidean geometries and the proof of the fact that the consistency of non-Euclidean geometries is equivalent to the consistency of Euclidean geometry gave rise to the question of which is the geometry of physical space, on which the development of infinitesimal calculus (and, hence, of mathematical analysis, in general) was based. Therefore, many mathematicians turned their attention to the concepts of a function and a real number, and

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[581] Thom, *Structural Stability and Morphogenesis*, p. 6.

they attempted to achieve a rigorous foundation of mathematical analysis by means of these concepts (as I explained in Chapter 2). The rigorous foundation of mathematical analysis was initiated by Cauchy (with his "epsilon-delta" definitions of the limit and the continuity of a function and with the use of the characteristic phrases "as little as one wishes" and "the variable approaches its limit"), and it was developed further by Bolzano, Abel, Dirichlet, and Weierstrass (all these scientific endeavors and concepts were studied in Chapter 2). Due to the research works of such mathematicians as Cauchy, Bolzano, Abel, Dirichlet, and Weierstrass, we can present qualitative results in a rigorous way, because, in the context of modern mathematical analysis, we know how to define a "form," and we can determine whether two functions have the same form.

3.5.3. Logicism[582]

The attempts of nineteenth-century mathematicians to found mathematical analysis in a rigorous way was based on real numbers, which also needed a rigorous foundation. The first mathematician who attempted to study real numbers in a systematic way and interpreted them in terms of infinite decimals was the Flemish mathematician, physicist, and military engineer Simon Stevin (1548–1620). In his book *De Thiende* (*The Tenth*), Stevin introduced the idea that we could get away with doing arithmetic only with integers, so that, instead of having to deal with difficult fractions, we could use only decimal numbers, provided that we are ready to think of numbers extending indefinitely (for instance, the repeated fraction $\frac{11}{7}$ could be interpreted as something going on forever with the same pattern repeated over and over again, namely, $1.5714285714 28\ldots$; similarly, $\frac{11}{6}$ could be interpreted as $1.833333333333\ldots$, etc.; and several irrational numbers can be expressed in terms of infinite decimal numbers, such as $\sqrt{2} = 1.414213\ldots$, $\pi = 3.1415926\ldots$, $e = 2.718281\ldots$, etc.). But Stevin did not define a real number in a rigorous way, nor did he clarify the exact manner in which the system of real numbers works (for instance, he did not give rigorous answers to such questions as how do we do arithmetic with infinite decimals, how do we verify the laws of arithmetic with infinite decimals, etc.). However, in the nineteenth century, mathematicians realized that the theory of real numbers was at the core of most foundational problems of mathematical analysis, and Weierstrass argued that we cannot understand continuous functions unless we have a rigorous theory of the arithmetic continuum, namely, a rigorous way of thinking of numbers as being on a number line (see section 2.7).

Whereas, from the perspective of ancient mathematicians, numbers are things by means of which we count, Cartesianism, which is based on the algebraization of geometry, gave rise to the idea that numbers can be thought of as positions on the number line. Fusing geometry and arithmetic is an arduous task. In order to understand the difficulties that originate from fusing geometry and arithmetic, let us consider, for instance, the famous irrational number $\sqrt{2}$, which was discovered by Pythagoreans when they attempted to compute the length of a diagonal of a unit square. The Pythagoreans realized that the diagonal of a unit square is not

582 Kneebone, *Mathematical Logic and the Foundations of Mathematics*; Leng, *Mathematics and Reality*; Moore, *The Infinite*; Potter, *Set Theory and Its Philosophy*; Struik, *A Concise History of Mathematics*. The origins of the logicist project, according to which mathematics is reducible to logic, can be traced in the work of Leibniz.

commensurable with the side of the given square, but, by keeping geometry and arithmetic separate from each other (that is, by refusing to identify numbers with lengths of straight line segments), ancient Greek mathematicians could argue as follows: given a straight line segment whose length is one, we can construct a straight line segment whose length is $\sqrt{2}$, and, in general, irrational numbers are geometrically constructible (and, hence, geometrically explicable and manageable), even though, from the perspective of arithmetic, irrational numbers are ideal quantities, in the sense that the calculation of irrational numbers (e.g., $\sqrt{2}$) is an infinite process (i.e., irrational numbers have infinitely many decimal digits). On the other hand, in the nineteenth century, having endorsed the Cartesian approach to mathematics, mathematicians realized that they had to clarify some still ambiguous fundamental concepts (such as that of a real number), to formulate new methods of doing mathematics in a logically rigorous way, and to create a rigorous theory of the arithmetic continuum, specifically, a rigorous theory of real numbers and their arithmetic.

As I explained in Chapter 2, Dedekind developed his theory of the so-called Dedekind cuts as a theoretical framework for the arithmetic continuum, or, in other words, for establishing a rigorous theory of real numbers and their arithmetic. Dedekind recognized that the major difficulty in establishing a rigorous theory of real numbers and their arithmetic consists in the transition from rational numbers to irrational numbers, or, equivalently, in the transition from the realm of discreteness to the realm of continuity. In particular, according to Dedekind, we can define a real number in terms of rational numbers by thinking about cuts of rational numbers into two subsets that partition the whole set of rational numbers, so that, for instance, $\sqrt{2}$ can be defined as the set $\{a \in \mathbb{Q} | a^2 < 2 \; or \; a < 0\}$. Finally, Dedekind attempted to articulate a rigorous foundation of rational numbers, namely, of the set $\mathbb{Q}$, and, hence, ultimately, of natural numbers, namely, of the set $\mathbb{N}$, and he argued that the only stable foundation in order to achieve this goal is logic. In particular, Dedekind attempted to use purely logical principles and concepts, such as the concept of a "class" (a precursor of the concept of a set), in order to achieve his goal. By being based on the concept of the set $\mathbb{Q}$ of all rational numbers and on the concept of a subset of $\mathbb{Q}$, and by interpreting rational numbers as infinite sets, Dedekind's theory solves particular problems, but it creates new ones, because it requires a prior rigorous formulation of set theory and of the concept of infinity, and, therefore, Dedekind's theory *per se* is incomplete. Furthermore, there is an important difference between defining a subset of $\mathbb{Q}$ by "choice" and defining a subset of $\mathbb{Q}$ by an "algorithm," and Dedekind failed to address this difference.

Even though it is incomplete, Dedekind's research work in the foundations of mathematics is the precursor of a *corpus* of mathematical research works that espouse the thesis that the principles of logic are *a priori*, independent of the physical world, history, and society, and that mathematics can be entirely reduced to logic and, thus, acquire an absolute character. Dedekind's approach to mathematics, which is known as "logicism," was similar to Cantor's approach to mathematics. As I explained in Chapter 2, Cantor attempted to articulate a rigorous theory of sets and of infinity and to define the concept of a number by means of his theory of sets. In fact, in the early 1870s, Dedekind and Cantor spent much time in mathematical discussions with each other. However, as I explained in Chapter 2, Cantor's theory of sets was lacking in rigor, for which reason another great German mathematician, Kronecker (who headed the Department of Mathematics at the University of Berlin until his death in 1891), went as far as to call Cantor a "corrupter of youth" for teaching, prematurely, his set theory to a younger generation of mathematicians.[583] In fact, Cantor's definition of a set (which I studied in section 2.1.1), leaves the following questions unanswered: (i) What kind of objects does his definition allow? (ii) How should one tackle self-referential issues? Specifically, is the collection of all conceivable sets a set? As I explained in section 2.1.1, if we accept $U = \{x|xisaset\}$ as a set, then U contains itself, and, therefore, as Russell has shown, set theory produces contradictions.

In the late nineteenth century, working in parallel with Cantor, Gottlob Frege attempted to extricate modern mathematics from the ambiguities and the logical deficiencies of Dedekind's theory. Frege wanted to develop a logically rigorous mathematical language that would make possible the formalization of every definition and every proof within a symbolic system precluding every ambiguity. Frege outlined his plan in his book entitled *Begriffsschrift (Concept-script)*, published in 1879, and he formulated it in a more mature way in his two-volume book entitled *Grundgesetze der Arithmetik (Basic Laws of Arithmetic),* whose first edition was published in 1893, and its second edition was published in 1903. In other words, Frege's system, which has been characterized as a term logic (since every complete expression denotes terms), is a predicate calculus. In particular, in Frege's term logic, sentences denote terms that denote truth values. Frege distinguished two truth values, The True and The False, which he interpreted as objects. The basic sentences of Frege's logical system are constructed by using the expression

"$( ) = ( )$"

which signifies a binary function that maps a pair of objects x and y to The True if x is identical to y, and maps x and y to The False if x is not identical to y. Moreover, according to Frege, a "concept" is a function that maps every relevant argument to a truth value. Thus, Frege extended his system to the representation of non-mathematical thoughts and predications, paving the way for the development of predicate calculus. In Frege's system, complex and general statements are expressed by means of the following four special functional expressions:

Statement: The function that maps The True to the True, and maps every other object to The False (it is used in order to express the thought that the argument of the function is a true statement).

Negation: The function that maps The True to The False, and maps every other object to The True.

Condition: The function that maps a pair of objects to The False if the first is The True and the second is not The True, and maps every other pair of objects to The True.

Generality: The second-level function that maps a first-level concept Φ to The True if Φ maps every object to The True, while, otherwise, it maps Φ to The False.

In his *Grundgesetze der Arithmetik* (*Basic Laws of Arithmetic*), Frege attempted to expand the domain of objects of his logical system as follows: given any function (or concept) f in his system, he associated an object, which he called "the course-of-values of f," with f. The course-of-values of a function (or a concept) f consists in a record of the values of f for each argument. Frege symbolized the course-of-values of a function f with a Greek letter

[583] Quoted in: Dauben, "Georg Cantor and Pope Leo XIII," p.89.

epsilon with a smooth breathing mark above it, namely: $\dot{\varepsilon}(f)$. Using this notation, Frege formulated his basic law of extension as follows: the course-of-values of a concept f is identical to the course-of-values of a concept g if and only if f and g agree on the value of every argument, that is, if and only if, for every object x, f(x) = g(x); symbolically:

$$\dot{\varepsilon}(f) = \dot{\alpha}(g) \equiv \forall x\big(f(x) = g(x)\big).$$

Therefore, Frege's system departed from what was until then the ordinary way of using mathematical concepts, and, on the basis of his logical system, Frege attempted to prove that mathematical concepts can be defined in terms of purely logical concepts, and that mathematical principles can be derived from the laws of logic alone. It goes without saying that, in the context of Frege's logical system, argumentation is extended into an impressive subtlety that precludes evidentness and is underpinned by and addressed to logic alone. Nevertheless, in June 1902, as Frege was preparing the proofs of the second volume of the *Grundgesetze der Arithmetik*, he received a letter from Russell, informing him that he had discovered a contradiction in set theory that was also present in the first volume of Frege's *Grundgesetze.* In particular, as I mentioned in section 2.1.1, Russell pointed out the following paradox in Cantor's set theory: if U is the set of all those sets which are not members of themselves, symbolically, $\mathrm{U} = \{\mathrm{X} | \mathrm{X} \notin \mathrm{X}\}$, then both the statement that $\mathrm{U} \in \mathrm{U}$ and the statement that $\mathrm{U} \notin \mathrm{U}$ contradict the definition of U. By analogy, regarding Frege's logical theory, Russell framed the aforementioned paradox first in terms of the predicate P = "being a predicate which cannot be predicated on itself," and then in terms of the class of all those classes which are not members of themselves. Hence, Frege's logical system was proved to be logically deficient. Even though Frege's system is unsuitable for the foundation of mathematics, it is far from a useless or failed system. As I shall explain in section 3.6, the concept of a fallacy should be clearly distinguished from the concept of an irrationality, and, as I argued in section 1.3.3 (regarding the over-statements and the over-simplifications of skepticism), partial knowledge should not be semantically equated with ignorance, and an approximation of truth should not be equated with falsehood. Frege's work reinforced and clarified the philosophical "school" of logicism, according to which logic is a suitable and sufficient foundation for mathematics. In fact, the philosophical and mathematical work of Russell and Whitehead was based on Frege's work, and it marks the culmination of logicism.

Russell and Whitehead formulated their viewpoints and their own variety of logicism in their book entitled *Principia Mathematica (Principles of Mathematics),* which was first published in three volumes in 1910, 1912, and 1913, while a second edition appeared in 1925 (Volume I) and in 1927 (Volumes II and III). Before the publication of this book, Alfred North Whitehead was already a distinguished mathematician and philosopher (and, in 1903, he was elected a Fellow of the Royal Society as a result of his work on universal algebra, symbolic logic, and the foundations of mathematics), while Bertrand Russell was a lecturer at Trinity College, University of Cambridge. The viewpoints of Russell and Whitehead regarding the foundations of mathematics and philosophy consist essentially of the basic theses of logicism, which are also found in the works of Dedekind, Cantor, and Frege. However, the symbolic language that Russell and Whitehead used in their *Principia Mathematica* was based on the work of the Italian mathematician Giuseppe Peano, who was the first mathematician who constructed a symbolic language for mathematics and formulated the needs of mathematics in terms of a symbolic language. As I explained in section 2.2.1, Peano, in his seminal research work entitled *Formulaire de Mathématiques*, formulated the axiomatic system of the set ℕ of all natural numbers. The major difference between Peano and the logicists is that Peano treats symbolic logic and the axiomatic method as means to a logically rigorous foundation of mathematics, whereas logicism attempts to philosophically explain mathematics in terms of logic. From this perspective, Peano's research work was a precursor of Hilbert's formalism, to which I shall refer later in this section.

Russell believed that the foundational problems of mathematics could be solved and overcome by reformulating Aristotelian logic in a "Platonic" way, in order, in this way, to equip mathematics with epistemologically and ontologically robust underpinnings. Thus, Russell argued that mathematical truths are reducible to logical truths, and that logic is equivalent to Plato's world of ideas. In other words, at the core of Russell's logicism, is Russell's decision to interpret Platonic ideas as logical substances, and, thus, to assert the universality and the necessity of logic. In view of the arguments that I put forward in section 1.2.2, Russell's aforementioned interpretation of Plato's ontology and Aristotle's logic represents a significant departure from both original Platonism and original Aristotelianism, and it is a *logicist* synthesis between Plato's ontology and Aristotle's logic. According to the *Principia Mathematica*, mathematics is an ideal system of propositions, or, in other words, it is the formalization of mathematical propositions in the context of mathematical theories, which they are ultimately underpinned by logic. From the perspective of the *Principia Mathematica*, the substance of mathematics consists in logically perfect theories that mathematicians formulate.

In order to understand the manner in which Russell and Whitehead attempted to reduce all mathematics to logic and to the concept of a set, we have to understand the manner in which they defined the concept of a number as a set of mutually equivalent sets. For instance, Russell and Whitehead defined the number five as the set of all sets that are equivalent to the set of the elements of a natural 5-tuple. However, because, in the aforementioned example, the number five has an empirical content, whereas Russell and Whitehead want mathematics to be founded on logic alone, the manner in which numbers are ultimately defined in the context of the *Principia Mathematica* is the following: Zero (0) is the class of all classes that are equivalent to the class of all those objects which are not identical to themselves: such a class is empty, because, otherwise, the principle of identity would not hold; and, because all classes that have the same members are identical to each other, there is only one class that is empty. The uniqueness of the zero class underpins the definition of the number one. Thus, in the context of the *Principia Mathematica*, the number one (1) is the class of all classes that are equivalent to the class whose only member is zero (0). By analogy, the number two (2) is the class of all classes that are equivalent to the class whose members are zero and one, and this process can be repeated indefinitely. In this way, Russell and Whitehead defined natural numbers on the basis of the logical principle of identity, the logical concept of a class, the logical concept of a member of a class, and the logical principle of equivalence between classes.

Given that Russell pointed out the contradictory nature of Cantor's set theory through his famous Russell's Paradox, the *Principia Mathematica* uses set-theoretical concepts in an alternative way, exactly in order to keep away from logical contradictions. In particular, the *Principia Mathematica* expounds and proposes the so-called theory of types, which I explained in section 2.1.1. In simple terms, the theory of types rigorously and systematically classifies every set-theoretical concept into hierarchical levels that are called types. In this way, elements belong to the type 0, sets of elements belong to the type I, sets of sets of elements belong to the type II, etc. Having thus rigorously and systematically classified concepts into types, Russell and Whitehead proceed with the study of the relations between members of types, and they postulate that relations must relate only members of specific types to each other. In particular, "inclusion," denoted by $\subseteq$, is a logically legitimate set-theoretical concept if it relates members of type I (i.e., sets) to each other; and "belonging," denoted by $\in$, is a logically legitimate set-theoretical concept if there is a member of type 0 to the left of $\in$, and if there is a member of type I to the right of $\in$. Thus, according to Russell's and Whitehead's theory of types, $\subseteq$ refers to members of the same type, whereas $\in$ refers to members of successive types. By postulating the use of $\in$ in the aforementioned way, Russell and Whitehead preclude any statement of the form $X \in X$, and, therefore, they manage to overcome Russell's Paradox and to equip set theory with a very high level of logical rigor. However, the contradictory nature of Cantor's set theory can be overcome through an axiomatization of set theory that does not need to impose logical constraints as strict as those imposed by Russell and Whitehead.

3.5.4. Axiomatic Set Theory and Category Theory[584]

As I mentioned in section 2.1.1, before the first rigorous axiomatization of set theory by Ernst Zermelo, Cantor's set theory was based on his (intuitive) definition of the set and on the General Comprehension Principle, according to which, given any condition expressible by a formula $\varphi(x)$, it is possible to form the set of all sets x meeting that condition. Cantor endorsed the General Comprehension Principle mainly because it was in agreement with his intuition about sets, but Bertrand Russell stated the well-known Russell's Paradox, which implies that the General Comprehension Principle is not valid.

Almost simultaneously with the development of Russell's and Whitehead's theory of types, Ernst Zermelo proposed a different way to overcome the antinomies of Cantor's set theory, namely, to replace Cantor's intuitions with axioms. Zermelo regards set theory as a fundamental theory, in the sense that, according to Zermelo, set theory investigates mathematically the fundamental concepts of a number, an order, and a function. As I mentioned in section 2.1.1, in Zermelo's axiomatic system, it is assumed that there exist a "universe of objects" U, some of which are sets, and some "definite conditions and operators" in U, the basic of which are the following:

$$x = y \Leftrightarrow the\ object\ x\ is\ identical\ to\ y,$$
$$Set(x) \Leftrightarrow x\ is\ a\ set,$$
$$x \in y \Leftrightarrow Set(y) \&\ x\ belongs\ to\ y.$$

The objects that are not sets are called "atoms."

[584] Eves, *Foundations and Fundamental Concepts of Mathematics*; Grattan-Guinness, *The Search of Mathematical Roots: 1870–1940*; Kneebone, *Mathematical Logic and the Foundations of Mathematics*; Leng, *Mathematics and Reality*; Moore, *The Infinite*; Potter, *Set Theory and Its Philosophy*; Struik, *A Concise History of Mathematics*.

The axiomatization of set theory that has been proposed by Zermelo contains eight axioms.[585] More precisely, there are two versions of Zermelo's axiomatization of set theory: Zermelo's axiomatic system ZAC, consisting of axioms I–VII and AC, as well as Zermelo's axiomatic system ZDC, consisting of axioms I–VII and DC, where:

i. Axiom of Extensionality: If A and B are two arbitrar
ii. y sets, then
$A = B \Leftrightarrow (\forall x)[x \in A \Leftrightarrow x \in B]$.
In other words, any two sets that contain exactly the same members are the same set, and, thus, a set is defined by its members.
iii. *Axiom of Empty Set:* There exists a conventional set that contains no elements. *Remark:* The Axiom of Extensionality implies that the empty set, $\emptyset$, is unique.
iv. *Axiom of Pairing:* For any objects x and y, there exists a set $A = \{x, y\}$ whose elements are exactly x and y, so that
$z \in A \Leftrightarrow [z = x \vee z = y]$.
Remark: The Axiom of Extensionality implies that there exists exactly one set A such that A contains x and y and, for any z, $z \in A \Leftrightarrow [z = x \vee z = y]$. If $x = y$, then $\{x, x\} = \{x\}$ is said to be the "singleton" of the object x.
v. *Axiom of Separation:* For every set A and for every definite condition P of one variable x, there exists a set B such that $x \in B \Leftrightarrow [x \in A \wedge P(x)]$.
Remark: The Axiom of Extensionality implies that such a set B is unique. The Axiom of Separation is a restriction of the General Comprehension Principle, and it is helpful in order to define important concepts, such as the intersection and the difference of two sets: $A \cap B = \{x \in A | x \in B\}$ and $A - B = \{x \in A | x \notin B\}$. In order to circumvent Russell's paradox, the Hungarian-American mathematician and computer scientist John von Neumann (1903–57) distinguished between two kinds of collections: classes and sets.[586] For any set A, the set $R = \{x \in A | x \notin x\}$ is not a member of A, and, therefore, the collection of all sets is not a set, namely, there exists no set U such that $x \in U \Leftrightarrow Set(x)$, because R is a set by the Axiom of Separation, and, if $R \in A$, then $R \in R \Leftrightarrow R \notin R$, which is a contradiction. According to von Neumann, *a class u (i.e., any collection of any definite, distinguishable objects of perception or thought conceived as a whole) is a set if there is a class v that entirely contains u.*
vi. *Axiom of Power Set:* For any set A, the power set $\wp(A)$ of A is also a set. *Remark:* The Axiom of Extensionality implies that the power set $\wp(A)$ of any set A is unique.
vii. *Axiom of Union:* For every object $\mathbb{A}$, there exists a set B such that
$x \in B \Leftrightarrow (\exists X \in \mathbb{A})[x \in X]$,
namely, the members of B are the members of the members of $\mathbb{A}$. *Remark:* By the Axiom of Extensionality, such a set B is unique and is called the "union" of $\mathbb{A}$, denoted by $\cup \mathbb{A} = \{x | (\exists X \in \mathbb{A})[x \in X]\}$.
viii. *Axiom of Infinity:* There exists a set A such that
$\emptyset \in A$ and $(\forall x)[x \in A \Rightarrow \{x\} \in A]$.

[585] Zermelo, "Untersuchungen über die Grundlagen der Mengenlehre I."
[586] Neumann, "Eine Axiomatisierung der Mengenlehre."

Remark: Notice that $x \neq \{x\}$.

ix. *Axiom of Choice (AC):* For any binary relation $T \subseteq A \times B$, where A and B are arbitrary sets,
$(\forall x)(\exists y \in B)[xTy] \Rightarrow (\exists f: A \to B)(\forall x \in A)[xTf(x)]$.
Remark: By a "partial function" $f: A \to B$ from a set A to a set B, we mean any function whose domain is a subset of A and its range is B. Then an equivalent way to state AC is the following: For any set A, there exists a partial function $f: \wp(A) \to A$ such that
$\emptyset \neq X \subseteq A \Rightarrow [X \in Domain(f) \wedge f(X) \in X]$;
such a partial function is called a "choice function." Another equivalent way to state AC is the following: If $\mathbb{E} = \{A_i, i \in I\}$ is a non-empty family of non-empty pairwise disjoint sets, then there exists a set A consisting of exactly one element from each A_i. A well-known example that clarifies the significance of Zermelo's Axiom of Choice is due to Bertrand Russell. Let A be a set containing pairs of shoes, $B = \cup A$, and $xTy \Rightarrow y \in x, \forall x, y | x \in A \; and \; y \in B$. Then the function
$f(x) = the \; right \; shoe \; of \; x$
chooses exactly one shoe from each pair $x \in A$, namely:
$(\forall x \in A)[xTf(x)]$.
Thus, according to Russell, for any (even infinite) collection of pairs of shoes, one can pick out the left shoe from each pair to obtain an appropriate selection; in this case, "pick out the left shoe" is a "choice function." But, if A is a set of pairs of socks, so that each pair of socks contains two identical objects, then no function $f: A \to \cup A$ can be defined in such a way that it chooses one sock out of each pair. Thus, appeal to Zermelo's Axiom of Choice is necessary in order to overcome the problems that are caused by the existence of subsets that one would like to consider but that cannot be described by any property, as it happens in the aforementioned example.
Countable Principle of Choice (CC): Given a set A and a binary relation $T \subseteq \mathbb{N} \times A$, where $\mathbb{N}$ is the set of all natural numbers,
$(\forall n \in \mathbb{N})(\exists t \in A)[nTt] \Rightarrow (\exists f: \mathbb{N} \to A)(\forall n \in \mathbb{N})[nTf(n)]$.

x. *Axiom of Dependent Choices (DC):* Given an arbitrary set A and a binary relation $T \subseteq A \times A$,

xi. $\big[a \in A \wedge (\forall x \in A)(\exists y \in A)[xTy]\big]$
$\Rightarrow (\exists f: \mathbb{N} \to \mathrm{A})\big[f(0) = a \wedge (\forall n \in \mathbb{N})[f(n)Tf(n+1)]\big]$.
In other words, for any non-empty set A and for any binary relation T defined on A in the aforementioned way, there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ in A such that $x_n T x_{n+1} \; for \; all \; n \epsilon \mathbb{N}$. This axiom was introduced by the Swiss mathematician Paul Bernays in 1942,[587] but, even without DC, for any $n \epsilon \mathbb{N}$, we can use mathematical induction in order to form the first n terms of the aforementioned sequence. However, DC asserts that in this way we can form a countably infinite sequence.[588]
Theorem[589]*:* (i) $AC \Rightarrow DC$; (ii) $DC \Rightarrow CC$.

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[587] Bernays, "A System of Axiomatic Set Theory III."
[588] See: Illari, Russo, and Williamson, eds., *Causality in the Sciences*.
[589] Bernays, "A System of Axiomatic Set Theory III."

Proof:
(i) Given the choice function

$$F: \wp(A) - \{\emptyset\} \to A$$

and the hypothesis of DC, the required function in order to establish the conclusion of DC is $f: \mathbb{N} \to A$ defined by
$f(n) = a$,
$f(n+1) = F(\{y \in A | f(n)Ty\})$.
(ii) Given the hypothesis of CC, let $B = \mathbb{N} \times A$ and $b = (0, a)$ with $a \in A$ and $0Ta$. If we define the relation
$(n, x)T^*(m, y) \Leftrightarrow (m = n + 1 \wedge mTy)$
on B, then the function $f: \mathbb{N} \to \mathbb{N} \times A$, which follows from the conclusion of DC for b and T^*, gives $f(n) = \big(g(n), h(n)\big)$ with $g(0) = 0$ and $h(0) = a$ for adequate functions g and h, and, for any n, $g(n+1) = g(n) + 1$ and $g(n+1)Th(n+1)$. Therefore, for any n, $g(n) = n$ and $nTh(n)$.■

xii. Axiom of Replacement: As I have already mentioned, John von Neumann has made a rigorous distinction between the terms "class" and "set." Thus, the German-Israeli mathematician Abraham Fraenkel has proposed the "Axiom of Replacement": If A is a set, and if $f: A \to \mathbb{A}$ is a definite operator in one variable, then the image $f(A) = \{f(x) | x \in A\}$ of A by f is a set. The axiomatic system that consists of ZDC plus the Axiom of Replacement is denoted by $ZFDC$, while the axiomatic system $ZFAC$ consists of ZAC plus the Axiom of Replacement.

However, during the complicated history of set theory, several other concepts and rules have been developed.[590] For instance, another restrictive hypothesis that is very often used in set theory is the

Principle of Purity: There exist no atoms (i.e., every object with which we are concerned is a set).

An object x is said to be "ill-founded" if there exists a function f with domain $\mathbb{N}$ such that $x = f(0) \ni f(1) \ni f(2) \ni \cdots$; and, if an object is not ill-founded, then it is "well-founded." If $A \in A$, then $A \ni A \ni A \ni A \ni \cdots$, and, therefore, A is ill-founded. Hence, a well-founded set is not a member of itself. In other words, a binary relation T is said to be "well-founded" on a class X if every non-empty subset $A \subseteq X$ has a least element with regard to T, that is, an element k not related to aTk for any $a \in A$; symbolically, T is well-founded if

$$(\forall A \subseteq X)[A \neq \emptyset \Rightarrow (\exists k \in A)(\forall a \in A)\neg(aTk)].$$

Equivalently, assuming the Axiom of Dependent Choices, a relation is well-founded if it contains no countable infinite descending chains, that is, if there is no infinite sequence $x_0, x_1, x_2, \ldots$ of elements of X such that $x_{n+1}Tx_n$ for any $n \in \mathbb{N}$.

Examples of well-founded relations:

[590] See: Ferreirós, *Labyrinth of Thought*; Kleene, *Introduction to Meta-Mathematics*.

a) The natural numbers $\{1,2,3,\dots\}$ with the order defined by $a < b$ if and only if a divides b and $a \neq b$.
b) The set of all finite strings over a fixed alphabet with the order defined by $u < v$ if and only if u is a proper substring of v.
c) The set $\mathbb{N} \times \mathbb{N}$ of pairs of natural numbers ordered by $(n_1, n_2) < (m_1, m_2)$ if and only if $n_1 < m_1$ and $n_2 < m_2$.

Examples of ill-founded relations:

a) The negative integers $\{-1,-2,-3,\dots\}$ with the usual order, because any unbounded subset has no least element.
b) The set of strings over a finite alphabet with more than one element, under the usual order, because the sequence $\mathrm{B} > \mathrm{AB} > \mathrm{AAB} > \mathrm{AAAB} > \cdots$ is an infinite descending case.
c) The rational numbers under the standard ordering, because, for instance, the set of positive rationals lacks a minimum.

The Russian-Swiss mathematician Dmitry Mirimanoff[591] and the American logician Dana Scott[592] have equipped the axiomatization of set theory with the

Principle of Foundation: Every set is well-founded.

Remark: The Principle of Foundation is valid if and only if, for every non-empty set A, there exists some $u \in A$ such that $u \cap A = \emptyset$, that is, $\forall x$, $x \in A \Rightarrow \neg(x \in u)$. In other words, each non-empty set must contain "atoms" u, which form its "foundation," so that the following two conditions hold: (i) No non-empty set can be a member of itself. (ii) If A and B are distinct non-empty sets, then it is not possible that both $A \in B$ and $B \in A$ are true.

The most widely accepted system of axioms for sets is denoted by ZDC and consists of $ZFDC$, AC, the Principle of Purity, and the Principle of Foundation.

Intimately related to the development of set theory is the development of category theory, which is concerned with the very structure of mathematics. The development of set theory by Cantor, Zermelo, Gödel, von Neumann, and others was considered to be the solution to the problem of finding an appropriate infrastructure for the elaboration of structures needed in any branch of mathematics. The fundamental concepts of "structure" and "operation on structures" were not regarded as "primitive," but they were "reduced" to the more fundamental concepts of set and membership. However, from the 1930s onward, especially as a result of the development of abstract algebra, this attitude has become less firm, because it has been observed that certain mathematical fields are characterized by a level of universality and necessity that is not directly dependent on their set-theoretical background. This is the case, for instance, in abstract algebra, where we study such basic algebraic structures as groups (see section 2.1.4).

[591] Mirimanoff, "Les Antinomies de Russell et de Burali-Forti et le Problème Fondamental de la Théorie des Ensembles."
[592] Scott, "Axiomatizing Set Theory."

According to Samuel Eilenberg and Saunders Mac Lane,[593] a "category" E consists of two classes: the members of the first class are called "objects" ("structures"), and they are denoted by $X, Y, \ldots$, while the members of the second class are called "arrows" ("morphisms"), and they are denoted by $f, g, \ldots$ In other words, from the perspective of category theory, functions are arrows between objects, and types are objects whose properties are defined by arrows. In this way, category theory tries to speak about different kinds of sets without speaking about their structure or about the nature of their elements, but by speaking only about interactions between different kinds of sets, namely, about "arrows." In a sense, the underlying reasoning of category theory is closely related to the underlying reasoning of the proverb "a man is known by the company he keeps." Following the logic of functional programming (namely, the method of creating software by applying and composing pure functions) and trying to cope with the constraints and the needs of functional programming (namely, avoiding shared state (i.e., any variable, object, or memory space existing in a shared scope), mutable data, and side-effects), category theory describes properties of objects (i.e., of kinds of sets) merely in terms of arrows that are incoming into these objects and arrows that are outgoing from these objects (in the context of functional programming, a function can be thought of as a program that takes input in the form of an argument and produces some output in the form of a result, and an object is an argument that is passed to a function which it operates on, so that the function is considered the operator, and the object is considered the operand).

Each arrow f is assigned an object X, the "domain" of f, and an object Y, the "codomain" of f, indicated by writing $f: X \to Y$. If $g: Y \to Z$ is an arrow with domain Y, the codomain of f, then there is an arrow $g \circ f: X \to Z$ called the "composition" of f and g. Any arrows $f: X \to Y$, $g: Y \to Z$, and $h: Z \to W$ are assumed to satisfy the following axioms:

identity: for every object X, there exists a morphism $id_X X \to X$ called the "identity arrow" (or the "identity morphism") on X such that, for every arrow $f: X \to Y$, we have $id_Y \circ f = f = f \circ id_X$; and
associativity: $h \circ (g \circ f) = (h \circ g) \circ f$.

Remarks: In simple terms, a "category" is an assemblage (not necessarily a "set" in the strict sense) of objects, and, therefore, it enables us to think at a higher level of abstraction than that of traditional set theory. In fact, according to Mc Lane himself,

> Category theory starts with the observation that many properties of mathematical systems can be unified and simplified by a presentation with diagrams of arrows . . . Many properties of mathematical constructions may be represented by universal properties of diagrams.[594]

Moreover, notice that the preceding formal definition of a category is formulated in a first-order language leading to "first-order category theory."

Functors are structure-preserving mappings between categories. Eilenberg and Mac Lane[595] considered a ("covariant") "functor" F from a category C to a category D to be

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[593] Eilenberg and Mac Lane, "General Theory of Natural Equivalences."
[594] Mac Lane, *Categories for the Working Mathematician*, p.1.
[595] Eilenberg and Mac Lane, "General Theory of Natural Equivalences."

composed of a pair of functions (both denoted by F), one from the class of objects of C to that of D, and another from the class of arrows of C to that of D, sot that, for each object x in C, there exists an object $F(x)$, and, for each arrow $f: x \to y$ in C, there exists an arrow $F(f): F(x) \to F(y)$, such that the following properties hold:

$F(id_x) = id_{F(x)}$ for every object x in C, and
$F(g \circ f) = F(g) \circ F(f)$ for all arrows $f: x \to y$ and $g: y \to z$.

A "contravariant functor" $F: C \to D$ is like a covariant functor except that it reverses all arrows, that is, it is a covariant functor from the "opposite category" C^{op} to D. According to Eilenberg and Mac Lane, the usefulness of the concept of a category derives from "the precept that every functor should have a definite class as domain and a definite class as range," so that "the categories are provided as the domains and ranges of functors."[596] Thus, category theory can be considered as the study of structures by means of functors. An important observation of category theory is that large parts of mathematics can be "internalized" in any category (and, thus, defined in terms of category theory) with sufficient structure.

A category C is said to be "closed" if, for any pair a, b of objects of C, the collection of arrows (morphisms) from a to b can be regarded as forming itself an object of C. A category is said to be "Cartesian closed" if any arrow (morphism) defined on a product of two objects can be naturally identified with an arrow defined on one of the factors. In a more rigorous way, we say that a category C is "Cartesian closed" if it satisfies the following axioms:

it has a terminal object, in the sense that there exists an element T such that, for every object X in C, there exists precisely one morphism $X \to T$;
any two objects X and Y of C have a product $X \times Y$ in C;
any two objects Y and Z of C have an exponential Z^Y in C.

These categories are very important in mathematical logic and computer programming, because their "internal language" is the simply typed "λ-calculus" ("lambda-calculus"), which consists of "λ-abstractions" ("lambda-abstractions") and their applications. A λ-abstraction is the process of interpreting a formula for a function (or an operation) as defining an actual function, specifically, a function $f: A \to B$ of sets A and B that respects the binary operation (for every a, b in A, $f(a * b) = f(a) * f(b)$). Let p be an expression of some sort involving a free variable x. Then the λ-abstraction, denoted by

$\lambda x. p,$

represents the function that takes one input (argument), and its output consists in substituting that input for x in p. For instance, the λ-abstraction $(\lambda x. (x + 1))$ is the function that adds one to its input. Thus, by the term "application," we mean the manner in which we "undo" abstraction by applying a function to an input. The application of the function f to the input p is denoted by fp or $f(p)$. In general, "application" is considered to associate to the left, in the

[596] Ibid, p. 247.

sense that, for instance, rst denotes the application of r to s followed by the application of the result (assuming that it is again a function) to t. In this way, functions of multiple variables can be represented in terms of functions of one variable. This method was originally developed by the Russian logician and mathematician Moses Schönfinkel (1889–1942), and it was studied further by the American mathematician and logician Haskell Curry (1900–82).[597] It is worth noticing that, in computer science, λ-calculus is inextricably linked to "functional programming": programming in "assembly language," which underpins the so-called "Turing machine," is the process of building software by telling the computer what to do in a precise and imperative way (as follows: take a thing x from memory, put x into the register, use it as an address, and then jump, etc.). Functional programming, developed by the American mathematician and logician Alonzo Church (1903–95) on the basis of λ-calculus, is declarative rather than imperative, and it constructs programs by applying and composing functions, that is, by thinking in terms of mathematical transformations (thus, functional programming expresses the logic of a computation without describing its control flow).[598]

It is evident that category theory presupposes that we have some class or set theory. However, in this case, Zermelo–Fraenkel set theory is inadequate, because, for instance, it does not allow a class of all sets, which is necessary in order to define the category "Set," whose objects are all sets, and arrows are all (set) functions. Moreover, Gödel–Bernays–von Neumann set theory does not allow any classes having proper classes as elements, which are necessary in order to define a category of categories. By Gödel–Bernays–von Neumann set theory, we mean a conservative extension of Zermelo–Fraenkel set theory, in the sense that Gödel–Bernays–von Neumann set theory introduces the concept of a "class," which is a collection of sets defined by a formula whose quantities range over sets.[599]

A solution to the aforementioned problem is given by the addition of Grothendieck Universes to Zermelo–Fraenkel set theory. By a Grothendieck Universe, we mean a set U that satisfies the following axioms:

if x is an element of U, and, if y is an element of x, then y is also an element of U;
if x and y are both elements of U, then $\{x, y\}$ is an element of U;
if x is an element of U, then the power set $\wp(U)$ of U, is also an element of U; and
if $\{x_k\}_{k\in I}$ is a family of elements of U, and, if I is an element of U, then the union $\cup_{k\in I} x_k$ is an element of U.

However, by adding Grothendieck Universes (specifically, sets closed under the usual set-theoretical operations) to Zermelo–Fraenkel set theory, we encounter the complication of having different categories of all sets for each universe. A more refined method, proposed by Horst Herrlich and George E. Strecker, involves only one such universe, which is added to the Gödel–Bernays–von Neumann set theory.[600] But, even in this case, we encounter a new complication, namely, the fact that we have three basic kinds of collections: sets, classes, and conglomerates. Hence, the restrictions that were imposed on the formation of classes and on

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[597] See: Bimbó, *Combinatory Logic*.
[598] See: Hindley and Seldin, *Introduction to Combinators and λ-Calculus*.
[599] See: Mendelson, *Introduction to Mathematical Logic*, Chapter 4.
[600] Herrlich and Strecker, *Category Theory*.

the operations to which these classes are subject in order to avoid set-theoretical antinomies have a considerable impact on category theory.

The American philosopher and mathematician Solomon Feferman approached category theory from the perspective of "combinatory logic" (or "calculus of combinators"), which is synonymous with "λ-calculus," which, as I have already mentioned, is an internal language for Cartesian closed categories, and it signifies an attempt to eliminate the need for quantified variables in mathematical logic. In this way, Feferman managedto form a very large class of objects including all functions and relations as well as his morphisms.[601] However, understandably, Feferman's system uses a very large number of new primitive concepts. A more refined system using combinatory logic was proposed by M. W. Bunder in the 1980s.[602]

According to Bunder, pure combinatory logic can be thought of as a concrete category where the objects and the functions corresponding to the morphisms form the same class. In Bunder's notation, the value of (the function) X applied to Y is denoted by (XY), so that morphisms are triplets of the form $(Y, X, (XY))$. Then (X, Y, Z) is said to be a morphism of the pure combinatory logic category if $Z = (YX)$. The objects of this category (i.e., combinatory logic) include three basic combinators K, S, and I defined as follows:

i. $KXY \equiv X$, or, equivalently, in simply typed λ-calculus,
$K: A \rightarrow (B \rightarrow A)$;
ii. $SXYZ \equiv (XZ)(YZ)$, or, equivalently, in simply typed λ-calculus,
$S: \big(A \rightarrow (B \rightarrow C)\big) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$;
iii. $Ix \equiv x$, or, equivalently, in simply typed λ-calculus,
$I: A \rightarrow A$.

Moreover, other combinators can be defined, such as B and C, where:

$BXYZ \equiv X(YZ)$, or, equivalently, in simply typed λ-calculus,
$B: (B \rightarrow C) \rightarrow (A \rightarrow B) \rightarrow (A \rightarrow C)$,

which has the same type as the composition operation in a closed category;

$CXYZ \equiv (XZ)Y$, or, equivalently, in simply typed λ-calculus,
$C: \big(A \rightarrow (B \rightarrow C)\big) \rightarrow (B \rightarrow (A \rightarrow C))$,

which has the type of a symmetry for a closed category.

Notice that, according to the aforementioned notation, the combinator I can be defined as CKK. By the term "linear combinatory logic," we mean the version of combinatory logic that includes the aforementioned combinators B, C, and I, and, for this reason, it is usually referred to as the "BCI logic." By the term "affine combinatory logic," we mean the version of combinatory logic that includes the aforementioned combinators B, C, K, and I, and, for this reason, it is usually referred to as the "BCKI logic." "Traditional combinatory logic" could be called "KSI logic."

[601] Feferman, "Categorical Foundations and Foundations of Category Theory."
[602] Bunder, "Category Theory Based on Combinatory Logic."

However, if combinatory logic is extended with a second primitive predicate⊢, then it cannot be represented as a category (except in the formulation of its terms), because predicate calculus cannot be formulated as a category, since it is used in order to express category theory.

3.5.5. Intuitionism[603]

It should be clear by now that Russell's discovery of contradictions in Cantor's set theory imperiled the entire discipline of mathematical analysis, since the latter was founded on set-theoretical concepts. Russell and Whitehead eliminated the contradictions and the ambiguities of Cantor's set theory through the theory of types, and Zermelo eliminated the contradictions and the ambiguities of Cantor's set theory through the axiomatization of set theory. On the other hand, the philosophico-mathematical "school" of intuitionism has followed a substantially different approach to the foundational problems of mathematics. In 1907, the Dutch mathematician Luitzen Egbertus Jan Brouwer completed his dissertation *On the Foundations of Mathematics* (under the supervision of D. J. Korteweg at the University of Amsterdam), initiating a systematic intuitionist approach to mathematics, and articulating an acute criticism of modern mathematical analysis and especially of Russell's viewpoints.

According to Brouwer and his advocates, the foundational problems of modern mathematical analysis are deeper than the contradictions of Cantor's set theory. In contrast to Russell, Brouwer argued that modern mathematical analysis was formulated on the basis of an insufficient logic, and, therefore, its foundation was not as rigorous as it should be. Thus, Brouwer discarded Russell's and Whitehead's *Principia Mathematica* and especially Russell's attempt to underpin Aristotelian logic with Platonic metaphysics. In general, intuitionism maintains that Aristotelian logic, even after its modification by Russell and Whitehead, cannot be the ultimate foundation of mathematics. In particular, according to intuitionism, classical logic was an outgrowth of abstraction within a specific system of knowledge of a particular historical era, namely, within the context of ancient Greek mathematics. Furthermore, intuitionism maintains that a major attribute of ancient Greek mathematics is its finite geometric character, in the sense that ancient Greek mathematicians did not interpret geometric figures as point sets, they kept geometry and arithmetic separate from each other, and, in general, they treated the realm of continuity and the realm of discreteness as two equally primal areas and aspects of mathematics, whereas, with the development of infinitesimal calculus, a major attribute of mathematics from the seventeenth century onward is the endorsement and the systematic use of the concept of infinity in the form of the continuum (namely, the set $\mathbb{R}$ of all real numbers) and in the form of cardinal arithmetic. Therefore, intuitionists emphatically argue that Aristotelian logic, as an outgrowth of abstraction within the context of ancient Greek mathematics, is unsuitable for the rigorous manipulation of such concepts as infinity, the continuum, a real number (involving infinite decimals), and an infinite set. As a consequence of the aforementioned arguments,

[603]Eves, *Foundations and Fundamental Concepts of Mathematics*; Grattan-Guinness, *The Search of Mathematical Roots: 1870–1940*; Kneebone, *Mathematical Logic and the Foundations of Mathematics*; Leng, *Mathematics and Reality*; Moore, *The Infinite*; Potter, *Set Theory and Its Philosophy*; Struik, *A Concise History of Mathematics*.

intuitionists refuse to accept set theory as a suitable foundation for mathematical analysis, since set theory is characterized by an extensive use of the concept of infinity. Intuitionists perceive set theory as part of combinatorics (an area of mathematics that is primarily concerned with counting, including graph theory, coding, cryptography, and probability), but they believe that, if set theory is used as a foundation of mathematics, then set theory, with its extensive use of the concept of infinity, poses serious epistemological risks.

In view of the foregoing, the first major principle of intuitionism is the following: classical logic cannot rigorously manipulate the concept of infinity in the context of modern mathematical analysis, and, since set theory is characterized by an extensive use of the concept of infinity, set theory is not suitable for the foundation of mathematical analysis. Thus, in 1908, at the Fourth International Conference of Mathematicians in Rome, Brouwer argued against the general validity of the principle of the excluded middle (according to which, given any proposition P, either P or its negation, symbolically, $\neg P$, holds; and, in Aristotle's terms, "there cannot be an intermediate between contradictions"[604]). In particular, according to Brouwer, the principle of the excluded middle can be imposed and is valid in the general case of finite sets, because, in this case, we can determine whether a proposition or its negation holds by applying an algorithm, namely, a finite sequence of well-defined instructions. In other words, according to Brouwer, the validity of the principle of the excluded middle is *not a priori*, but it is proven by the application of an algorithm to the solution of specific problems. However, Brouwer maintains that the application of the principle of the excluded middle should not be allowed in the general case of infinite sets, unless one proves its validity.

The second major principle of intuitionism is the thesis that, in mathematics, existence is equivalent to constructability, that is, something exists in mathematics if and only if it can be constructed. The two major principles of intuitionism are strongly logically interconnected, since, in the context of a mathematical system that is determined by finite processes (and, hence, it is not based on the concepts of infinity and the continuum), "existence" can be easily identified with "constructability." In order to understand the intuitionists' principle of constructability, we should consider the following issue: In 1904, Zermelo proved the so-called "Well-Ordering Theorem," which is equivalent to the Axiom of Choice, and states that an arbitrary set can be equipped with an ordering relation such that every non-empty subset of the given set has a least element (i.e., an element of the given set that is smaller than any other element of the given set) under the given ordering. Let us consider the interval $(0,1)$. Obviously, the standard order of numbers cannot give us the least element of $(0,1)$. In general, no ordering relation can give us the least element of $(0,1)$, but Zermelo's Well-Ordering Theorem assures us that the interval $(0,1)$ has a least element. Thus, great mathematicians, such as Henri Poincaré, Émile Borel, and Hermann Weyl, started inquiring into and debating about the significance of a theorem that speaks about the existence of something that is beyond any specification. For instance, the Axiom of Choice tells us that something exists without telling us what it is or how to describe it. On the other hand, intuitionism maintains that it is not epistemologically legitimate to argue that a mathematical entity exists if we cannot find it, and if it is not present before us. Intuitionists maintain that such mathematical propositions as the Axiom of Choice and the famous Bolzano–Weierstrass Theorem (i.e., "every infinite and bounded subset of $\mathbb{R}$ has at least one accumulation point,"

[604] Aristotle, *Metaphysics*, 1011b23–24.

as proven in section 2.3.4) are insignificant metaphysical perceptions of the existence of a mathematical entity, since such propositions declare/demonstrate the existence of mathematical objects without telling us how to find them.

From the perspective of intuitionism, mathematics is primarily a human activity, originating from experience and the free, creative operation of consciousness, and, therefore, intuitionism maintains that the existence of a mathematical entity is equivalent with its constructability, that is, with the finding of a method that determines the given mathematical entity. Intuitionists accept the value of symbolic logic and the axiomatic method, but they refuse to accept symbolic logic as an ultimate foundation of mathematics, and they counter-argue that symbolic logic has historically originated from mathematics as a product of high-level abstraction. According to the philosophico-mathematical "school" of intuitionism, there is only one reliable, *a priori* foundation of mathematics, namely, the concept of a natural number. In particular, intuitionism maintains that the concept of a natural number is self-evident, in the sense that it is innate in consciousness, and there is no need to be reduced to anything else. Brouwer argues that the very fact that the human being can distinguish two different objects implies that the number two and, hence, every other natural number are innate in the human intellect.

The French semi-intuitionists Poincaré and Borel maintain that logic plays an indispensable role in a mathematical argument, but logic on its own can yield only tautologies, and a "mathematical intuition" is necessary. However, Brower, the acknowledged founder of mathematical intuitionism, was the first mathematician who specified intuition, namely, he described exactly what is intuited and how this intuition underpins mathematics. According to Brouwer, the primal intuition ("ur-intuition") of mathematics is the system of natural numbers, and, from the intuitive counting "one, two, three . . .," what Brower calls the "main theorem" of arithmetic (i.e., the statement that the number of elements of a finite set is independent of the order in which they are counted) can be deduced. In contrast to Dedekind, Brouwer argues that complete induction is neither a theorem that requires a proof nor an axiom, but a natural mathematical act. Thus, when Brouwer argues that the primal intuition ("ur-intuition") of mathematics is the system of natural numbers, he does not refer to the natural numbers as a "set," since, intuitionism discards the concept of a set in its general form. From the perspective of intuitionism, since natural numbers increase indefinitely, they cannot be regarded as a given, "completed" totality (and, therefore, they cannot be regarded as a "set"), but they represent a possibility (similar to Aristotle's concept of a potential infinity), in the sense that, irrespective of how many steps we have taken, we can always find a new natural number. According to Brouwer, the fundamental property of natural numbers that actualizes their potential infinity is the principle of complete induction, which is an essentially constructive property, since we can always find the successor of any natural number by adding one.

In order to understand the controversy between logicism and intuitionism, one has to delve into their fundamental assertions. Logicism asserts that mathematical entities are completely defined in the language of symbolic logic, and it equips symbolic logic with ontological content and ontological underpinnings. Intuitionism asserts that mathematical entities are mental constructs, and it equips what Brouwer has called the primal intuition ("ur-intuition") with ontological content and ontological underpinnings. Both logicism and intuitionism recognize the reality of the world, but they do so in different ways: logicism recognizes the reality of the world by identifying it with symbolic logic and, thus, arguing that the reality of the world is ω-complete, and that truth in the reality of the world implies provability in accordance with symbolic logic (specifically, the *Principia Mathematica*); whereas intuitionism, denying that the reality of the world is ω-complete, recognizes the reality of the world by identifying it with intuition and, thus, arguing that truth in the reality of the world implies provability in accordance with the "ur-intuition." Consequently, for logicism, mathematical truth is based on the logical structure of mathematics itself, whereas, for intuitionism, mathematical truth is based on the logical structure of the mathematician's mind.

Brouwer's awareness of the significance of the knower in the development of mathematics led him to such excesses as the belief that only those statements are true which are known, specifically, constructively proven, today, and intuitionism tends to make mathematics extremely complex and difficult to use, since the basis of the constructivist methodology of intuitionism is very restrictive, namely, the system of natural numbers alone. Moreover, there are logicist excesses, too, such as the extreme claim that set theory is not part of logic. However, both logicism and intuitionism can be reformulated in more moderate ways in the context of the dialectic of rational dynamicity, which I expounded in section 1.3. Highlighting and analyzing the structural continuity between the reality of the world and the reality of consciousness, the dialectic of rational dynamicity overcomes the antithesis between (moderate varieties of) logicism and intuitionism: the system of natural numbers (namely, the process-system of counting) can be postulated as the primal mathematical intuition according to Brower's intuitionism, but, since thinking is an ontological attribute of the human being (as I explained in section 1.1), logic (namely, the science of correct reasoning), at a fundamental level, namely, as a tendency to organize and evaluate thinking in a systematic and methodical way, can be postulated as part of the primal philosophico-mathematical intuition, thus giving rise to a synthesis between Brouwer's intuitionism and Russell's and Whitehead's type theory. In this way, the dialectic of rational dynamicity epistemologically legitimizes infinity, the continuum, and proofs that do not comply with intuitionism's constructivist requirements, and, simultaneously, the dialectic of rational dynamicity epistemologically legitimizes intuitionists' attempt and ambition to specify the location of mathematical entities in the mathematical universe and to find algorithms through which one can know not only the existence of a mathematical entity *in abstracto* but also its existence *in concreto*, whenever this is possible.

The manner in which radical intuitionism developed in the second half of the twentieth century and in the beginning of the twenty first century—namely, by discarding infinity, every infinite process, and the continuum, and by equating existence with constructability (and, hence, by equating mathematical modelling with algorithmization)—suits and reflects the *modus operandi* and the structure of artificial intelligence more than the *modus operandi* and the structure of human intelligence. By the term "artificial intelligence," we refer to artificial neural networks (ANNs), namely, computer algorithms that imitate particular functions of the human brain.[605] An artificial neural network (ANN) consists of virtual neurons ("nodes," or "computational units") that are arranged in interconnected layers, and

[605] See: Rojas, *Neural Networks*. In mathematical terms, a neural network is an algorithm that computes, from an input x, an output y, that is, such an algorithm defines a function $y = f_s(x)$, and the computer program that calculates this function is made up of a sequence of several stages, each of which performs elementary calculations.

they transmit information, thereby performing calculations, much like the neurons of the human brain. In the context of an ANN, virtual neurons are actually numbers in the corresponding code, typically having values between 0 and 1. In other words, virtual neurons are encoded in bites and strings on hard disk drives or silicon chips. Moreover, in the context of an ANN, the connections between virtual neurons are also associated with numbers, and these numbers are called “weights.” These weights determine the importance of one lawyer’s information to the next layer. Thus, the free parameters of an ANN are the values of the virtual neurons and the weights of the connections between virtual neural layers. The use of ANNs consists in “training” an ANN in order to find those values of the free parameters that minimize a certain function that is called the “loss function.” In other words, ANNs solve optimization problems, in the context of which “backpropagation” takes place. By the term “backpropagation,” we mean that, if the output of an ANN is not scientifically satisfactory enough (in terms of the corresponding “loss function”), then we go back and change the free parameters, and, thus, the ANN is said to “learn” through trial and error. In other words, as training in the context of ANNs begins with random free parameters, and the goal is to adjust them so that the output error will be minimal, the purpose of the backpropagation algorithm is to reduce this error until the ANN “learns” the training data. In ANNs, each layer of virtual neurons is usually fully connected to its previous layer and to its next layer, but the human brain does not have lawyers, and it relies on an *a priori* structure (so that not all regions of the human brain are fully interconnected, and each region of the human brain is specialized for certain purposes). In ANNs, the layers of virtual neurons are precisely ordered according to the successor relation, but the human brain does a lot of parallel processing, and it is not constrained by any particular ordering relation. Furthermore, it is important to mention that an ANN starts from the beginning, without using anything that already exists, each time, whereas the human brain has an important, complex structure wired into its connectivity, and it utilizes models that have been proven useful during evolution. Finally, as I explained in Chapter 1, the human brain is the most complex entity in the entire (known) universe, and, in contrast to ANNs, its learning mechanism is not exhausted in the method of trial and error. As a result of the aforementioned attributes of the human brain, the latter is capable not only of identifying and classifying patterns but also of creating models of the world. By contrast, ANNs are unable to create models of the world, and they can only “learn” to identify and classify patterns, and their pattern recognition operations can fail with only small changes.

From the perspective and in the context of artificial intelligence, the ultimate epistemological values and goals are precision and the formulation and application of consistent finite processes (specifically, algorithmization). For this reason, several proponents of radical intuitionism defend the constructivist methodology of intuitionism by invoking the requirements of artificial intelligence and the need for more efficient interplays between humans and computing machines. On the other hand, as I have already argued, from the perspective and in the context of human intelligence, there are other epistemological values and goals, besides precision and the formulation and application of consistent finite processes, such as the creation of models of the world itself, the understanding of the world as a “whole,” and the elucidation of the objects of consciousness (i.e., the clarification of what one means by claiming that one understands or explains something). For this reason, human intelligence is a proper superset of artificial intelligence, and, in fact, human intelligence transcends artificial intelligence; after all, artificial intelligence has been created by human intelligence in order to imitate particular aspects and functions of human intelligence and serve particular human goals.

It is important to recall the ancient Greek term "epopteia," which means having seen an object in a comprehensive way. "Epopteia" can creatively combine logic, deduction, and the axiomatic method, which underpins Euclid's *Elements*, with the concepts of infinity, universality, and transcendence. This combination is intimately related to the dialectic of rational dynamicity.

3.5.6. Formalism[606]

As I explained in section 2.2.1, Peano attempted to equip mathematics with more rigor by forming symbolic logic and formalizing proofs (especially in the area of mathematical analysis), thus reinforcing the axiomatic method, whose origin can be traced to Euclid's research work. The axiomatic method was further reinforced by the great German mathematician David Hilbert (1862–1943). Hilbert's book entitled *Grundlagen der Geometrie* (*Foundations of Geometry*), published in 1899, was the first book after Euclid's *Elements* that provided a logically rigorous formulation of geometry. Hilbert's axiomatic model of Euclidean geometry consists of the following primitive concepts and axioms:

Primitive concepts of Hilbert's axiomatic model of Euclidean geometry:

- *Three undefined primitive terms:*
  Point;
  Line;
  Plane.
- *Three primitive relations:*
  "Betweenness": a triadic relation linking points;
  "Lies on" ("Containment"): three binary relations, one linking points and straight lines, one linking points and planes, and one linking straight lines and planes;
  "Congruence": two binary relations, one linking line segments and one linking angles, each denoted by an infix $\cong$.

Axioms of Hilbert's axiomatic model of Euclidean geometry:

i. *Axioms of Incidence:*
   1. For every two points A and B, there exists a straight line l that contains both of them. Then we write $AB = l = BA$. Apart from using the term "contains," one may say that "A lies upon l," "l goes through A and through B," "l joins A to B," etc.
   2. For every two points, there exists no more than one straight line that contains both of them.
   3. There exist at least two points on a straight line, and, given any straight line, there exists at least one point not on it.

[606] Eves, *Foundations and Fundamental Concepts of Mathematics*; Grattan-Guinness, *The Search of Mathematical Roots: 1870–1940*; Kneebone, *Mathematical Logic and the Foundations of Mathematics*; Leng, *Mathematics and Reality*; Moore, *The Infinite*; Potter, *Set Theory and Its Philosophy*; Struik, *A Concise History of Mathematics*.

4. For every three points A, B, and C not lying on the same straight line, there exists a plane P that contains all of them. Every plane contains at least one point.
5. For every three points A, B, and C not lying on the same straight line, there exists only one plane P that contains all three points.
6. If two points A and B of a straight line l lie on a plane P, then every point of l lies on P.
7. If two planes P and Q have a point A in common, then they have at least a second point B in common.
8. There exist at least four points not all contained in the same plane.

ii. *Axioms of Order:*

1. If a point B lies between points A and C, then B is also between C and A, and there exists a straight line containing the distinct points A, B, and C.
2. If A and C are two points of a straight line, then there exist at least one point B lying between A and C and at least one point D so situated that C lies between A and D.
3. Of any three points situated on a straight line, there is always exactly one that lies between the other two.
4. Any four points A, B, C, and D situated on a straight line can always be so arranged that B will lie between A and C as well as between A and D, and, furthermore, that C will lie between A and D as well as between B and D.[607]
5. *Pasch's Axiom*[608]*:* Given any three points A, B, and C that are not situated on the same straight line, and given a straight line l contained in the plane ABC but not containing any of the points A, B, and C, it holds that, if l contains a point on the segment AB, then l also contains a point on the segment AC or on the segment BC. In other words, if a straight line l meets side AB of the triangle ΔABC but contains none of the given triangle's vertices, then l meets side BC or side AC but not both.

iii. *Axiom of Parallels (Euclid's Axiom):* In a plane P, consider an arbitrary straight line l and an arbitrary point A that does not lie on l. Then there exists exactly one straight line that passes through the point A and does not intersect the straight line l.

iv. *Axioms of Congruence*[609]*:*

1. Given two points A and B as well as a point A' situated on a straight line l, there exist exactly two points C and D such that A' is between C and D, and $AB \cong A'C$ and $AB \cong A'D$. Every segment is congruent to itself; symbolically: $AB \cong AB$.
2. If a segment AB is congruent to the segment $A'B'$ and to the segment $A''B''$, then the segment $A'B'$ is congruent to the segment $A''B''$.
3. Let AB and BC be two segments of a straight line l that have no points in common aside from the point B, and let $A'B'$ and $B'C'$ be two segments of the

[607] The American mathematicians Eliakim Hastings Moore and Robert Lee Moore have independently proved that this axiom is redundant (thus reducing the number of the axioms of Hilbert's axiomatic model of Euclidean geometry to twenty).

[608] This statement is known as Pasch's Axiom, because, even though it was implicitly used by Euclid, the nineteenth-century German mathematician Moritz Pasch discovered its essential role in plane geometry.

[609] Two figures/objects are "congruent" if they have the same shape and size.

same or of another straight line l' having no point other than B' in common. Then, if $AB \cong A'B'$ and $BC \cong B'C'$, it holds that $AC \cong A'C'$.

4. Given an angle $\angle ABC$ and a ray (i.e., a straight line having one defined endpoint and extending endlessly in one direction) $B'C'$, there exist exactly two rays $B'D$ and $B'E$ such that $\angle DB'C' \cong \angle ABC$ and $\angle EB'C' \cong \angle ABC$.
5. If, in two triangles ΔABC and $\Delta A'B'C'$, the congruences
   $AB \cong A'B'$, $AC \cong A'C'$, and $\angle BAC \cong \angle B'A'C'$
   hold, then
   $\Delta ABC \cong \Delta A'B'C'$.

v. *Axioms of Continuity:*

1. *Axiom of Archimedes:* Given a straight line segment CD and a ray AB, there exist points $A_1, \ldots, A_n$ on AB such that $A_i A_{i+1} \cong CD$, where $1 \leq i \leq n$, and B is between A_1 and A_n. Equivalently, we can say that, given two arbitrary straight line segments AB and CD, there is a natural number n such that the sum of n copies of AB will be greater than CD.
2. Axiom of Completeness: To a system of points, straight lines, and planes, it is impossible to add other elements in such a manner that the system thus generalized shall form a new geometry obeying all the five aforementioned groups of axioms (I–V). In other words, if we regard the five aforementioned groups of axioms (I–V) as valid, then the elements of geometry form a system that is not susceptible to extension.

Hilbert's *Foundations of Geometry* is not only an axiomatization of Euclidean geometry, but also a research program of "formal axiomatics." What Hilbert thought that he had demonstrated was that Euclidean geometry provides a true representation of space and that the creation of non-Euclidean geometries neither threatened mathematical truth itself nor clashed with the existence of Euclidean geometry, provided that the consistency of Euclidean and non-Euclidean geometries could be assured. Furthermore, Hilbert attempted to refute Kronecker's argument that geometry is not part of pure mathematics. Thus, in 1904, faced with the discovery of contradictions in set theory, Hilbert insisted that it must be possible to develop a logically rigorous and completely mathematically satisfying foundation for the concept of number that, contra Kronecker's claims, encompasses both irrational and rational numbers. According to Hilbert, if mathematics is "formalizable," then the consistency problem is reducible to the derivability of a formula in a formal system that expresses both a statement and its negation. In addition, according to Hilbert, a consistency proof could demonstrate that a property of the axioms—which Hilbert called "homogeneousness"—is passed on by the rules of the axiomatic system to all of the theorems that are deduced from the given axiomatic system.

Based on the idea that, in analytic geometry, whose axioms are those of real numbers, geometric axioms can be logically and mathematically justified, Hilbert attempted to prove the consistency of Euclidean geometry within the context of the theory of real numbers, and, therefore, he interpreted the "points" and the "lines" of the system of Euclidean geometry as pairs of real numbers and linear equations, respectively. In other words, according to Hilbert's research program of "formal axiomatics," Euclidean geometry has as much claim to truth as the theory of real numbers. Thus, Hilbert's research program of "formal axiomatics" defended geometry against Kronecker's viewpoints, and it defended the system of real numbers against the critics of the continuum and set theory, but it gave rise to new questions regarding the truth of pure mathematics itself.

In 1905, responding to Hilbert's research work in the foundations of mathematics, Poincaré argued that the problem with Hilbert's attempt to secure the foundations of mathematics with consistency proofs for formal systems is that these must refer to *every* proof in the system (and, hence, since they must employ mathematical induction, they cannot be used in order to justify mathematical induction). In 1927, Hilbert qualified his position by distinguishing two kinds of induction: "contentual" and "formal." In particular, Hilbert argued that, according to his theory,

> . . . two distinct methods that proceed recursively come into play when the foundations of arithmetic are established, namely, on the one hand, the intuitive construction of the integer as numeral . . . that is, contentual induction, and, on the other hand, formal induction proper, which is based on the induction axiom and through which alone the mathematical variable can begin to play its role in the formal system.[610]

Hilbert's priority was to eliminate skepticism from mathematics without jeopardizing the possibility of future discoveries, and, therefore, he resorted to metamathematics. Consequently, in the context of Hilbert's research program of "formal axiomatics," the truth of the axioms of a given system is based on and derives from a metamathematical consistency proof, and the "interplay" between the given mathematical object and metamathematical systems is a dialectical process that accounts for the development of mathematical thought: a mathematical object provides the "raw material," specifically, a system of contentual propositions, which are formalized in a metamathematical system, yielding "ideal propositions." Therefore, the concept of "formalization" should not be confused with the concept of "axiomatics," since the former is only one aspect of the latter: whereas axiomatics (i.e., the axiomatic method) originated in the research works of ancient Greek mathematicians, Hilbert maintains that formalization (i.e., the formalist method) aims to render the concept of an axiomatic theory more precise by introducing the concept of a formal system as the next stage in the development of axiomatics. In particular, formalism shows that mathematical theories themselves can be treated as precise mathematical objects and investigated with regard to their consistency and completeness, and it gives rise to a general theory of such theories, namely, to a so-called "metatheory."

Whereas logicism maintains that mathematical entities can be defined in the language of symbolic logic, and whereas intuitionism maintains that mathematical entities are mental constructs, formalism maintains that mathematical entities, irrespective of any question about their essence, can be studied as terms of a formal language modulo the equivalence relation of "provable equality" (i.e., two terms are equivalent, or denote the same entity, if the formula obtained by putting an equality sign between them is provable in the given formal language). In other words, according to formalism, truth is equivalent to the deduction of theorems from a consistent and complete axiomatic system, and, if an axiomatic system is consistent and complete, then it is epistemologically legitimate (an it can coexist with other such systems). The very term "formalism" implies that one focuses more (or almost exclusively) on the

[610] Hilbert, "The Foundations of Mathematics," pp. 471–2.

formal qualities of a mathematical theory over its substantial content and intuitions, for which reason, according to Hilbert, mathematical theories rest not only on axioms and rules of inference (by means of which theorems are deduced from axioms), but also on undefined (primitive) notions. Hilbert's formalism, like intuitionism, maintains that mathematics is an activity of human consciousness, which is free to choose the axioms of the theories that it articulates, but this freedom is not absolute, because the truth of a formalized axiomatic theory consists in proving that the given formal system is consistent and complete. Thus, in the 1920s, Hilbert undertook to formalize the entire discipline of mathematics and, thus, transform it into an inventory of provable formulas.

In the 1930s, the great Austrian mathematician and logician Kurt Gödel undertook to evaluate the logical rigor of formalism, specifically, the two most comprehensive formal systems that had been developed until that time, namely, Russell's and Whitehead's system of *Principia Mathematica* (henceforth *PM*) and the Zermelo–Fraenkel axiomatic system of set theory (further developed by John von Neumann). In 1931, referring to these two formal systems, Gödel pointed that they "are so comprehensive that in them all methods of proof today used in mathematics are formalized, that is, reduced to a few axioms and rules of inference."[611] Therefore, Gödel contended, one could "conjecture that these axioms and rules of inference are sufficient to decide *any* mathematical question that can at all be formally expressed in these systems."[612] However, Gödel proved that "this is not the case, that on the contrary there are in the two systems mentioned relatively simple problems in the theory of integers that cannot be decided on the basis of the axioms."[613] Furthermore, Gödel proved an even stronger claim, namely, that the existence of "formally undecidable propositions" is not due to the special nature of *PM* and the Zermelo–Fraenkel axiomatic system of set theory, but it "holds for a wide class of formal systems; among these, in particular, are all systems that result from the two just mentioned through the addition of a finite number of axioms," provided that they are consistent.[614] In particular, in 1931, Gödel proved

Gödel' Incompleteness Theorem[615]*:* If a theory is consistent (i.e., if it neither contains nor produces contradictions), and if it is comprehensive enough to contain elementary arithmetic as the latter has been encoded by Peano's axioms for natural numbers, then it is not complete, that is, we can prove that there is a statement that is undecidable (i.e., it can be neither proved nor disproved) in the given theory.

Sketch of Proof (for simplicity, we shall consider the case of PM): First of all, as Gödel has observed, we have to keep in mind the following: (i) the formulas of a formal system, say *PM*, are formalized as finite sequences of primitive signs (i.e., variables, logical constants, and parentheses or punctuation dots), and, thus, it can be easily determined with complete precision whether a sequence of primitive signs is meaningful; and (ii) in the context of a formal system, say *PM*, proofs are formalized as finite sequences of formulas with certain specifiable properties. As regards metamathematics, it does not matter what objects are chosen as primitive signs, and, thus, Gödel assigned natural numbers to this case, so that a

[611]Gödel, "On Formally Undecidable Propositions of Principia Mathematica and Related Systems," p. 17.
[612] Ibid.
[613] Ibid.
[614] Ibid.
[615] Ibid, pp. 17–47.

formula will be a finite sequence of natural numbers, and a proof array will be a finite sequence of finite sequences of natural numbers. In this way, the metamathematical concepts (i.e., propositions) become concepts (specifically, propositions) about natural numbers or sequences of natural numbers, and, therefore, they can, at least in part, be expressed by the symbols of the formal system *PM* itself (i.e., in this way, we obtain an isomorphic image of the formal system *PM* in the domain of arithmetic). In particular, the concepts of a formula, a proof array, and a provable formula can be defined in the formal system *PM*, since we can, for instance, find a formula $F(v)$ of *PM* with one free variable v (of the type of a number sequence) such that, if $F(v)$ is interpreted according to the meaning of the terms of *PM*, then $F(v)$ says: "v is a provable formula." In view of the foregoing, Gödel constructed an undecidable proposition of the formal system *PM*, namely, a proposition A for which neither A nor $\neg A$ (i.e., the negation of A) is provable, in the following way.

If a formula of *PM* has exactly one variable, and if its variable is of the type of the natural numbers (i.e., a class of classes), then the given formula is said to be a "class sign." Then Gödel assumes that the class signs have been arranged in a sequence in some way, he denotes the nth one by $R(n)$, and he observes that the concept of a class sign and the ordering relation R can be defined in the formal system *PM*. Furthermore, Gödel assumes that α is an arbitrary class sign, he uses the symbol $[\alpha; n]$ in order to denote the formula that results from the class sign α when the free variable is replaced by the sign denoting the natural number n, and he observes that the ternary relation $x = [y; z]$ is also definable in the formal system *PM*. Therefore, a class K of natural numbers can be defined as follows:

$$n \in K \equiv \overline{Bew}[R(n); n], \tag{1}$$

where $Bewx$ means that x is a provable formula, and the bar denotes negation. Given that the concepts that occur in the definiens are all definable in *PM*, the concept K formed from them is also definable in *PM*, namely: there is a class sign S such that the formula $[S; n]$, interpreted according to the meaning of the terms of *PM*, states that the natural number n belongs to K. Since S is a class sign, it is identical to some $R(q)$, so that

$$S = R(q)$$

for a certain natural number q. Hence, Gödel managed to prove that the proposition $[R(q); q]$ is undecidable in the formal system *PM* as follows: If the proposition $[R(q); q]$ is provable, then it is also true. But, in that case, according to the aforementioned definitions, q would belong to K, and, by (1), $\overline{Bew}[R(q); q]$ would hold, which contradicts the assumption that the proposition $[R(q); q]$ is provable. If, on the other hand, the negation of $[R(q); q]$ is provable, then $\overline{q \in K}$, and, by (1), $Bew[R(q); q]$ would hold. But, in that case, both $[R(q); q]$ and its negation would be provable, which again is impossible.■

Remark: The undecidable proposition $[R(q); q]$ states that $q \in K$, and, therefore, by (1), that $[R(q); q]$ is not provable. In this way, Gödel constructed a proposition that says about itself that it is not provable in *PM*, without involving the fallacy of circular reasoning, since, initially, the given proposition asserts that a certain well-defined formula (specifically, the one obtained from the qth formula in the lexicographic order by a certain substitution) is unprovable, and only subsequently does it turn out that this formula is precisely the one by which the original proposition was expressed. In other words, broadly speaking, Gödel considered a statement of the type

$P =$ "This statement is false,"

which leads to the following complicated situation: if $P =$ "This statement is false" is true, then it is false, but the sentence asserts that it is false, and, if it is, indeed, false, then it must be true, and so on. The earliest study of problems pertaining to self-reference in logic is due to the seventh-century B.C. Greek philosopher and logician Epimenides, who formulated the classical "liar paradox."[616] Gödel's Incompleteness Theorem shows that such complicated situations can occur in any theory that is consistent and comprehensive enough to contain elementary arithmetic as the latter has been encoded by Peano's axioms for natural numbers. Consequently, logic is complete for itself, but it cannot ensure the completeness of any theory that is consistent and comprehensive enough to contain elementary arithmetic as the latter has been encoded by Peano's axioms for natural numbers.

According to Gödel, human consciousness, in general, and thought processes, in particular, are not merely algorithmic. Gödel established the following argument mathematically: "Either . . . the human mind (even within the realm of pure mathematics) infinitely surpasses any finite machine, or else there exist absolutely unsolvable Diophantine problems."[617] Thus, according to Gödel, logic is a necessary but not a sufficient underpinning of mathematics, since mathematical truth is not totally formalizable. Hao Wang, studying Gödel's theorems, points that "the analysis of concepts is essential to philosophy. Science only combines concepts and does not analyze concepts . . . Analysis is to arrive at what thinking is based on: the inborn intuitions."[618]

3.5.6. Conclusions

As I mentioned in Chapter 1, philosophy aims to articulate a *general method* and a *general criterion* for the explanation of every object of philosophical research; ontology is the branch of philosophy that seeks to inquire into and ascertain the reality of its object and, especially, the reality of being; and epistemology is the branch of philosophy that is preoccupied with the validity of knowledge and the different ways in which one can obtain valid knowledge. The aforementioned three "schools" of mathematical philosophy, namely, logicism, intuitionism, and formalism, express the need of every thorough and conscientious mathematician to analyze, evaluate, and found one's object of mathematical research in a rigorous way.

Scientists cannot be fully aware of what they do and of how they do it, unless they are sensitive to ontology and epistemology, that is, unless they philosophize on the object of their

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[616] See: Fowler, *The Elements of Deductive Logic.*

[617] Gödel, "Some Basic Theorems on the Foundations of Mathematics and Their Implications," p. 310. By the term "Diophantine problems," we refer to equations with rational solutions (in the third century A.D., the Greek mathematician Diophantus of Alexandria published a book entitled *Arithmetica*, in which he studied such problems and paved the way for important advances in number theory).

[618] See: Wang, *A Logical Journey*, p. 273.

research work. As I argued in Chapter 1, mathematics and philosophy are homomorphic, in the sense that mathematics plays a deeply philosophical role in science: mathematics equips science with criteria and methods of reasoning, computing, and, generally, conducting research as well as with suitable means of scientific expression and formalization. Thus, mathematics represents the highest level of abstraction vis-à-vis science, but, when mathematics reflects on itself in order to fully know itself and to organize itself in a rigorous way, mathematics ascends to an even higher level of abstraction, namely, to a level of abstraction that transcends mathematics itself, and it is called “mathematical philosophy.”

As I have already shown, neither logicism, nor intuitionism, nor formalism can stand as a general theory of mathematical philosophy: the major weaknesses of Russell’s and Whitehead’s *Principia Mathematica* are that it suffers from intellectual stiffness, it endows Aristotelian logic (further developed by Russell and Whitehead) with Platonic ontological authority, and it does not provide a proof of the assumed consistency of the theory of types; the major weaknesses of Brower’s and his advocates’ intuitionism are that it is characterized by unnecessarily extreme complexity, its constructivist methodology (specifically, the argument that a mathematical object should only be viewed as real if it can be explicitly constructed) entails the rejection of major parts and aspects of set theory, topology, and mathematical analysis, and it cultivates a restrictive mindset (for which reason, paraphrasing Hilbert, one could fairly argue that, doing mathematics merely under the constraints of constructivism is like trying to box without using one’s fists); the major weakness of Hilbert’s formalism is that its attempt to found the truth of the system *Principia Mathematica* and of the Zermelo–Fraenkel axiomatic system of set theory (further developed by John von Neumann) on the principle of consistency and, thus, achieve the total formalization of mathematical truth was proved to be necessarily (i.e., intrinsically) ineffective by Gödel. As Edna E. Kramer has summarized Gödel’s research work in the foundations of mathematics, Gödel showed that “there must be ‘undecidable’ statements in *any* system, that is, propositions which can neither be proved nor refuted using the rules of the system, and that *consistency* is one of these undecidable propositions.”[619] However, this state of affairs does not imply that the aforementioned great mathematicians’ research works in the domain of mathematical philosophy were fruitless or unsuccessful, because, on the contrary, they paved new ways of thinking about mathematics, and they gave rise to new awarenesses; in particular: they clarified the epistemological potential and the limits of the axiomatic method as a way of systematizing mathematics, they managed to found set theory and other mathematical areas (such as topology) in a rigorous way, and they gave rise to the development of metamathematics as a systematic way of reflecting on and evaluating mathematical work.

By the second half of the twentieth century, none of the aforementioned three “schools” of mathematical philosophy existed separately, in its original, pure form, and the borders between logicism, intuitionism, and formalism became blurred. Thus, from the middle of the twentieth century onward, mathematicians became preoccupied with new, broader epistemological debates, which, in fact, have prevailed in every scientific discipline (both in the natural sciences and in the social sciences), and they center around the following two issues: (i) the difference between “truth as a discovery” and “truth as an invention”; and (ii)

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[619] Kramer, *The Nature and Growth of Modern Mathematics*, p. 686.

the determination of the degrees of truth and the difference between "correctness" and "fallacy." I shall study these two epistemological debates in sections 3.7 and 3.8.

3.6. The Problem of Empirical Relevance in the Context of Science

As I mentioned in section 1.2.3, the modern scientific worldview implies that a sentence makes a cognitively significant assertion if and only if it is either L-determinate or non-L-deterimate, where L stands for the relevant formal language: the truth value of an L-determinate statement is determined in L by an interpretation of the symbols in L; on the other hand, a non-L-deterimate statement is called true or false not only on the basis of the rules of interpretation in the relevant deductive system, but also on the basis of a rule of disposition by reference to empirical data (non-L-deterimate statements for which a rule of disposition by reference to empirical data has been established are called "factual statements," and the deductive systems in which they appear are called "applied"). Therefore, from the aforementioned perspective, the formulation of particular criteria of empirical significance depends on the aforementioned general principle of cognitive significance. Furthermore, in order to be able to study particular criteria of empirical significance, one must be, first of all, aware of certain requirements that must be met by any criterion of cognitive significance, namely, one must formulate some condition of adequacy for criteria of cognitive significance. The following necessary (though not sufficient) condition (here called "Condition A") of adequacy for criteria of cognitive significance is originally due to Carl Gustav Hempel[620]:

Condition A: Let C be a criterion of cognitive significance. If, under C, a sentence S is non-significant (i.e., S cannot be significantly assigned a truth value), then so must be all truth-functional compound sentences containing S.

Corollary A1: If, under C, S is non-significant, then so must be $\neg S$ (i.e., the negation of S).

*Corollary A2:*If, under C, S is non-significant, then so must be $S \wedge S'$ and $S \vee S'$, where S' is any sentence, significant or non-significant under C.

Based on the above preliminaries, we can now study and evaluate different criteria of empirical significance. The oldest such criterion is the verifiability criterion:

The Criterion of Complete Verifiability in Principle (CV)[621]*:* A necessary and sufficient condition that a sentence has empirical meaning is that it is not L-determinate and follows logically from some finite and logically consistent class of observation sentences (these observation sentences may be false, since the criterion refers to testability "in principle").

The above criterion, however, has many defects. First of all, it is worth mentioning that, originally, (CV) had restricted the permissible evidence to what is observable by the speaker

[620] Hempel, *Aspects of Scientific Explanation.*
[621] See: Ayer, *Language, Truth and Logic*; Russell, *Human Knowledge.*

or one's fellow beings during their lifetimes.[622] In that form, (CV) had an important defect, namely, under such a criterion, all statements about the distant future or the remote past are cognitively meaningless.[623] This defect can be overcome if the concept of verifiability in principle is construed as referring to *logically possible* evidence as expressed by observation sentences, so that the class of statements that are verifiable in principle includes statements about the distant future, the remote past, and, generally, about phenomena that are not observable by the speaker or one's fellow beings (for instance, the sentence "Mars and the Antarctic existed before humanity discovered them").

Nevertheless, even after the above refinement, the verifiability criterion still has serious defects, as has been argued, among others, by Hempel.[624] Following Hempel's reasoning, let us, first, assume that the properties of being a cat and of having a tail are both observable characteristics and that the former does not *logically* entail the latter. Then the sentence

"All cats have a tail" (S^*)

is neither L-determinate nor contradictory, and, furthermore, it is not deducible from any finite set of observation sentences. Thus, under (CV), the above sentence is devoid of empirical significance and so are all other sentences expressing general laws. But, because sentences of the above type constitute a significant part of a scientific theory, it follows that (CV) is too restrictive. Second, the negation of (S^*), namely, the sentence

"There exists at least one cat that has not a tail" ($\neg S^*$)

is cognitively significant under (CV), but (S^*) is not, and this contradicts Corollary A1. Third, if S is a sentence that does and S' a sentence that does not satisfy (CV), then S is deducible from some set of observation sentences, so that $S \vee S'$ is deducible from the same set (i.e., $S \vee S'$ is cognitively significant), which contradicts Corollary A2.

Moreover, Karl R. Popper criticizes the verifiability criterion on the grounds that the positivist argument that empirical science is a system of statements satisfying certain logical criteria does not make provisions for what Popper considers to be the major distinguishing feature of empirical statements, namely, their susceptibility to revision.[625] In other words, Popper's approach to the question of empirical meaninglessness calls for a systematic study of the manner in which science advances and a choice is made between conflicting systems of theories. Thus, Popper proposes the falsifiability criterion as an alternative to the verifiability criterion:

The Criterion of Complete Falsifiability in Principle (CF)[626]*:* A necessary and sufficient condition that a sentence has empirical meaning is that it its negation is not L-determinate and follows logically from some finite and logically consistent class of observation sentences.

However, (CF) indirectly contains (CV), since (CF) qualifies a sentence as empirically meaningful if its negation satisfies (CV). Therefore, Hempel has pointedly argued that (CF) has similar defects with (CV).[627] Indeed, (CF) has the following defects: (i) it rules out purely

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[622] Ibid.
[623] Ibid.
[624] Hempel, *Aspects of Scientific Explanation.*
[625] Popper, *The Logic of Scientific Discovery.*
[626] Ibid.
[627] Hempel, *Aspects of Scientific Explanation.*

existential hypotheses (for instance, the statement "There exists at least one cat that has not a tail") as cognitively insignificant; (ii) if P is an observation predicate, then the assertion that all things satisfy P is significant under (CF), but its negation—being a purely existential hypothesis—is not significant under (CF), and this contradicts Corollary A1; (iii) if a sentence S does and a sentence S' does not satisfy (CF), then $S \wedge S'$ does satisfy (CF) (since, if $\neg S$ is entailed by a class of observation sentences, then $\neg(S \wedge S')$ is entailed by the same class), and this contradicts Corollary A2.

Given that both (CV) and (CF) have been proved to be too restrictive and susceptible to serious defects, the British philosopher Sir Alfred Ayer attempted to formulate a criterion of confirmability that avoids the defects of (CV) and (CF) by construing the testability criterion as consisting in a partial and possibly indirect confirmability of empirical hypotheses by observational evidence.[628] In particular, Ayer's *confirmability criterion* states that a sentence S is empirically significant if S in conjunction with suitable auxiliary hypotheses imply observation sentences that cannot be derived from the auxiliary hypotheses alone. However, Ayer himself, in the second edition of his book *Language, Truth and Logic* (1946), recognized that the previous confirmability criterion is too liberal.[629] For instance, if S is the sentence "The totality is everything," and if one chooses as an auxiliary hypothesis the statement "If the totality is everything, then the cat is black," the following observation sentence can be deduced: "The cat is black." Therefore, Ayer restricted the auxiliary hypotheses mentioned in the initial version of his confirmability criterion to sentences that either are L-determinate or can independently be shown to be testable in the sense of the refined confirmability criterion.[630] Nevertheless, not even this refinement of the confirmability criterion is enough; for, as Hempel has pointed out, it allows empirical significance to any conjunction $S \wedge S'$ where S does and S' does not satisfy Ayer's criterion (e.g., S' is a sentence such that "The totality is everything").

A general remark that applies to all the above-mentioned criteria of empirical significance is that they are all based on an attempt to define the concept of empirical significance in terms of certain logical connections that should hold between a significant sentence and suitable observation sentences. Moreover, all these criteria have been proved to have serious defects. Therefore, one might reasonably attempt to avoid the defects of the above-mentioned criteria by proposing an alternative way of explicating the concept of empirical significance. Such an alternative approach may be based on the characterization of cognitively significant sentences by certain conditions that their own constituent terms must satisfy.[631] In particular, all extralogical terms[632] in a significant sentence must have empirical content, and, therefore, their meanings must be explicable by reference to observables only.[633] In other words, the aforementioned testability criteria of meaning (i.e., (CV), (CF), and Ayer's confirmability criterion) were based on an attempt to characterize cognitively significant sentences by means of certain *logical connections* in which they must stand to

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[628] Ayer, *Language, Truth and Logic* (both first edition: 1936 and second edition: 1946).
[629] Ayer, *Language, Truth and Logic,* second edition (1946), chapter 1.
[630] Ibid.
[631] Hempel, *Aspects of Scientific Explanation.*
[632] By an "extralogical term," we should always understand a term that does not belong to the specific vocabulary of logic. For instance, the following phrases and those definable by means of them are logical terms: "or," "and," "if . . . then . . .," "all," "some," "is an element of class . . .," etc.
[633] Hempel, *Aspects of Scientific Explanation*.

some observation sentences, whereas this alternative approach aims to specify the *vocabulary* itself (i.e., the constituent elements) that may be used in order to form significant sentences. This vocabulary, the class of significant terms,[634] is characterized by the condition that each of its elements is either a logical term or a term with empirical significance.[635] In this way, the defects of the previous criteria can be overcome (e.g., if S is a significant sentence, then so is $\neg S$).

Nevertheless, the last conclusion cannot end the discussion about significance, since another important question remains open: Which are the appropriate logical connections between empirically significant terms and observation terms that can give rise to an adequate criterion of cognitive significance (notice that "adequate" means that it satisfies Condition A)? In the empiricist literature, a well-known attempt to answer the aforementioned question consists in the criterion of definability.[636]

Criterion of Definability (CD)[637]*:* Any empirically significant term must be explicitly definable by means of observation terms.

The criterion of definability is too stringent, since it rules out many important scientific and prescientific terms that are not explicitly definable by means of observation terms. For instance, the German-American philosopher Rudolf Carnap argues that the attempt to provide explicit definitions in terms of observables fails when it encounters disposition terms, such as "soluble," "malleable," etc.[638]

Carnap proposes an alternative to the criterion of definability (CD). He introduces the concept a reduction sentence, namely, a sentence that, unlike definitions, specifies the meaning of a term only conditionally or partially.[639] In order to understand the difference between a definition and a reduction sentence, let us consider, for instance, the word "elastic." One can define elastic behavior as follows: An object x is elastic if and only if, at any time t that it is deformed (e.g., when x is stretched), the deformation is reversible at time t'. If the statement connectives of the previous definition are construed truth-functionally, then the given definition can be written symbolically as follows:

$$Ex \equiv (t)(Dxt \supset Rxt').$$

But then one faces the following problem: if y is any object that is not elastic but such that it has never been deformed during its existence, then Dyt is false, and, therefore, it holds that $Dyt \supset Ryt'$ for any t, so that the observation predicate E (elastic) is true in the case of Ey even though y is not elastic. To remedy that defect, one can follow Carnap's theory of reduction sentences, so that the term "elastic" can be expressed by the following reduction sentence:

$$(x)(t)[Dxt \supset (Ex \equiv Rxt')],$$

[634] Any term contained in a cognitively significant sentence is said to be a "cognitively significant term."
[635] Hempel, *Aspects of Scientific Explanation.*
[636] This criterion has been critically studied by Hempel in his book *Aspects of Scientific Explanation.*
[637] Ibid.
[638] Carnap, "Testability and Meaning."
[639] Ibid.

which states that, if x is deformed at any time t, then x is elastic if and only if that deformation of x is reversible at time t'.

On the other hand, reduction sentences cannot account for the use of theoretical constructs, which play an important role in the construction of scientific theories. For instance, in classical physics, the length in meters between two points may assume any positive real number as its value. But one cannot use observables in order to formulate a sufficient condition for the applicability of such an expression as “x has a length of 10^{-30} m,” or “x has a length of 10^{30} m,” namely, extremely small or extremely large numbers.

Theoretical constructs should be construed as being stated in the form of hypothetico-deductive systems. The extralogical terms of deductively developed axiomatized systems are of two kinds: “primitive terms” (or “basic terms”), which are not defined within the theory, and “defined terms,” which are explicitly defined by means of the primitives. The primitive and the defined terms together with the terms of logic constitute the vocabulary in terms of which all the sentences of a given theory are construed. Moreover, the statements of a theory are of two kinds: “axioms” or “postulates,” which are not derived from any other statements in the theory, and “derived statements,” which follow from the postulates by logical deduction. Empiricism maintains that such deductively developed systems can constitute empirical scientific theories if they have gained empirical content. As I have already argued, an empirical science presupposes the assignment of a meaning in terms of observables to certain terms or sentences of a given deductive system (i.e., an interpretation of the given deductive system). An interpretation may take the form of a partial assignment of meaning. For instance, the rules for the measurement of weight by means of a standard weight may stand as a partial empirical interpretation of the term “the weight, in grams, of an object x.” However, in the aforementioned example, the suggested method of measuring weight is applicable to weights ranging within a certain interval, and, furthermore, it cannot be regarded as a full interpretation, since it does not constitute the only way of measuring weight.

Consequently, one should not focus one’s methodological research on the “empirical content” of specific terms or sentences; for, usually, no individual statement in a scientific theory implies any observation sentences. In fact, a sentence can entail the occurrence of certain observation phenomena only if it is conjoined with other auxiliary hypotheses (namely, observation sentences and provisionally accepted theoretical statements). In particular, Hempel argues that the empirical significance of a given expression U is related to the language L to which U belongs (L contains the rules of inference) and to the theoretical context in which U occurs (the theoretical context of U consists of the statements in L that may stand as auxiliary hypotheses).

Conclusively, a criterion of cognitive significance should refer to an entire theoretical system formulated by means of a well-defined language. Additionally, the basis of cognitive significance in such a system is the possibility of its interpretation in terms of observables; such an interpretation may be formulated by means of (bi)conditional sentences connecting non-observable terms of the system with observation terms in the given language. Yet, the requirement of partial interpretation is too liberal, since it can be satisfied by a system consisting of an empirical theory, say modern physics, with some set of isolated sentences, even if the latter have no empirical interpretation. Notice that an “isolated sentence” is defined to be a sentence that is neither a purely formal truth or falsehood nor does it have any empirical content. In other words, isolated sentences can be construed as sentences of speculative metaphysics, where "metaphysics" refers to doctrines about the fundamental nature of substances, or about tautological matters, or about our relation to external objects. The following criterion deals with the problem of isolated sentences:

Criterion of Cognitive Significance (CCS)[640]*:* A necessary and sufficient condition that a theoretical system is cognitively significant is that it is partially interpreted to such an extent that in no system equivalent to it at least one postulate is isolated.

Nevertheless, it is not direct observation of phenomena that can lead to the formulation of generalizations of great scope or rigor. Such generalizations need theoretical constructs. In fact, properly defined theoretical constructs provide the framework within which new general connections may be discovered, which otherwise (i.e., if one adopts a strict phenomenalist or positivist approach implied by (CCS) and, thus, rules out certain terms and sentences because of (CCS)) would remain in the dark.

Hempel has conjectured that no successful alternative to (CCS) can be found, and that, therefore, one cannot formulate a precise criterion by means of which those partially interpreted systems whose isolated sentences might be said to have a significant function can be separated from those in which the isolated sentences are redundant.[641] Hence, instead of trying to modify (CCS), one should recognize that cognitive significance in a theoretical system varies. In fact, significant systems range from those all of whose extralogical terms consist of observation terms, through systems that depend heavily on theoretical constructs, on to systems whose empirical relevance is marginal. For instance, positive economics, dealing with facts and behavior in an economy, does not consist of pure deductive systems, whereas normative economics, dealing with what "ought to be" in an economy, does consist of pure deductive systems.

Therefore, instead of espousing a sharp dichotomy between significant and non-significant systems, one should compare different theoretical systems with respect to the following characteristics that have been originally formulated by Hempel[642]:

(C1) the level of accuracy that characterizes the manner in which a theory is formulated and the manner in which the logical relationships of its elements to each other and to observation sentences have been made explicit;
(C2) the ability of a theory to explain and/or predict observable phenomena;
(C3) the formal simplicity of a theory in terms of which explanation and prediction will take place;
(C4) the extent to which a theory has been empirically confirmed.

3.7. Truth as a Discovery and Truth as an Invention

Truth, in general, can be defined as a structure, specifically, as a set of relations, say $\{R_1, R_2, \dots, R_n\}$, which determine if and the extent to which the representation of reality

[640] For a thorough study of this criterion, see: Hempel, "The Concept of Cognitive Significance: A Reconsideration."
[641] Hempel, *Aspects of Scientific Explanation.*
[642] Ibid.

within consciousness, namely, the knowledge of reality, is in concordance with the presence of reality itself, namely, with the nature of reality.

The way in which Plato defines propositional truth in *Sophist*, 261e–262d, can be summarized and interpreted as follows: a sentence stating that "a is x" is true if and only if it states about a things as they *are*, that is, a is really x; otherwise, the given statement is false. Thus, from Plato's perspective, "truth" signifies the concordance between a being or thing and its idea (i.e., the corresponding beingly being), so that a being or thing is true if and to the extent that it is in concordance with its idea. Given that Plato's conception of truth is intimately related to a metaphysical intuition, which underpins Plato's theory of philosophical vision, Plato says nothing about "correspondence" or about "facts." It was Thomas Aquinas who, interpreting Aristotle's philosophy, defined truth as "the correspondence between the intellect [of the knower] and the thing [the known]" ("adaequatio rei et intellectus"), as I mentioned in section 1.2.1. During the Middle Ages and the Renaissance, most of the philosophies that affirmed the possibility of obtaining valid knowledge endorsed Aquinas's correspondence theory of truth. According to Aquinas's Aristotelianism, there is a gap between reality and consciousness, and this gap can be interpreted as the distance that determines the reflection of reality in consciousness. However, the aforementioned Thomistic perception of truth is deficient, because it does not clarify whether the aforementioned reflection gives rise to the existence of a reversed (or, generally, distorted) image of reality in consciousness.

Descartes and, in general, Cartesianism oppose Aquinas's Aristotelianism, by rejecting the definition of truth as a relation and by identifying truth with reality. According to Descartes, in particular, understanding (or intellection) is the basic reality, and it is activated by conceiving itself (hence, Descartes's famous *dictum*: "cogito ergo sum"). According to Malebranche, who is one of the most important Cartesian philosophers, and, simultaneously, he espouses various elements of Augustine's and Thomas's philosophies, truth does not merely exist within the absolute, but it is identified with the absolute, and, therefore, we partake of truth to the extent that we partake of the absolute. Thus, in his *Treatise Concerning the Search after Truth*, Malebranche thinks of truth as a transcendent object, existing independently of consciousness, and he argues that consciousness can know truth by identifying itself with truth, specifically, either through the absorption of truth by consciousness or through the absorption of consciousness by truth. In fact, the aforementioned approach to the problem of truth can give rise to various syntheses between Cartesianism and mysticism.

The French epistemologist Bachelard's approach to the problem of truth is similar to Malebranche's approach to the same problem, but Bachelard emphasizes the process of objectification that takes place in the context of science. In particular, Malebranche refers to a truth that is being increasingly approached by consciousness as the latter is trying to remove the border between itself and its object, a border that has been drawn by consciousness itself in order to help consciousness to take distance from the world and, thus, develop a rational stance toward the world and achieve its scientific goals. Bachelard argues that humanity's initial contact with the world is grounded in humanity's primitive drives, which induce reverie and dream, preceding any kind of reflection: "We are being faithful to a primitive human feeling, to an elemental organic reality, a fundamental oneiric temperament."[643] In

[643] Bachelard, *Water and Dreams*, p. 5.

Bachelard's view, this oneiric aspect of humanity's primary encounter with the reality of the world is characteristic of the way in which the reality of everyday life is constructed. According to Bachelard, only when humanity overcomes the aforementioned oneiric state can a rational stance toward the world come into being. Thus, Bachelard maintains that rationality is a continuous process of overcoming primary impulses, and, in particular, he argues as follows: "In point of fact I see no solid basis for a natural, direct, elemental rationality . . . Rationalist? That is what we are trying to *become*."[644]

In view of the foregoing, we realize that, in the context of modern philosophy, the conception of truth as a discovery does not imply a static conception of truth, but, on the contrary, it is inextricably linked to a dynamic cognitive process. This dynamic approach to truth characterizes the conception of truth as an invention, too. Truth can be conceived as a creation, that is, as a reality that is continuously under formation and reformation by consciousness. The conception of truth as a creation can be further clarified by Bachelard's notion of an "objective meditation": "subjective [Cartesian] meditation is bent on attaining clear and definitive knowledge; objective mediation differs from this by the very fact that it makes progress, by its intrinsic need always to go further, to extend the limits of the known."[645] Thus, Bachelard characterizes science as a *dynamic* process both guided by and striving for *rationality*. According to Descartes, the changing qualities of the wax force us to dismiss the trustworthiness of sensory-sensuous knowledge, but, according to Bachelard, it is precisely the experimental revealing of the morphological diversity of the wax that allows for its objectification, and, therefore, scientific consciousness should be continuously open to experience new objects, which are constituted by the manner in which different aspects of them are experimentally revealed. Truth as an invention can be regarded as a possibility that is being increasingly actualized and specified due the interplay between consciousness and the reality of the world. In this case, the interplay between consciousness and the reality of the world is similar but not identical to some Thomistic/Aristotelian notion of a relation (or "correspondence") regarding truth: from the perspective of any Thomistic/Aristotelian notion of a relation (or "correspondence") regarding truth, consciousness is a passive mirror of reality, whereas, according to the aforementioned conception of truth as an invention, consciousness, due to its rationality, plays a much more active and responsible role in the acquisition of valid knowledge.

In view of the arguments that I have already put forward in this section and in section 3.6, truth is neither a pure essence nor a pure relation (or "correspondence"), but it is a dynamic and rational contemplation of the world and of consciousness as consciousness (re)integrates itself into the world. Therefore, truth should be construed neither as a discovery alone nor as an invention alone, but as the outcome of the contact and the interaction between consciousness and the reality of the world. The integration of consciousness into the world is both a volitional act and an existential necessity. However, when conscious beings integrate themselves into the world, they do not only accept the reality of the world as a substantial presence, but also they attempt to understand and interpret the reality of the world. Even when consciousness cannot enter into and partake of the reality of a particular aspect the world or of a particular situation, consciousness can create a pertinent concept. Hence, as I explained in section 3.6, theoretical constructs play a necessary and substantial role in

[644] Ibid, p. 7.
[645] Bachelard, *The New Scientific Spirit*, p. 171.

science. Moreover, Kant has masterfully proved that scientific laws are neither connatural to reality nor innate in it, but they are kinds of relations, specifically, hypothetico-deductive systems, through which consciousness understands and interprets reality. During the process of scientific explanation, the consciousness of a scientist creates new, more complete systems of relations (i.e., hypothetico-deductive systems) in order to improve one's understanding and interpretation of reality, thus replacing older, scientifically degenerating systems of relations with new ones, which have a broader explanatory domain (this reasoning underpins Hempel's approach to the problem of significance, which I studied in section 3.6).

When consciousness establishes a correspondence between the intellect and the thing, it is not passive, but active. In particular, consciousness conceives structures that concur with its own structure, and it (re)integrates itself into the world in accordance with these structures. Thus, the philosophy of rational dynamicity discards both the idealist argument that reality is a mere extension of consciousness and the pragmatist argument that one should merely opportunistically seek for a congenial, even temporal, way of settling the contradiction between "success" and "failure." Furthermore, the philosophy of rational dynamicity discards any variety of philosophical realism that assigns a passive role to consciousness in the context of the correspondence theory of truth. According to the philosophy of rational dynamicity, consciousness does not only observe reality, but also it structures and restructures reality, and, for this reason, a theory is not a set of observed occurrences and recorded associations, but it is instead an explanation of them. In particular, the transition from causal speculations based on factual studies to theoretical formulations requires the following methods: (i) isolation, namely, viewing particular factors and forces with certain *ceteris paribus* assumptions (i.e., assuming that other things remain equal); (ii) abstraction; (iii) aggregation, namely, grouping data together according to the criteria of the corresponding theory; and (iv) idealization, namely, conceiving an ideal state or a state in which a limit has been reached. Consequently, truth should be construed as the specification of the intentionality of consciousness as the latter attempts to impose an interpretation of the world and, thus, to (re)structure the world, in accordance with the goals of consciousness.

In the context of the Thomistic-Aristotelian variety of the correspondence theory of truth, consciousness plays a mainly passive role. By contrast, the philosophy of rational dynamicity reverses the Thomistic-Aristotelian correspondence between reality and consciousness, and it highlights not only the dynamic role of consciousness, but also the potential adaptation of reality to the intentionality of consciousness. In particular, according to the philosophy of rational dynamicity, consciousness aims to (re)structure reality not according to some arbitrary idealist vision, but according to the dialectic of rational dynamicity, which I expounded in section 1.3.3. Consequently, truth is continuously being created by the contact and the interaction between consciousness and reality, and, even though the so-obtained truth is relative and partial, it does not prohibit humanity from struggling for the attainment of the absolute truth in any area of reality.

3.8. Degrees of Truth

As I have already argued, truth has a triple nature: it has an epistemological nature (as a rigorous creation of consciousness), an axiological nature (as a judgment and, specifically, as a logical value), and an ontological nature (as a correspondence). The process of understanding and articulating truth is complex and arduous, and it includes tasks that are susceptible to distortion and error. In ancient times, Plato argued that the divergence from the correct path of truth is due to ignorance, in the sense that one "does not willingly err."[646] In modern times, Descartes partially endorsed the aforementioned Platonic argument. According to Plato, consciousness may be led to a fallacy due to a mechanical intellectual process, which Plato described in *Theaetetus*, 197c–d, as follows: one who possesses knowledge may not really have it, just as, for instance, a man who keeps pigeons in an aviary "has acquired power over them, since he has brought them under his control in his own enclosure," but "he has none of them," in the sense that, due to the turmoil that prevails in the aviary, when he tries to catch one of these pigeons, he catches another instead of another. However, according to Descartes, consciousness may be led to a fallacy due to the distortive effects of an intervention of volition in pure thinking.[647] Thus, Plato interprets ignorance in terms of mechanism, whereas Descartes interprets ignorance in terms of dynamism. Moreover, Aristotle has distinguished between those fallacies which are involuntary logical sins and those fallacies which are sophisms, namely, deliberate distortions of truth.[648]

Francis Bacon, one of the most prominent scholars of the British Enlightenment, analyzed the distortions that consciousness may undergo as a result of mental "idols," namely, deeply rooted fallacies, which govern human societies and obstruct human consciousness from understanding real situations with which it has to deal and of which it has to form intellectual representations.[649] In particular, Bacon divides the aforementioned "idols" into the following four categories: (i) "Idols of the tribe": they have their origin in the tendency of consciousness to focus its attention on favorable approaches to a problem and to beautify and simplify particular things of the world (e.g., things pertaining to one's own nation-state, social group, vested interests, etc.); (ii) "Idols of the cave": they refer to Plato's myth of the cave,[650] and they are due to the preconditioned system of every person, comprising education, custom, and/or contingent experiences. (iii) "Idols of the Market

[646] Plato, *Republic*, 589c.

[647] See: Cottingham, ed., *The Cambridge Companion to Descartes.*

[648] Aristotle, *On Sophistical Refutations.*

[649] See: Bacon, *The Advancement of Learning.*

[650] In his work *Republic*, 514a–520a, Plato narrates the following myth, which is known as the "myth of the cave," and it symbolizes humanity's relationship with the good-in-itself as a process of education and psychological remolding: In the depths of a gloomy, underground chamber like a cave, are men who have been prisoners there since they were children. They are fastened in such a way that they cannot turn their heads. Some way off, behind and higher up, a fire is burning. Between the fire and the prisoners and above them runs a road, in front of which a curtain-wall has been built, like the screen at puppet shows between the operators and their audience. Furthermore, there are men carrying all sorts of gear along behind the curtain-wall, projecting above it. Hence, due to the way in which the prisoners' legs and necks are fastened, the only things that the prisoners can see are the shadows of the objects carried along the road (these shadows are thrown by the fire on the wall of the cave opposite them). Suppose that one of these prisoners were let loose, suddenly compelled to stand up and turn his head and look and walk toward the fire and that, ultimately, he were forcibly dragged up the steep and rugged ascent and not let go until he had dragged out into the sunlight. The previous process would be a painful one, to which the prisoner would much object. However, during his march toward the sun, the prisoner would realize that, apart from the shadows, there were other things, too, such as a burning torch and several objects carried along the road, and, when, at last, he would manage to get out of the cave, he would see things in the upper world outside the cave, and, finally, he would manage to look at the sun itself. Later on, he would come to the conclusion that, when he was in the cave, he was looking at the shadows of things and not at things themselves.

Place": they are based on false conceptions that derive from social and political interactions between humans. (iv) "Idols of the theater": they stem from sophistry and arbitrary mysticism, and they manifest themselves as arbitrary generalizations and abstractions, errantly drawn analogies, and irrational meditations.

In view of the arguments that I put forward in sections 1.3.3, 3.4, and 3.7, the problem of logical values cannot be adequately understood as a sharp distinction between truth and falsehood. From the perspective of classical logic, there are only two logical values, namely, truth and falsehood, but modern logic has shown that infinitely many logical values can be defined, thus giving rise to n-valued logics, where $n \geq 2$ represents the corresponding number of logical values, two of which are the classical logical values of truth and falsehood. Therefore, one must study the problem of logical values in a more careful way, utilizing the dialectic of rational dynamicity, instead of being intellectually trapped in a simplistic logical dualism.

First of all, we should clarify the difference between the terms "truth," "correctness," and "fallacy." The term "correctness" signifies the definition of the final result of a process aimed to obtain the most general conception of truth, and, therefore, the correct truth *per se* is a unique truth. However, the attempt to obtain the most general conception of truth includes "fallacies." The term "fallacy" signifies an approximation of correctness (i.e., of the correct truth).[651] It is worth mentioning that the Greek term for "fallacy" is "plāne" ("πλάνη"), which also means a wandering, a straying about (in fact, the Greek term "planētes" (i.e., planet) derives from "plāne"). Therefore, apart from referring to an error, the term "fallacy" also refers to the free and rational wandering of thinking consciousness as it seeks truth freely and decisively. Consequently, the aforementioned conception of truth-as-fallacy is substantially different from Nietzsche's argument that "man's truths . . . are his irrefutable errors."[652]

In contrast to the notions of a falsehood and an irrationality, a fallacy is susceptible to correction, since it is an imperfect step toward the knowledge of truth-as-correctness. The difference between the correct truth and anyone of its approximations is the measure of fallacy of the corresponding approximate truth. In other words, a fallacy can be construed as an approximate model of correctness that consists of a set of terms that can be determined *a priori* and of a set of terms that can only be determined *a posteriori*, since the context in which a model holds cannot be completely determined beforehand. For instance, consider the expression $w = (a, b)$, where w is a member of the power set $\wp(W)$ of the states of the world W under consideration, a is the interpreted component of w, and b is the uninterpreted component of w. Moreover, let us assume that

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[651] The awareness that the correct truth can be gradually and rationally approximated through fallacies was inculcated in me by Michael Nicholson, with whom I cooperated in 1998 at the University of Sussex, where he was teaching epistemology and methodology of international relations. Michael Nicholson was one of the most eminent European scholars in the formal analysis of war and conflict and the possibility of peaceful resolution of disputes, and he supported Imre Lakatos's methodology of scientific research programs. Summarizing Lakatos's methodology of scientific research programs, Nicholson has written the following: "According to Lakatos, a theory is not refuted in the abstract merely because of a few counter-instances . . . A theory is refuted only in terms of another which explains all the other did and more besides . . . A research program should continue, provided it is still 'progressive' . . . In the competition between research programs, which according to Lakatos is the usual state of affairs, a degenerate program slowly gives way to a progressive program" (Nicholson, *Causes and Consequences in International Relations*, pp. 72–4).

[652] Nietzsche, *The Joyful Wisdom*, aphorism 265.

$w = (w_1, \dots, w_i, \dots, w_n)$,

where n is an arbitrary positive integer,

$a = (w_{1'}, \dots, w_{i'}, \dots, w_{n'})$, and
$b = (w_{1''}, \dots, w_{i''}, \dots, w_{n''})$,

such that

$n = n' + n''$, and $n \neq 0$.

Then the quotient

$\frac{n'}{n} = g$, where $0 < g \leq 1$,

is the measure of generality of expression w; and, for $g = 1$, the measure of fallacy of expression w is equal to zero, so that we obtain an expression representing correctness. Consequently, the "scientific" work of both the natural scientists and the social scientists can be construed as an attempt to construct expressions whose measure of generality is as close to 1 as possible. Inherent in the aforementioned statement is the awareness that, far from signifying opposition to truth, a fallacy may be construed as a subset of truth, from which, finally, correctness emerges. This dynamic and rational process, according to which fallacy operates as the midwife of truth, is governed by the dialectic of rational dynamicity, which was expounded in section 1.3.3, and it is in agreement with Imre Lakatos's concept of "sophisticated falsificationalism" and his "methodology of scientific research programs."[653] As Lakatos has pointedly argued, even a bad research program is better than nothing.

The term "falsehood" signifies, specifically, the exact opposite of "correctness," not the exact opposite of "truth" in general. Moreover, we should clarify the difference between the terms "falsehood," "error," "absurdity," and "irrationality." The term "error" signifies that a syllogism is intrinsically (that, is structurally and automatically) mistaken, and that it is the result of a misinterpretation and/or a misapplication of the rules of logic. An errant syllogism cannot be corrected, and, therefore, it can only be dismissed. Falsehood is the expression of an error. The term "absurdity" signifies that a syllogism is definitively mistaken (as is the case with an "error"), but it can operate as a criterion for the correction of a series of syllogisms into which it has been deliberately inserted, according to the form of argument that is known as *reductio ad absurdum*.[654] The term "irrationality" signifies the output of an inconsistent series of syllogisms. When a series of irrationalities is taken to its ideal natural conclusion, it converges to falsehood, whereas, when a series of fallacies is taken to its ideal natural conclusion, it converges to correctness. Therefore, an irrationality can be construed as the exact opposite of a fallacy, which is an approximation of correctness.

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[653] Lakatos and Musgrave, eds., *Criticism and the Growth of Knowledge.*
[654] *Absurdity:* $(p \rightarrow c) \rightarrow \neg p$,
where p denotes a proposition, and c a "contradiction."
Reductio ad absurdum: $(p \rightarrow q) \leftrightarrow [(p \wedge \neg q) \rightarrow c]$,
where p and q stand for propositions, and c stands for "contradiction."

The above definitions can be summarized as follows:

i. Oppositional concepts at the highest level of generality:
Correctness versus Falsehood.
ii. Oppositional concepts in the context of approximations:
Fallacies versus Irrationalities,
where fallacies are approximations of correctness, and irrationalities are approximations of falsehood.

René Thom has emphasized the importance of the qualitative, structural aspect of the approximation of correctness through fallacy.[655] In particular, he has considered the following case[656]: Let us suppose that the experimental study of a phenomenon Φ gives an empirical graph g whose equation is $y = g(x)$, and that a theorist attempting to explain Φ has available two theories, say θ_1 and θ_2. In Figure 3.3, we see the empirical graph $y = g(x)$ of the phenomenon Φ, the graph $y = g_1(x)$ of theory θ_1 and the graph $y = g_2(x)$ of theory θ_2. Neither the graph $y = g_1(x)$ nor the graph $y = g_2(x)$ fits the graph $y = g(x)$ well. As shown in Figure 3.3, the graph $y = g_1(x)$ fits better *quantitatively*, in the sense that, over the interval considered, $\int |g - g_1|$ is smaller than $\int |g - g_2|$. On the other hand, Figure 3.3 clearly shows that the graph $y = g_2(x)$ fits better *qualitatively*, in the sense that it has the same shape and appearance (e.g., more specifically, similar monotonicity and similar curvature) as $y = g(x)$. Hence, René Thom argues that, "in this situation, one would lay odds that the theorist would retain θ_2 rather than θ_1 even at the expense of a greater quantitative error, feeling that θ_2, which gives rise to a graph of the same appearance as the experimental result, must be a better clue to the underlying mechanisms of Φ than the quantitatively more exact θ_1."[657]

Being an integral part of the structuralist philosophical tradition, the dialectic of rational dynamicity implies and underpins a rational and dynamic approach toward both fallacious deduction and fallacious induction. In particular, by the term "fallacious deduction," statisticians refer to "errors frequently made by imputing to each member of a group the 'average' behavior of the group,"[658] and, by the term "fallacious induction," statisticians refer to the fact that "induction—moving from a part to the whole—is a fruitful source of error in the collection and analysis of quantitative data."[659]

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[655] Thom, *Structural Stability and Morphogenesis.*
[656] Ibid, p. 4.
[657] Ibid.
[658] Neiswanger, *Elementary Statistical Methods*, p. 34. In particular, Neiswanger gives the following example: given that the U.S. does not export more than ten per cent of its national output, foreign trade is not vital to the U.S. economy, but, "if reference is made to particular industries, it is found, as Jerome B. Cohen has pointed out, that about 50 per cent of the cotton crop is exported, 35 per cent of the tobacco crop, and 25 per cent of the wheat crop" (ibid).
[659] Ibid, p. 37. In particular, Neiswanger has pointed out that, "if the part of the data brought forth for analysis is carefully chosen by modern methods of statistical sampling, one may be as well or even better off than though he attempted to work with the entire mass (population) from which the sample came," but "serious trouble enters . . . when some perhaps unsuspected influence colors the selection so that only certain parts—a chunk—of the population are heard from, resulting in wrong estimates of the whole" (ibid).

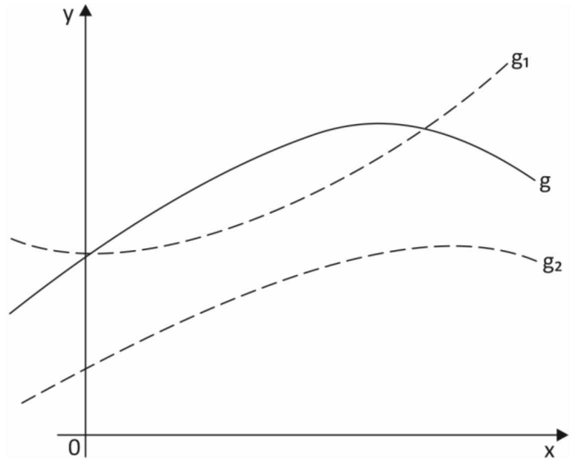

Figure 3.3. Approximations.

Conclusively, we realize that, according to the dialectic of rational dynamicity, the domain of truth consists of both correctness and fallacy, since correctness refers to a unique expression of truth increasingly approximated by a series of fallacies. Truth is not something definitively given, a crystallized datum, but it is something implicitly discerned, gradually emerging, adjustable, revisable, and susceptible to restructuring according to the intentionality of consciousness. The rationalism and the dynamism of this intentionality are expressed according to the dialectic of rational dynamicity.

3.9. From Logical Values to Moral Values: Ethics and Social Theory from the Perspective of Rational Dynamicity

As I argued in Chapter 1, if action is as an autonomous activity, and not the result of coercion, then it is guided by a purpose, which has been adopted by consciousness. Moreover, this purpose is underpinned by the intentionality of consciousness. The transition from the conception of an action by consciousness to the practical implementation, the actualization of that action is determined by the value system of consciousness. Thus, the value system of consciousness is the major attractor of conscious free action. In this way, conscious free action has a quantitative aspect and a qualitative aspect: the quantitative aspect of conscious free action consists in a fact, and the qualitative aspect of conscious free action consists in a value, namely, in a judgment.

As the French philosopher Louis Lavelle (one of the greatest French metaphysicians of the twentieth century) has argued, every value is an object of a desire and of a judgment.[660] Hence, even though, as Kant has argued in his *Critique of Practical Reason*, values are mainly created by consciousness, they are completely experienced by consciousness only when they are objectified. According to Lavelle, consciousness distances itself from values in the context of their objectification, and, in this way, consciousness reaffirms its creation,

[660] See: Smith, *Contemporary French Philosophy*, Part 1.

namely, values, by intentionally moving toward them. The aforementioned dialectical process allows one to overcome the contradiction between the subjectivist theory of values (according to which values are determined by the way in which an acting individual characterizes something in relation to the achievement of one's desired ends and according to one's judgment) [661] and the objectivist theory of values (according to which each value is a particular mode of being, thus qualitatively enriching the "basic modes of being," which are studied in ontology[662]). Moreover, through the aforementioned dialectical process, consciousness creates and manipulates values, and, therefore, it imposes itself as an autonomous value that organizes every other particular value into a value system. The return of consciousness to the values that it has created in the context of the aforementioned dialectic expresses and is determined by the rational dynamicity of consciousness. Hence, the determination of values depends on the rational dynamicity of consciousness, which was expounded in Chapter 1.

As I mentioned in section 1.1, when the consciousness of existence operates as a judge, it is called moral consciousness. In other words, moral consciousness is not a distinct kind of consciousness, but a particular operation of the consciousness of existence itself. Moral consciousness is a function of the following variables: sentiments, volition, and reason (see also section 1.1). Sentiments are emotions equipped with judgments, and, therefore, they are the strongest underpinnings of moral consciousness, since they can help moral consciousness to make decisions even when one's will falters and/or when one's intellect is irresolute. Volition is guided by the principle of pleasure, which expresses attraction to life. Reason expresses the power of consciousness to control itself. From the perspective of the philosophy of rational dynamicity, all the aforementioned partial aspects of moral consciousness can and should be synthesized with each other into a unified moral criterion according to the dialectic of rational dynamicity. In this way, morality expresses the intentionality of consciousness, as the latter seeks a better life and utilizes every aspect of morality in order to achieve its goals.

Intimately related to the development of moral consciousness is a state of inner vigilance. Due to the development of moral consciousness, humanity becomes increasingly vigilant, and, therefore, it becomes increasingly able to discern the difference between *being* intelligent and merely *demonstrating* intelligent behavior, between *having* certain merits and merely *demonstrating* meritorious behavior, as well as between *having* certain conscious qualities and merely *simulating* them. It is worth mentioning that Gregory of Nyssa (a Byzantine theologian who served as the bishop of Nyssa in the fourth century A.D.) has cited the following example, paraphrasing a similar story originally narrated by the ancient satirist and rhetorician Lucian:

> An animal trainer in Alexandria taught a monkey to skillfully impersonate a female dancer on stage. The spectators at the theatre praised the monkey, which was dressed as a female dancer and danced to the beat of the music. But while the viewers were occupied

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[661] As an advocate of the subjectivist theory of value, Sartre, for instance, maintains that one's own freedom is the only foundation of one's values, and that, apart from one's freedom, nothing else can justify the adoption of any value or any system of values (*Sartre, Being and Nothingness*).

[662] This interpretation of the objectivist theory of values was formulated by Gabriel Marcel in his book *Les Hommes Contre l'Humain*. Moreover, advocating the objectivist theory of value, René Le Senne, in his book *La Destinée Personnelle*, argues that the ego cannot produce values, since the very fact that it seeks for values implies that the ego cannot provide itself with values.

> observing such a novel spectacle, some comedian decided to show everyone that a monkey is nothing more than a monkey. While they all shouted and applauded at the skill of the monkey, the comedian threw some sweets onto the stage that monkeys particularly like. As soon as the monkey saw the sweets, it forgot the dance, the applause, and the elaborate costume, and dashed around, groping with its paws for the sweets; and since its dress interfered, it began to tear it apart with its nails, attempting to remove it. And in place of praise and amazement, laughter broke out among the spectators.[663]

The value of moral consciousness has been scrutinized and disputed by a multifaceted current of moral skepticism. The origins of moral skepticism can be traced in the reasoning of ancient Greek sophists, such as Gorgias and Thrasymachus, as well as in the reasoning of such modern philosophers as Michel de Montaigne and Blaise Pascal. Furthermore, moral skepticism was reinforced by theorists and practitioners of *Realpolitik* in the nineteenth and the twentieth centuries, and it was provided with postmodern intellectual underpinnings in the late twentieth century and in the early twenty-first century, not only by postmodern scholars but also by politicians who understand and practise politics more in terms of rhetoric, public relations, and a trade of legislation[664] than in terms of principles and convictions.

Indeed, moral consciousness can be judged according to the stability or the instability of its manifestations; and, indeed, a student of world history and, generally, any careful observer of social life can identify cases in which moral consciousness remains unvaried and other cases in which moral consciousness exhibits variations in its operation. These changes in the state of moral consciousness have led skeptical philosophers and social theorists to articulate negative evaluations of moral consciousness. However, moral skepticism, in general, has failed to recognize that, even though moral consciousness is subject to change, the changes that take place in the field of moral consciousness are not arbitrary, but they are characterized by progressive patters. Hence, it is important to find answers to a number of difficult methodological and epistemological questions that have always bedeviled the social sciences, and they are raised in acute form when attention is drawn to the subject matter of historical sociology. At the heart of these questions is the problem associated with the relationship between agency and structure, which lies at the core of the philosophy of rational dynamicity.

According to the distinguished American social anthropologist and social theorist Clyde Kluckhohn, (social) values play a substantial role in the integration and the maintenance of the social body and of the personality of the human being, thus underpinning both social identity and individual identity.[665] This awareness echoes Augustine of Hippo's germane argument that a commonwealth (a "people") is made of rational humans united by love of a common thing.[666] Thus, civilizations can be judged according to their fundamental underpinning values. Moreover, in this way, we can explain why, when scientific and

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[663] *Patrologia Graeca*, vol. 46, 240C.

[664] In 2015, Thomas Ferguson (Professor of Political Science at the University of Massachusetts), Paul Jorgensen (Assistant Professor of Political Science at the University of Texas Rio Grande Valley), and Jie Chen (University Statistician at the University of Massachusetts) co-authored and published a rigorous analysis of the strong correlation between campaign funding and electability to the U.S. Congress (Ferguson, Jorgensen, and Chen, "How Money Drives US Congressional Elections"). Moreover, Michael Parenti has rigorously analyzed the foreign policy of the United States in relation to the system of class power within the United States (Parenti, *Power and the Powerless*).

[665] Kluckhohn, "Values and Value-Orientation in the Theory of Action."

[666] Augustine, *De Civitate Dei*, Book 19.

technological achievements are used by agents that represent morally superior civilizations, they produce and underpin good historical results, whereas, when scientific and technological achievements are used by agents that represent morally inferior civilizations, they produce and underpin evil historical results.

In the history of civilization and political theory, the term "Enlightenment" refers to a movement of intellectual change that expanded throughout Europe (and America) during the eighteenth century. The essential goal of the Enlightenment was to emancipate human reason from the thraldom of prejudice and superstition (and especially from the established systems of political and religious despotism), and to apply it to the cause of social and political reform in accordance with the intentionality of rational human consciousness. The Enlightenment emphasized the presence of human consciousness in the world, and, from this perspective, it can be regarded as the major underpinning of modern Western civilization's superiority vis-à-vis other civilizations, which have not assimilated the essence of the Enlightenment. However, during industrialization, the modern Western world was gradually conquered by capitalism, due to which modern West ultimately betrayed and abandoned its greatest achievement, namely, the Enlightenment. As a result of the subjugation of modern West to capitalism, the Enlightenment's rationalism (as it has been explained and highlighted by Kant) was gradually substituted with the rationalism of the "homo economicus" ("economic man"), whose rationalism was, ultimately, subjugated to selfish passions, thus becoming a shadow of itself. Moreover, as a result of the subjugation of modern West to capitalism, the Enlightenment's arguments for the rational emancipation of humanity (as they have been put forward and explained by Kant) were gradually substituted with the emancipation of capital. On the other hand, in the twentieth century, the essence of the Enlightenment was preserved and defended in Europe by the socialist movement as it was represented by such scholars and ideologues as Alexander Bogdanov, Antonio Gramsci, Ágnes Heller, Karel Kosík, Alec Nove, etc., as well as by several social-liberation movements around the world. Moreover, the development of cybernetics in the Soviet Union and other socialist states that where strategic allies of the Soviet Union was a continuation of the spirit of the Enlightenment in the twentieth century, after the capitalist West's betrayal of the Enlightenment. In fact, the 22nd Congress of the Communist Party of the Soviet Union (1961) declared that cybernetics was one of the "major tools of the creation of a communist society," provoking opposition and subversive reactions from obscurantist Soviet factions and the Western capitalist "camp." From the perspective of the philosophy of rational dynamicity, and according to the arguments that I have put forward in my book *Taking the Bull by the Horns: Causes, Consequences and Perspectives in Politology and Political Economy* (originally published in Greek in Athens, Greece, in 2021, by ΚΨΜ Publications), socialism is both the perfection of modern liberalism and the dialectical transcendence of modern liberalism toward higher levels of rationalism and humanism.

A scientifically rigorous study of the history of civilization, in general, and of the history of politics, in particular, allows one to identify and analyze progressive patterns of development of moral consciousness. As Michele Moody-Adams has pointed out, "moral progress in belief involves deepening our grasp of existing moral concepts, while moral progress in practices involves realizing deepened moral understandings in behavior or social institutions."[667] Allen Buchanan's and Russell Powell's typology of moral progress is particularly helpful in order to identify and analyze progressive patterns of development of moral consciousness[668]: As regards moral concepts themselves, the understanding of the virtues, moral reasoning, and moral motivation, world history exhibits the following

[667] Moody-Adams, "The Idea of Moral Progress."
[668] Buchanan and Powell, *The Evolution of Moral Progress.*

progressive patterns: (i) an increasing *specification* of morality, in the sense that moral rules tend to be increasingly differentiated from religion and law; (ii) an increasing *internalization* of morality, in the sense that, as Kant has pointedly argued, moral consciousness evaluates actions not only on the basis of their consequences, but also on the basis of the agent's motives; (iii) an increasing *individualization* of morality, in the sense that, in addition to group rights, individual rights are increasingly highlighted, esteemed, and protected, especially as people mature psycho-spiritually; (iv) an increasing *expansion* of morality, in the sense that human rights (namely, "rights we have simply because we exist as human beings—they are not granted by any state"[669]) are increasingly highlighted, esteemed, and protected, especially as modernity is consolidated and develops. As regards the understanding of moral standing, moral statuses, and justice, world history exhibits the following progressive pattern: humanity's increasing desire and efforts to create and impose new institutions in order to achieve higher levels of justice. The abolition of slavery and the development of international law on the basis of the International Bill of Human Rights[670] are two characteristic cases in point. As regards the proper moralization of humanity and the understanding of the nature of morality, world history exhibits the following progressive pattern: several manifestations of inhuman and degrading treatment that were considered to be normal in previous societies (e.g., child abuse, torture, gender discrimination, various forms of political and spiritual despotism, etc.) are unacceptable to and morally condemned by the modern human being, and, even though, in the contemporary world, human rights abuses continue to take place, the authorities that are responsible for or involved in such types of immoral behavior try to find justifications for them, and they usually do not dare to commit human rights abuses in a blatant way.

However, the identification of the above progressive patterns of development of moral consciousness can only partially refute the arguments of moral skepticism. The history of humanity is characterized by both cases of moral progress and cases of moral setback. In fact, one can discern whole segments of historical space-time that are overwhelmed by morally negative and unacceptable situations, such as those caused by capitalist oligarchies during the "Long Depression" (1873–96) and the "Great Depression" (1929–39), the twentieth-century fascist/Nazi regimes, the twentieth-century regimes of bureaucratic socialism, etc. Therefore, neither the viewpoint that is focused on progressive patterns of development of moral consciousness nor the viewpoint that is focused on the instability of moral consciousness and on cases of moral setback can lead to a comprehensive and rigorous way of understanding the dynamics of moral consciousness.

In order to obtain a comprehensive and rigorous way of understanding the dynamics of moral consciousness, one must extricate oneself from both the intellectual shackles of moral progressivism (i.e., the viewpoint that is focused on progressive patterns of development of moral consciousness) and the intellectual shackles of moral skepticism (i.e., the viewpoint that is focused on the instability of moral consciousness and on cases of moral setback), and to search for those structural elements of moral consciousness that enable one to argue that moral consciousness is characterized by structural stability. By maintaining that one should inquire into the structural stability of the operation of moral consciousness, I mean that one should inquire into the qualitative features of moral consciousness that are recurrent. In particular, if we inquire into the contents of moral values, then we realize that, in different

[669] This is the basic definition of "human rights" according to the United Nations: https://www.ohchr.org/en/issues/pages/whatarehumanrights.aspx.

[670] See: https://www.ohchr.org/en/issues/pages/whatarehumanrights.aspx.

segments of historical space-time, different values were placed at the apex of the corresponding "moral pyramid."

For instance, as regards the prevailing moral criterion, the study of the history of the European and the modern American civilizations implies the following: in early Antiquity, the prevailing moral criterion was bravery, and the corresponding anthropological ideal was a hero; in classical Antiquity, the prevailing moral criterion was education, and the corresponding anthropological ideal was a wise person or a philosopher; in late Antiquity, the prevailing moral criterion was sanctity (in the sense of psychical beauty); in the Middle Ages, the prevailing moral criterion was chivalry (with its integrated religious, moral, and social code); in the seventeenth-century French society, the prevailing moral criterion was honesty (paving the way for the conception of the modern nation-state as the social space in which "honesty" is manifested and becomes meaningful and, thus, underpinning nationalism and the French notion of the "human of the State" ("homme d'État")); in the nineteenth-century British society, the prevailing moral criterion was social success, and the corresponding anthropological ideal was a person who complies with the Victorian model of social discipline and control (underpinning Great Britain's capitalist system and imperial policy); in the nineteenth-century American society, the prevailing moral criterion was individual success, namely, success that originates from and is based on an individual's own actions, thoughts, and will, and the corresponding anthropological ideal was a "self-made individual" (in particular, the phrase "self-made man" was coined on 2 February 1842 by Henry Clay Sr. in the United States Senate to describe individuals whose success was an entirely individual achievement, and, by the mid-1950s, "success" in the U.S.A. generally implied "business success," underpinning the United States' capitalist system and neo-imperial policy, ideologized in a systematic and radical way by the American economist Milton Friedman).

Additionally, nationalism has significantly contributed to the relativization of many people's perceptions of morality and rationality. The age of nationalism in its most precise sense is usually dated from the eighteenth century, and it is intimately related to the American and the French revolutions. However, the European system of nation-states had already emerged from the European wars of religion, which began in the sixteenth century (after the Protestant Reformation). In particular, the aforementioned system was a system of sovereign princes whose cultural rivalries were kept in check by the principle "whosoever's territory, his religion" ("cuius region eius religio"), and whose political rivalries were kept in check by a system that is known as the "balance of power." The system of balance of power was weakened but not destroyed by the revolutions that broke out in the nineteenth century, and it was restored by the so-called Holy Alliance after the defeat of Napoleon Bonaparte in 1815. However, nationalism and its states-system were internationalized in the aftermath of the First World War. The principles of national sovereignty and national self-determination have been systematically used and invoked by nationalists in order to counter classical ontology, and especially its quest for universal values and principles, as well as in order to equip national bourgeois elites and state bureaucracies with ultimate authority over moral questions and with powerful means for the conduct of psychological operations (for instance, the rhetoric about "patriotism" and "national security" has often served as a pretext for the violation of human rights and liberties by national governments and for the development of the industry of war).

Even though, in different segments of historical space-time, different values were placed at the apex of the corresponding "moral pyramid," certain values, such as "veracity," "uprightness," "accountability," "strength," and "perseverance," irrespective of the particular ways in which they are interpreted by different human communities, seem to have been exerting indisputable moral authority over humanity throughout its known history. In addition, if we inquire into the forms of moral values, and if we approach morality in a formalist way, then we realize that, in every segment of historical space-time, humanity makes a fundamental distinction between "good," perceived as moral positivity, and "evil," perceived as moral negativity. Therefore, by inquiring into the contents of moral values and into the forms through which moral values are manifested, we realize that moral consciousness has some recurrent qualitative features (meaning that it is characterized by structural stability).

It goes without saying that the social system exerts *significant influence* over moral consciousness, and the latter internalizes and reflects social values. But the social system *does not create* moral consciousness itself, and moral consciousness can always judge and change the established system of values, instead of passively complying with it. Hence, moral consciousness seems to be an innate attribute of the human being. In particular, through a combination of sentiments, volition, and reason, moral consciousness obtains a conception of the "good," and it determines the conditions under which the "good" can be historically objectified and, thus, become historically meaningful.

Let us consider the transition from feudalism to capitalism, and the birth of socialism. Feudalism, the dominant system in medieval Europe, was a system characterized by a rigid social stratification, according to which everyone had a rigidly instituted position within an "organic whole," whose major constituent components were the class of the feudal lords, the class of the serfs, and the church, whose major social role was to maintain a balance between the feudal lords and the serfs through religion. By the late Middle Ages, the bourgeois class (namely, a social class of professionals who were neither feudal lords nor serfs) deprecated the political, economic, and spiritual despotism of the feudal system, it revolted against feudalism, and it proclaimed that the social position of an individual should not be determined by feudal institutions, but it should be freely determined by individual action and by the interaction between individuals in the context of a free and fair society. One of the most characteristic examples of a bourgeois revolution in the modern era is the French Revolution of 1789, whose major motto was "Liberty, Equality, Fraternity." However, the elite of the bourgeois class conceived capitalism as the embodiment of human freedom in the domain of economics, and, for this reason, after the displacement of feudalism by capitalism, the liberty and the rights of the human individual were gradually largely displaced by and subordinated to the liberty and the rights of the capital itself and the capitalist elite. By the middle of the nineteenth century, the European peoples realized that capitalism had displaced feudalism, but, instead of ushering in liberty, equality, and fraternity among the people, capitalism tends to replace the authoritarian and exploitative relationship between the feudal lords and the serfs with a new authoritarian and exploitative relationship, namely, that between the capitalist class and the proletariat (working-class).[671] Therefore, socialism emerged as a criticism of and a revolt against capitalism, just as the bourgeois ideology had previously emerged as a criticism of and a revolt against feudalism. In fact, the term "socialism" first appeared in 1832 in *Le Globe*, a liberal French newspaper of the French philosopher and political economist Pierre Leroux, and, by the 1840s, socialism had already become the object of rigorous social-scientific analysis by the German economist and sociologist Lorenz von

[671] See, for instance: Engels, *The Condition of the Working Class in England.*

Stein. Moreover, the English socialist intellectual and activist Thomas Hodgskin (1787–1869) articulated a thorough critical analysis of capitalism and of the labor class under capitalism, and his writings exerted a significant influence on subsequent generations of socialists, including Karl Marx. In particular, from the perspective of Thomas Hodgskin, socialism signifies an attempt to create a free and fair market, in the context of which production and exchange are based on the labor theory of value (freed from exploitative institutions) as part of natural right, which endows moral consciousness, the freedom of the individual, social justice, and social autonomy with ontological underpinnings (in accordance with Thomas Hodgskin's deism).

However, it is worth mentioning that one of the biggest mistakes of Karl Marx and the orthodox Marxist "school" of socialism is that they maintain that the human being is a product of historical becoming, in the sense that human consciousness reflects the historical environment. This argument contradicts the dynamic nature of the history of world civilization, and it implies that there is no moral barrier to molding moral consciousness anyway one wishes. If no element of moral consciousness, not even an inner drive to act freely and creatively, is innate to the human being, then there is no moral reason for instituting a free society, and then a ruling class (e.g, the Central Committee of the Communist Party, the ruling elite of a fascist state, the capitalist-corporate elite, a religious elite, etc.) can mold humans into being what it thinks they ought to be.

Furthermore, given that, in the era of advanced modernity, one of the most complex and heated debates is the debate about the positive and the negative consequences of artificial intelligence (i.e., of intelligent behavior demonstrated by machines), I should mention that the dialectic of rational dynamicity, as I expounded it in section 1.3.3, is an adequate method for relating the multiplication of choices that arises as a result of artificial intelligence to moral criteria and for determining the moral criteria and the norms that should govern the development and the implementation of artificial intelligence. The dialectic of rational dynamicity provides a creative and rigorous way of understanding and handling the relationship between agency and structure as well as the relationship between change and control. In particular, the application of the dialectic of rational dynamicity to artificial intelligence ensures the development of artificial intelligence according to the intentionality of human consciousness and not according to an autonomous inner momentum of artificial intelligence.

From the perspective of the philosophy of rational dynamicity, a society that operates according to the dialectic of rational dynamicity chooses its values not in order to be enslaved to these values, or to a tradition, but because its values are characterized by rational dynamicity, and they are susceptible to criticism according to the dialectic of rational dynamicity. In other words, a society that is founded on the dialectic of rational dynamicity lacks neither tradition nor values, but it refuses to accept all those traditions and values which are not characterized by rational dynamicity and are not susceptible to criticism. A society that is founded on the dialectic of rational dynamicity stresses the right and the ability of criticism neither as an end-in-itself nor as a pleasant habit, but as an expression of a human community's decision to implement the dialectic of rational dynamicity as a method of policy-making.

In the scholarly discipline of economics, the moral concept of a value is objectified and manifested as the economic concept of a price, and, in the scholarly discipline of politics, the moral concept of a value is objectified and manifested as the political concept of a norm or a law. In accordance with Louis Lavelle, I maintain that values are judgments, whereas prices, political norms, and laws are facts.

Every human organization (whether political or economic or cultural) is a deterministic nonlinear feedback system, because it is characterized by decision-making rules and by specific interpersonal relations between the people who belong to the same organization or to different organizations—in fact, this is what social scientists mean by the term "institutional framework." The feedback loops that are created by people when they interact with each other, that is, when they form a network, are nonlinear because of the following reasons: (i) in human systems, the actors' choices are based on subjective perceptions that lead to disproportionately big or small reactions; (ii) there are almost always many possible outcomes that can follow an action; (iii) due to the action of structural forces, group behavior is something more than the mere sum of individual behaviors; (iv) outcomes are usually individual; and (v) small changes can escalate and lead to outcomes of major significance. Hence, the assumption of nonlinearity is necessary in order to formulate models that can account for the five aforementioned characteristics of human systems. The linear models that are often used in the context of classical macroeconomics and in classical microeconomics are only approximations of the actual state of affairs in macroeconomics and in microeconomics within a specific historical context.

In any deterministic nonlinear feedback system, actors must necessarily move around nonlinear feedback loops, which are formed by the corresponding institutional framework, and it is exactly for this reason that the system within which actors act is deterministic. On the other hand, every time an actor moves around such a loop, one is free to transform, ignore, or even overthrow the given institutional framework, because actors follow decision-making rules and specific models of behavior, but these rules and these models allow freedom of choice, that is, they are subject to change (this is the reason why, for instance, human history includes scientific breakthroughs, business innovations, social revolutions, changes in legislation, changes in morals and customs, etc.). Therefore, on the one hand, social actors cannot escape from the fact that the interactions between them have the character of a nonlinear feedback system, nor can they escape from the consequences of this nonlinear feedback, but, on the other hand, economic actors can, indeed, change the rules and the patterns that govern their behavior on different occasions, in accordance with their intentionality. The consequences that free choice has for the system can be divided into the following three categories:

i. *Stable outcomes:* If all social actors accept a given set of decision rules and make their choices according to these rules, then the whole system will end up in a state of stable equilibrium (i.e., it will exhibit a "regular" behavior). In this case, the corresponding system operates on the basis of negative feedback, which underpins the exhibition of regular, predictable behavior.
ii. *Unstable outcomes:* If all social actors continuously change the rules that govern their behavior, then none of them will be able to depend on others, and the whole system will be attracted to a state of unstable equilibrium due to positive feedback. In other words, as the level of conflict ("social entropy") increases in a human system, then this system leaves a state in which it is attracted to stability and moves toward a state in which it is attracted to a behavior of unstable equilibrium.

iii. *A state of rational dynamicity:* The alternative to either stability or instability lies in the border between them—specifically, in a state of rational dynamicity—where both negative feedback and positive feedback, both stability and instability, operate simultaneously to cause the emergence of changing patterns of behavior. If an organization is attracted only to the state of behavior that we call stability, then it will cease to be creative; and, if an organization is attracted only to the state of behavior that we call instability, then it will be dissolved. When a nonlinear feedback system operates in a state characterized by rational dynamicity, then it contains and utilizes both forces of stability and forces of instability in a rational way, in accordance with the given system's entelechy and intentionality. Therefore, due to the operation of forces of instability, the behavior of such a system is not totally algorithmizable, but, due to the operation of forces of stability, there is an identifiable qualitative structure in such a system's behavior.

As I explained in Chapter 2, many mathematical models of physical and social systems consist of differential equations. The concept of a dynamical system is a generalization of the concept of a differential equation.[672] Let us consider a differential equation of the form

$$x' = f(t, x),$$

where $t \in \mathbb{R}$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ (i.e., a system of differential equations),

$$f: D \to \mathbb{R}^n$$

is a continuous function, and $D \subset \mathbb{R} \times \mathbb{R}^n$ is an open and connected set. A solution of the aforementioned differential equation is a function

$$x_0: (a_0, b_0) \to \mathbb{R}^n \text{ with } x_0'(t) = f(t, x_0).$$

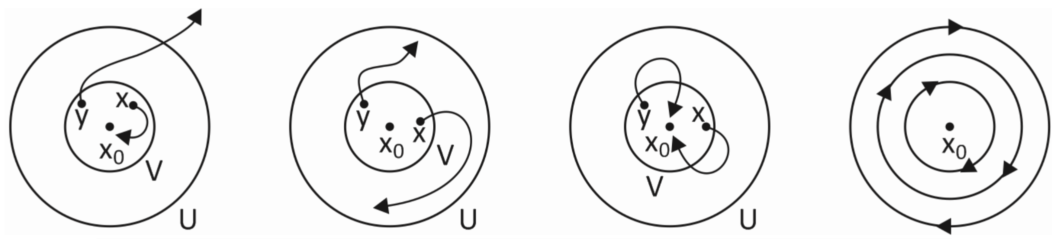
A solution that passes through the point $(t_0, v_0) \in D$, which represents the initial condition of the given system, has the property that $x_0(t_0) = v_0 \in \mathbb{R}^n$. The solution with initial condition (t_0, v_0) is defined in a "maximum" interval (a_0, b_0). A (topological) "dynamical system" is defined to be a 3-tuple $(\mathbb{R}, X, \varphi)$ where X is a locally compact metric space (i.e., for each $x \in X$, there exists a neighborhood N of x such that the closure $Cls(N)$ is compact; by definition, N is open), and $\varphi: \mathbb{R} \times X \to X$ is a continuous mapping (known as the "action mapping" of the given dynamical system) that has the following properties:

i. $\varphi(0, x) = x, \forall x \in X$, and
ii. $\varphi\big(t_1, \varphi(t_2, x)\big) = \varphi(t_1 + t_2, x), \forall t_i \in \mathbb{R} \,\&\, x \in X$.

The "orbit" of $x \in X$ is defined as $\mathbb{R}(x) = \{\varphi(t, x) | t \in \mathbb{R}\} \subseteq X$. In general, an "orbit" is a collection of points related by the action mapping of the given dynamical system. From the perspective of the theory of dynamical systems, "stability" refers to the behavior of the orbits

[672] See: Hirsch and Smale, *Differential Equations, Dynamical Systems, and Linear Algebra.*

of a dynamical system (resp. of the solutions of a differential equation) “around” a periodic orbit (resp. a solution). Let us now study the stability of a dynamical system, as shown in Figure 3.4.[673]



(a) (b) (c) (d)

Figure 3.4. The Stability of a Dynamical System.

Let $x_0 \in \mathbb{R}^n$ be a pointwise orbit of the dynamical system $(\mathbb{R}, X, \varphi)$, where $X \subset \mathbb{R}^n$ is open. Then x_0 is said to be a “Lyapunov-stable fixed point” of the given dynamical system if, for every neighborhood U of x_0, there exists a neighborhood $V \subset U$ of x_0 such that $x \in V \Rightarrow \mathbb{R}^+(x) \subset U$, as shown, for instance, in Figure 3.4(b); x_0 is said to be an “asymptotically stable fixed point” if it is Lyapunov-stable, and, additionally, it holds that $tx \to x_0$ as $t \to +\infty$ and for $x \in V$, as shown, for instance, in Figure 3.4(c); a fixed point $x_0$of a dynamical system may be Lyapunov-stable without being asymptotically stable, as shown, for instance, in Figure 3.4(d); otherwise (i.e., if x_0 is not stable), x_0 is said to be an “unstable fixed point,” as, shown, for instance, in Figure 3.4(a).

Inextricably linked to moral consciousness is intentionality. Intentionality gives an external direction to the states of consciousness, and, thus, intentionality can operate according to a hierarchy of relations that range from a minimum to a maximum. The levels of this hierarchy can be called “orders.” The intentionality chain is underpinned by rational dynamicity.

From the perspective of the philosophy of rational dynamicity, the reason of the beings that exist in the world consists in the way in which they participate in the corresponding species/form, in their entelechy, in the way in which they relate to each other in the context of the cosmic harmony and order, and in the way in which they express their intentionality. Therefore, “reason” includes both the concept of the efficient cause (which refers to one’s participation in the corresponding species) and the concept of the final cause (which refers to one’s entelechy). Furthermore, “reason” refers to the relationship of participation in the formation of the entire world as well as to the intentionality of consciousness. On 4 February 1923, the noted British-Indian geneticist and physiologist John Burdon Sanderson Haldane read a paper entitled “Daedalus, Or Science and the Future” to the Heretics Society at the University of Cambridge, in which he highlighted the development of reason as a liberating process by arguing that “the conservative has but little to fear from the man whose reason is the servant of his passions, but let him beware of him in whom reason has become the greatest and most terrible of the passions,” since “such men are interested primarily in truth as

[673] Ibid.

such, but they can hardly be quite uninterested in what will happen when they throw down their dragon's teeth into the world."[674]

The dialectic of rational dynamicity precludes and discards any fundamentalist, ossified response to *practical* questions about social organization. According to the philosophy of rational dynamicity, we should never short-circuit a *practical* question about social organization by having a fundamentalist, ossified mentality that a particular course of action is *necessarily and universally* the optimum course of action, while an alternative course of action should be *necessarily and universally* discarded. Thus, according to the philosophy of rational dynamicity, one should be *strategically immutable* (being firmly oriented toward specific existential goals and values) but *tactically flexible*, exactly in order to better serve one's strategic vision by applying discretion. The dialectic of rational dynamicity implies that social problems can be properly addressed only when one deals with the structure of an organization, irrespective of its formal characteristics and status. For instance, from the perspective of the dialectic of rational dynamicity, the dilemma of choosing between a market economy that is controlled by a private oligarchy and a planned economy that is controlled by an oligarchy of state officials is ultimately a deceptive dilemma, or at least an ill-posed dilemma, because both of these models are structurally similar, namely, they are oligarchies. One can argue that a private oligarchy aims to impose a social system characterized by Lyapunov-stability, as described in Figure 3.4(b), while a state oligarchy aims to impose a social system characterized by asymptotic stability, as described in Figure 3.4(c). As I have already explained, Lyapunov-stability means that a system starting in some ball of radius δaround the equilibrium will not leave a ball of radius ε around the equilibrium for some $\varepsilon > \delta$; and, in a system of private/liberal oligarchy, the corresponding parameter ε is determined and dictated by the interests, the perceptions, and the commands of the ruling private/liberal oligarchy. Moreover, as I have already explained, asymptotic stability means that a system starting in some ball of radius δ around the equilibrium will converge to the equilibrium; and, in a system of state/authoritarian oligarchy, the corresponding equilibrium, to which the established social system is supposed (and forced) to converge, is determined and dictated by the interests, the perceptions, and the commands of the ruling state/authoritarian oligarchy.

The ancient Greek society—especially the city of Athens from the eighth to the fifth century B.C.—is the first society in the history of humankind that founded its operation on a continuous critical evaluation and re-evaluation of its own self. It is the first society in the history of humankind that demands from itself to be able to explain (to itself and to others) why it is what it is. Thus, whereas all societies institutionalize mechanisms for the coercive imposition of their constitutive "logos," or reason-principle (specifically, their constitutive values and norms), on their members, the Athenian democracy is the first society in the history of humankind that institutionalized mechanisms of rational self-control (that is, a reflective political attitude), too. Additionally, contrary to twentieth-century diplomatic clichés, domestic and foreign policy are strongly interrelated, not only because, in a democracy, in explaining what needs to be done, a statesman needs to have the conscious approval of the people, but also because, as the distinguished U.S. statesman and scholar Joseph Nye, Jr. has pointed out, "it is tautological or at best trivial to say that all states try to

[674] Online: http://bactra.org/Daedalus.html.

act in their national interest," and "the important question is how leaders choose to define and pursue that national interest under different circumstances."[675]

Nye's aforementioned arguments are reminiscent of Plato's *Statesman*. In his *Statesman*, Plato maintains that the real acquisition of real political power requires a deep awareness of historical becoming, which, in turn, requires a deep awareness of the teleology of historical action. According to Plato, a real statesman is one who is aware that, even though one has to act in the context of historical becoming, one has to transcend the logic of historical becoming itself, since, according to Plato, the purpose of a real statesman is "poīesis" (genuine creation) and not merely managerial tasks. Thus, in his *Statesman*, 261a–b, Plato poses the following question: "Take the case of all those whom we conceive of as rulers who give commands: shall we not find that they all issue commands for the sake of producing something?" A real statesman acts within a world marked by necessities, but the ultimate purpose of a real statesman's political activity is the actualization of one's strategic vision, which transcends the corresponding historical context, and, according to Plato, it should be guided by values that transcend the logic of one's historical conditions. Moreover, the end of "poīesis" is known prior to action (in Greek "prāxis"), and, therefore, it is characterized by a mode of disclosing that Aristotle called "tēchne" (art proper), which can be construed as "authoritative knowledge" or "expertise." According to Aristotle,[676] "phrōnesis" is a bridging concept between theory and practice, and, in fact, it serves to mediate the gap between theory and practice in such a way that practice refers to a kind of objective knowledge (thus highlighting the reality of the world), and theory refers to a personalistic view of the world (thus highlighting the reality of consciousness). It goes without saying that the philosophy of rational dynamicity is intimately related to and in alignment with the aforementioned Aristotelian synthesis.

As we read in Thucydides's *History of the Peloponnesian War*, 2:35–2:46, Pericles (ca. 495–29 B.C.), one of the most prominent and influential statesmen and generals of Athens, argued that, in the Athenian political system, the principles of freedom and respect for law are united, and the activities of the city are based on the voluntary co-operation and on the critical thought of the citizens. Democracy, as it was invented and institutionalized in classical Athens, is neither a technical procedure nor a bourgeois political ritual, but a humanistic political culture, which is based on the premise that every law and every social institution are continuously and freely subject to rational criticism as regards their merits and their historical relevance. Thus, each law that was passed by the assembly of classical Athens bore the following clause: "It seemed good to the *dēmos* [people] and the *Boulē* [Council]." The philosophy of rational dynamicity advocates and reinforces the aforementioned classical democratic ethos by equipping its partakers and practitioners with the rationality and the dynamism that are necessary in order to make such a model of social organization historically meaningful and effective.

A being exists truly if and to the extent that it is united with its reason, it manifests its reason, and it practically affirms its reason. According to classical Greek political philosophy, the essence of politics consists in the provision and the maintenance of those existential conditions which allow, encourage, and help humans to exist truly in the aforementioned sense. In particular, the ancient Greek "polis"(city-state) has a unique characteristic on the

[675] Nye, Jr., "Why Morals Matter in Foreign Policy."
[676] Aristotle, *Nicomachean Ethics*, Book VI.

basis of which and due to which the institution of "polis" has been differentiated from other forms of organized collective behavior, and has given rise to the notions of "political art," "political virtue," and "political science." This unique characteristic of the ancient Greek conception of a "polis" consists in a collective attempt to institute a community whose "telos," or existential purpose, is not exhausted in the management of needs, but it is an attempt to live in harmony with the principle of truth, which, according to Aristotle's *Nicomachean Ethics*, X and II–VI, signifies the disclosure of reason. Therefore, from the perspective of the philosophy of rational dynamicity, the purpose of a polity should be to provide and maintain those existential conditions which allow and enable humans to live in a rationally dynamic way, specifically, according to the dialectic of rational dynamicity.

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